A COURSE OF MATHEMATICS.

IN TWO V LUMES.

COMPOSED E USE OF

THE ROYAL MILITARY ACADEMY.

BY

CHARLES HUTTON, LL.D. F.R.S.

FORMERLY PROFESSOR OF MATHEMATICS IN THAT INSTITUTION.

CONTINUED AND AMENDED BY

OLINTHUS GREGORY, LL.D. F.R.A.S.

VOL. I.

TWELFTH EDITION,

WITH CONSIDERABLE ALTERATIONS AND ADDITIONS,

BY

THOMAS STEPHENS DAVIES, F.R.S. L. & E. F.S.A.

ROYAL MILITARY ACADEMY.

LONDON:

LONGMAN, ORME, & CO.; J. M. RICHARDSON; J. G. F. & J. RIVINGTON; HAMILTON & CO.; WHITTAKER & CO.; DUNCAN & MALCOLM; SIMPKIN, MARSHALL, & CO.; J. SOUTER; COWIE & CO.; SMITH, ELDER, & CO.; ALLEN & CO.; HARVEY & DARTON; HOULSTON & STONEMAN; H. WASHBOURNE; L. A. LEWIS; C. DOLMAN; AND G. ROUTLEDGE. EDINBURGH: STIRLING & CO.

1841.

LONDON:
GILBERT AND RIVINGTON, PRINTERS,
ST. JOHN'S SQUARE.

THE

EDITOR'S PREFACE.

The circulation of nearly thirty thousand copies of Hutton's Course, sufficiently attests the estimation in which it has been held by mathematical teachers and students throughout the country.

"Long experience in all seminaries of learning," says the author in his original preface, "has shown that such a work was very much wanted, and would prove a great and general benefit; as for want of it, recourse has always been obliged to be had to a number of other books by different authors; selecting a part from one and a part from another, as seemed most suitable to the purpose in hand, and rejecting the other parts—a practice which occasioned much expense and trouble, in procuring and using such a number of odd volumes of various forms and modes of composition; besides wanting the benefits of uniformity and reference which are found in a regular series of composition."

Dr. Hutton's Course of Mathematics was greatly in advance, as to the manner of treating the subjects contained in it, of all works which had then appeared; and for many years the author continued to improve the successive editions, as new discoveries were made or new methods invented. At the close of his long, laborious, and useful life, he committed the work to the care of Dr. Gregory, who, by continual additions and modifications, endeavoured to assimilate it to the growing spirit of inquiry produced by a long period of general peace. Into the last edition, however, greater changes were introduced than had been made in the work since its first composition; and he did me the honor, soon after my appointment to this Institution, to request me to make the greater part of those alterations, under his editorship. It was, however, a matter of deep regret to both of us, that owing to the haste with which the work was urged through the press, adequate time was not given to complete our contemplated improvements. The same cause, also, gave rise to a great number of errata. I am now, however, not without a hope, that the present edition, whilst as free from errata as any mathematical work extant, will be found

to justify the views under which the alterations were commenced, and to give it that preference as a text-book for mathematical instruction which the original work so long enjoyed.

The state of the health of my lamented friend and coadjutor not allowing him to give the attention essential to the editorship of the work, has committed it wholly to my care, to carry out our joint views to the best of my ability. It was not, however, without some reluctance and much anxiety, that I undertook it: and for more than twelve months the present volume has been the unremitting object of my entire labour. Even yet, I am obliged to defer a few of our contemplated improvements for a future edition.

In the *arithmetic* very little alteration has been made, except a few occasional notes; and in the early part of the *algebra* comparatively few essential alterations have been made from the last edition. In the multiplication and division I have given prominence to the use of *detached coefficients* and the *synthetic method of division.* An elementary investigation of the latter process is annexed, as that of Mr. Horner is not easily understood except by students whose progress is considerably more advanced: but a still simpler and more direct one is given amongst the "Additions" at the end of the volume, and which I discovered since sheet K, (p. 128), was printed off. To the simpler operations of algebra, where the reason of the step is not apparent at once, investigations are annexed, to secure to the student a complete understanding of the logic of his processes.

In the chapters on *simple and quadratic equations*, the introductory remarks and suggestions, as well as the examples chosen for illustrating the methods by actual working, have been generally exchanged for others better adapted to show the true character of the operations. In the quadratics, the Hindû method of completing the square is enforced, as being generally superior, in respect of facility, to the Italian or common one.

The chapter on the *general resolution of numerical equations* has been wholly recomposed; and I hope it will be found free from those logical defects which are so liable to insinuate themselves into abbreviated treatises on subjects involving so many distinct principles as this does. The theory of equations, is, however, carried no further than is requisite for numerical solution: though to this extent, great pains have been taken to render it logically complete. Legitimate proofs, on *elementary principles*, are given of the criteria of De Gua and Budan, for detecting the imaginary roots of an equation; and as brief a form of investigating Sturm's criterion as I could devise, has also been added. Though I am as fully impressed as any

one can be of the great beauty and importance of Sturm's theorem, I have been led, I confess, to introduce it here more in accordance with the dictum of the mathematical public, than from my own conviction of its *practical utility* in reference to numerical solution—at least till some method less operose and practically embarrassing than is yet known, shall be discovered, of forming his successive auxiliary functions subsequent to the first derivative. Should such a method, in any way analogous to Horner's process for transformation, ever be invented, Sturm's theorem will become practically useful:—but not till then *.

Upon Horner's method of *continuous approximation to the roots of equations*, I have dwelt at sufficient length to render it easy of comprehension. As the first attempt ever made to compose an elementary treatise on this subject was made by myself in the previous edition of this work, my attention was naturally directed to it subsequently with sufficient precision to enable me to separate the essential and the useful part of that composition, from the parts which I found superfluous, and make such additions as experience might suggest during my professional use of the volume.

The chapters on *indeterminate coefficients*, *piling of balls*, the *binomial* and *exponential theorems*, and on *logarithms*, it will be seen are all written anew, and with especial reference to the order in which the subjects naturally present themselves in a systematic course of study. The same may be said of the chapters on *series* and *finite differences.*

The early part of the *geometry* is unaltered, though in a future edition I propose to remodel it entirely. The *doctrine of ratio* is put altogether in a new and, I persuade myself, a perfectly logical form; and the theorems depending on ratio are changed in their manner of demonstration, to be in accordance with the same principles. A few theorems of great practical value are added. The chapters on the *geometry of planes and solids* have also, for the most part, been modified and rewritten.

The *practical geometry* has been entirely recomposed, and in especial reference to the circumstances under which the problems themselves occur in practice. A number of constructions of this kind, which are believed to be new, and are adapted to peculiar exigences, have been introduced.

* This page was in type before I met with the elegant and instructive *Mathematical Dissertations* of Professor Young. In that volume, an important improvement is made in simplifying the numerical process, and especially the initial step of it. A few more such improvements would entirely remove the objections which, in a practical point of view, are urged against that important and beautiful process.

Independently of the discussion of the different points connected with Sturm's criterion, the volume of Mr. Young deserves the attention of the mathematical student, beyond any other work that I could suggest to him: as he will be at once led to study the logic of the processes involved in elementary mathematics, with more precision than in any other production with which I am acquainted.

The chapter on *practical geometry in the field* contains a series of problems of great importance to the military profession, to engineers and surveyors, and which form the substance of a course of lectures just delivered at the Royal Artillery Institution.

In the *plane trigonometry* nothing besides the examples for exercise, of the last edition, remains in this. To give every thing essential to elementary trigonometry investigated in a direct and simple manner, and entirely to exclude all matters of mere scientific curiosity, has been my guiding principle in the composition of these chapters. Trigonometry, therefore, instead of forming two separate treatises in two successive volumes, is now brought entirely into the first; and the examples that are changed in place have been marked by a quotation of the places in which they previously stood, for the convenience of those who wish to make reference to any works founded on the preceding edition. The few additional examples here given, will offer no difficulty to those who have fully mastered those of that edition.

In the *mensuration* the investigations are, for the sake of continuity, classed differently, but the problems have the same order as before; and a few easy additional examples are added, as the paucity of these in the former editions has often been a subject of complaint. In the *artificers' work* and *land surveying* no changes are made in this edition, but a complete revision of them will be given hereafter,—and, but for practical obstacles, would have been given now, as no parts of the entire course require it more than these.

The figures in this edition are nearly all newly-cut, and every attention has been paid to the arrangement of each page, both for convenience of reading and reference, and of losing no space that could possibly be filled up with useful matter. Much of the phraseology, and the entire notation, of former editions has been modernised, and an attempt has been made to render it, with the exceptions already specified, consistent and systematic throughout.

T. S. DAVIES.

Royal Military Academy, Woolwich,
10th February, 1841.

CONTENTS.

THE alterations made in this edition have caused the paging of the work to be altered; and as in some works references are made to the last edition, it has been considered advantageous to give, besides the pages of this edition, the corresponding pages of the preceding one: the first column refers to this, and the second to the preceding edition. The few articles brought from the second volume are marked with an asterisk.

ARITHMETIC.

ALGEBRA.

*

Ex. 7, wi…
Remar…
p. 195

× Double P…

† Reciproc…
Equation…
p 212

THEORETICAL GEOMETRY.

PRACTICAL GEOMETRY.

PLANE TRIGONOMETRY.

MENSURATION.

A

COURSE

OF

MATHEMATICS,

&c.

DEFINITIONS.

1. QUANTITY, or MAGNITUDE, is that which admits of increase or decrease. Those kinds of magnitude only which are capable of estimation in comparison with some unit of the same kind, are the proper subjects of mathematical study.

2. *Arithmetic* is conversant with numbers only in their abstract state. *Algebra* contemplates the subjects of arithmetic in a more general form; and generally (among other objects) furnishes the rules for the more complex arithmetical operations. *Geometry* treats of space, as of the forms, magnitudes, and positions of figures. The differential and integral calculus, the calculus of functions, &c. are also branches of Algebra, but of which no definite idea could be conveyed till the student's progress is considerably extended.

3. The sciences of Arithmetic and Geometry are styled the *Pure Mathematics*: whilst all applications of them to physical, civil, or social inquiries, (as Mechanics, Astronomy, Optics, Life Insurance, Population, &c.) constitute what is termed the *Mixed Mathematics*.

4. In mathematics are several general terms or principles; such as, Definitions, Axioms, Propositions, Theorems, Problems, Lemmas, Corollaries, Scholia, &c.

5. A *Definition* is the explication of any term or word in a science; showing the sense and meaning in which the term is employed.—Every Definition ought to be clear, and expressed in words that are common, and perfectly well understood.

6. *A Mathematical Proposition* refers either to something proposed to be demonstrated, or to something required to be done; and is accordingly either a Theorem or a Problem.

7. *A Theorem* is a demonstrative Proposition; in which some property is asserted, and the truth of it required to be proved. Thus, when it is said that,

The sum of the three angles of a plane triangle is equal to two right angles, that is a Theorem, the truth of which is demonstrated by Geometry. A set or collection of such Theorems constitutes a *Theory.*

8. *A Problem* is a proposition or a question requiring something to be done; either to investigate some truth or property, or to perform some operation. As, to find out the quantity or sum of all the three angles of any triangle, or to draw one line perpendicular to another. *A Limited Problem* is that which has but one answer or solution. *An Unlimited Problem* is that which has innumerable answers. And a *Determinate Problem* is that which has a certain number of answers.

9. *Solution* of a Problem, is the resolution or answer given to it. *A Numerical* or *Numeral Solution* is the answer given in numbers. *A Geometrical Solution* is the answer given by the principles of Geometry. And a *Mechanical Solution* is one which is gained by trials.

10. *A Lemma* is a preparatory proposition, laid down in order to shorten the demonstration of the main proposition which follows it.

11. *A Corollary,* or *Consectary,* is a consequence drawn immediately from some proposition or other premises.

12. *A Scholium* is a remark or observation made upon some foregoing proposition or premises.

13. *An Axiom,* or *Maxim,* is a self-evident proposition; requiring no formal demonstration to prove its truth; but received and assented to as soon as mentioned. Such as, The whole of any thing is greater than a part of it; or, The whole is equal to all its parts taken together; or, Two quantities that are each of them equal to a third quantity, are equal to each other.

14. *A Postulate,* or *Petition,* is something required to be done, which is so easy and evident that no person will hesitate to allow it.

15. *An Hypothesis* is a supposition assumed to be true, in order to argue from, or to found upon it the reasoning and demonstration of some proposition.

16. *Demonstration* is the collecting the several arguments and proofs, and laying them together in proper order, to establish the truth of the proposition under consideration.

17. *A Direct, Positive,* or *Affirmative Demonstration,* is that which concludes with the direct and certain proof of the proposition in hand.

18. *An Indirect,* or *Negative Demonstration,* is that which shows a proposition to be true, by proving that some absurdity would necessarily follow if the proposition advanced were false. This is also sometimes called *Reductio ad absurdum;* because it shows the absurdity and falsehood of all suppositions contrary to that contained in the proposition.

19. *Method* is the art of disposing a train of arguments in a proper order, to investigate either the truth or falsity of a proposition, or to demonstrate it to others when it has been found out. This is either Analytical or Synthetical.

20. *Analysis,* or the *Analytic Method,* is the art or mode of finding out the truth of a proposition, by first supposing the thing to be done, and then reasoning back, step by step, till we arrive at some known truth. This is also called the *Method of Invention,* or *Resolution;* and is that which is commonly used in Algebra.

21. *Synthesis,* or the *Synthetic Method,* is the searching out truth, by first laying down some simple and easy principles, and then pursuing the consequences flowing from them till we arrive at the conclusion. This is also called the *Method of Composition;* and is the reverse of the Analytic method, as this proceeds from known principles to an unknown conclusion; while the other

goes in a retrograde order, from the thing sought, considered as if it were true, to some known principle or fact. Therefore, when any truth has been discovered by the Analytic method, it may be demonstrated by reversing the process or by Synthesis: and in the solution of geometrical propositions, it is very instructive to carry through both the *analysis* and the *synthesis*.

ARITHMETIC.

Arithmetic may be viewed as a subject of speculation, in which light it is a *science;* or as a method of practice, in which light it is an *art*.

As a *science*, its objects are the properties and relations of numbers under any assigned hypothesis respecting their mutual relations or methods of comparison and combination.

As an *art*, it proposes to discover and put into a convenient form, compendious methods of obtaining those results which flow from any given methods of combining given numbers; but which results could, in the absence of these compendious methods, only be ascertained by counting the numbers themselves into one single and continuous series.

When it treats of whole numbers, it is called *Vulgar*, or *Common Arithmetic;* but when of broken numbers, or parts of numbers, it is called *Fractions*.

Unity, or an *Unit*, is that by which every thing is regarded as one; being the beginning of number; as, one man, one ball, one gun.

Number is either simply one, or a compound of several units; as, one man, three men, ten men.

An *Integer*, or *Whole Number*, is some certain precise quantity of units; as, one, three, ten. These are so called as distinguished from *Fractions*, which are broken numbers, or parts of numbers; as, one-half, two-thirds, or three-fourths.

A *Prime Number* is one which has no other divisor than unity; as, 2, 3, 5, 7, 17, 19, &c. A *Composite Number* is one which is the product of two or more numbers; as, 4, 6, 8, 9, 28, 112, &c.

A *Factor* of a composite number, or simply a *Factor*, is any one of the numbers which enters into the *composition* of that composite number.

NOTATION AND NUMERATION.

These rules teach how to denote or express any proposed number, either by words or characters: or to read and write down any sum or number.

The Numbers in Arithmetic are expressed by the following ten digits, or Arabic numeral figures, which were introduced into Europe by the Moors, about eight or nine hundred years since; viz. 1 one, 2 two, 3 three, 4 four, 5 five, 6 six, 7 seven, 8 eight, 9 nine, 0 cipher, or nothing. These characters or figures were formerly all called by the general name of *Ciphers;* whence it came to pass that the art of Arithmetic was then often called *Ciphering*. The first nine are called *Significant Figures*, as distinguished from the cipher, which is of itself quite insignificant as a number.

Besides this value of those figures, they have also another, or *local value*,

which depends on the place they stand in when joined together; as in the following table:

.... Hundreds of Millions	Tens of Millions	Millions	Hundreds of Thousands	Tens of Thousands	Thousands	Hundreds	Tens	Units ....
9	8	7	6	5	4	3	2	1
	9	8	7	6	5	4	3	2
		9	8	7	6	5	4	3
			9	8	7	6	5	4
				9	8	7	6	5
					9	8	7	6
						9	8	7
							9	8
								9

Here, any figure in the first place, reckoning from right to left, denotes only its own simple value; but that in the second place, denotes ten times its simple value; and that in the third place, a hundred times its simple value; and so on: the value of any figure, in each successive place, being always ten times its former value.

Thus, in the number 1796, the 6 in the first place denotes only six units, or simply six; 9 in the second place signifies nine tens, or ninety; 7 in the third place, seven hundred; and the 1 in the fourth place, one thousand: so that the whole number is read thus, one thousand seven hundred and ninety-six.

As to the cipher, 0, though it signify nothing of itself, yet being joined on the right-hand side to other figures, it increases their value in the same ten-fold proportion: thus, 5 signifies only five; but 50 denotes 5 tens, or fifty; and 500 is five hundred; and so on.

For the more easily reading of large numbers, they are divided into periods and half-periods, each half-period consisting of three figures; the name of the first period being units; of the second, millions; of the third, millions of millions, or bi-millions, contracted to billions; of the fourth, millions of millions of millions, or tri-millions, contracted to trillions, and so on. Also the first part of any period is so many units of it, and the latter part so many thousands.

The following Table contains a summary of the whole doctrine.

Periods.	Quadril.;		Trillions;		Billions;		Millions;		Units.	
Half-per.	th.	un.	th.	un.	th.	un.	th.	un.	th.	un.
Figures.	123,	456;	789,	098;	765,	432;	101,	234;	567,	890;

And the whole may be thus read:—One hundred and twenty-three thousand, four hundred and fifty-six quadrillions; seven hundred and eighty-nine thousand, and ninety-eight trillions; seven hundred and sixty-five thousand, four hundred and thirty-two billions; one hundred and one thousand, two hundred and thirty-four millions; five hundred and sixty-seven thousand, eight hundred and ninety.

NUMERATION is the reading of any number in words that is proposed or set down in figures; which will be easily done by help of the following rule, deduced from the foregoing tables and observations; viz.

Divide the figures in the proposed number, as in the summary above, into periods and half-periods; then begin at the left-hand side, and read the figures with the names set to them in the two foregoing tables.

EXAMPLES.

Express in words the following numbers; viz.

34	15080	13405670
96	72033	47050023
380	109026	309025600
704	483500	4723507689
6134	2500639	274856390000
9028	7523000	6578600307024

NOTATION is the setting down in figures any number proposed in words; which is done by setting down the figures instead of the words or names belonging to them in the summary above; supplying the vacant places with ciphers where any words do not occur.

EXAMPLES.

Set down in figures the following numbers:

Fifty-seven.

Two hundred and eighty-six.

Nine thousand two hundred and ten.

Twenty-seven thousand five hundred and ninety-four.

Six hundred and forty thousand, four hundred and eighty-one.

Three millions, two hundred and sixty thousand, and one hundred and six.

Four hundred and eight millions, two hundred and fifty-five thousand, one hundred and ninety-two.

Twenty-seven thousand and eight millions, ninety-six thousand, two hundred and four.

Two hundred thousand and five hundred and fifty millions, one hundred and ten thousand, and sixteen.

Twenty-one billions, eight hundred and ten millions, sixty-four thousand, one hundred and fifty.

OF THE ROMAN NOTATION.

The Romans, like several other nations, expressed their numbers by certain letters of the alphabet. The Romans used only seven numeral letters, being the seven following capitals; viz. I for *one;* V for *five;* X for *ten;* L for *fifty;* C for a *hundred;* D for *five hundred*; M for a *thousand*. The other numbers they expressed by various repetitions and combinations of these, after the following manner:

1 = I
2 = II
3 = III
4 = IIII or IV
5 = V
6 = VI
7 = VII
8 = VIII
9 = IX
10 = X

As often as any character is repeated, so many times is its value repeated.

A less character before a greater diminishes its value.

A less character after a greater increases its value.

50 = L
100 = C
500 = D or IƆ

1000 = M or CIƆ
2000 = MM
5000 = $\overline{\text{V}}$ or IƆƆ
6000 = $\overline{\text{VI}}$
10000 = $\overline{\text{X}}$ or CCIƆƆ
50000 = $\overline{\text{L}}$ or IƆƆƆ
60000 = $\overline{\text{LX}}$
100000 = $\overline{\text{C}}$ or CCCIƆƆƆ
1000000 = $\overline{\text{M}}$ or CCCCIƆƆƆƆ
2000000 = $\overline{\text{MM}}$
&c. &c.*

For every Ɔ annexed, this becomes 10 times as many.
For every C and Ɔ placed one at each end, it becomes 10 times as much.
A bar over any number increases it 1000-fold.

EXPLANATION OF CERTAIN CHARACTERS.

There are various characters or marks used in Arithmetic and Algebra, to denote several of their operations and propositions †; the chief of which are as follow:

$+$ signifies *plus*, or addition.
$-$ - - *minus*, or subtraction.
$\times$ or . - multiplication.
$\div$ - - division.
$: \because :$ - proportion.
$=$ - - equality.
$\sqrt{\ }$ - - square root.
$\sqrt[3]{\ }$ - - cube root, &c.
$\backsim$ - - difference between two numbers when it is either not known, or not necessary to state, which is the greater.

Thus, $5 + 3$, denotes that 3 is to be added to 5.
$6 - 2$, denotes that 2 is to be taken from 6.
7×3, or $7 \cdot 3$, denotes that 7 is to be multiplied by 3.
$8 \div 4$, denotes that 8 is to be divided by 4.
$2 : 3 \because 4 : 6$, expresses that 2 is to 3 as 4 is to 6.
$6 + 4 = 10$, shows that the sum of 6 and 4 is equal to 10.
$\sqrt{3}$, or $3^{\frac{1}{2}}$, denotes the square root of the number 3.
$\sqrt[3]{5}$, or $5^{\frac{1}{3}}$ denotes the cube root of the number 5.
7^2, denotes that the number 7 is to be squared.
8^3, denotes that the number 8 is to be cubed.
$2 \backsim 6$ signifies the difference between 2 and 6.
&c. &c. &c.

See, farther, the definitions in Algebra.

* To those students whose taste leads them to inquire into the *History of Arithmetic*, reference is especially made to *Professor Leslie's Philosophy of Arithmetic*, to the *Rev. George Peacock's Treatise on Arithmetic*, in the *Encyclopædia Metropolitana*, to a paper by the celebrated *Humboldt*, read before the *Royal Academy of Berlin*, of which a translation is printed in the *Journal of the Royal Institution*, vol. xxix.; and to a paper in the *Bath and Bristol Magazine* for Oct. 1833 (No. viii.) by *Mr. Davies*. Other references will be found in those works which, for want of room, must be omitted here.

† All such symbols as designate operations to be performed, are called *symbols of operation*, and those which designate quantities of any kind are called *symbols of quantity*.

OF ADDITION.

Addition is the collecting or putting of several numbers together, in order to find their *sum*, or the total amount of the whole. This is done as follows :

Set or place the numbers under each other, so that each figure may stand exactly under the figures of the same value, that is, units under units, tens under tens, hundreds under hundreds, &c. and draw a line under the lowest number, to separate the given numbers from their sum, when it is found.—Then add up the figures in the column or row of units, and find how many tens are contained in that sum. Set down exactly below, what remains more than those tens, or if nothing remains, a cipher, and carry as many ones to the next row as there are tens.—Next, add up the second row, together with the number carried, in the same manner as the first : and thus proceed till the whole is finished, setting down the total amount of the last row.

TO PROVE ADDITION.

First Method.—Begin at the top, and add together all the rows of numbers downwards, in the same manner as they were before added upwards ; then if the two sums agree, it may be presumed the work is right.—This method of proof is only doing the same work twice over, a little varied.

Second Method.—Draw a line below the uppermost number, and suppose it cut off.—Then add all the rest of the numbers together in the usual way, and set their sum under the number to be proved.—Lastly, add this last found number and the uppermost line together; then if their sum be the same as that found by the first addition, it may be presumed the work is right.—This method of proof is founded on the plain axiom, that "The whole is equal to all its parts taken together."

Third Method.—Add the figures in the uppermost line together, and find how many nines are contained in their sum.—Reject those nines, and set down the remainder towards the right hand directly even with the figures in the line, as in the annexed example. Do the same with each of the proposed lines of numbers, setting all these excesses of nines in a column on the right hand, as here, 5, 5, 6. Then, if the excess of 9s in this sum, found as before, be equal to the excess of 9s in the total sum 18304, the work is probably right.—Thus, the sum of the right-hand column, 5, 5, 6, is 16, the excess of which above 9 is 7. Also the sum of the figures in the sum total 18304, is 16, the excess of which above 9 is also 7, the same as the former.*

EXAMPLE I.

	Excess of nines.
3497	5
6512	5
8295	6
18304	7

* This method of proof depends on a property of the number 9, which, except the number 3, belongs to no other digit whatever; namely, that "any number divided by 9, will leave the same remainder as the sum of its figures or digits divided by 9 :" which may be demonstrated in this manner.

Demonstration.—Let there be any number proposed, as 4658. This, separated into its several parts, becomes $4000 + 600 + 50 + 8$. But $4000 = 4 \times 1000 = 4 \times (999 + 1) = (4 \times 999) + 4$. In like manner $600 = (6 \times 99) + 6$; and $50 = (5 \times 9) + 5$. Therefore the given number $4658 = (4 \times 999) + 4 + (6 \times 99) + 6 + (5 \times 9) + 5 + 8 = (4 \times 999) + (6 \times 99) + (5 \times 9) + 4 + 6 + 5 + 8$; and $4658 \div 9 = (4 \times 999 + 6 \times 99 + 5 \times 9 + 4 + 6 + 5 + 8) \div 9$. But $(4 \times 999) + (6 \times 99) + (5 \times 9)$ is evidently divisible by 9, without a remainder; therefore if the given number 4658 be divided by 9, it will leave the same remainder as $4 + 6 + 5 + 8$ divided by 9. And the same, it is evident, will hold for any other number whatever.

In

OTHER EXAMPLES.

2.	3.	4.
12345	12345	12345
67890	67890	876
98765	9876	9087
43210	543	56
12345	21	234
67890	9	1012
302445	90684	23610
290100	78339	11265
302445	90684	23610

Ex. 5. Add 3426; 9024; 5106; 8890; 1204, together. Ans. 27650.

6. Add 509267; 235809; 72920; 8392; 420; 21; and 9, together. Ans. 826838.

7. Add 2; 19; 817; 4298; 50916; 730205; 9180634, together. Ans. 9966891.

8. How many days are in the twelve calendar months? Ans. 365.

9. How many days are there from the 15th day of April to the 24th day of November, both days included? Ans. 224.

10. An army consists of 52714 infantry*, or foot, 5110 horse, 6250 dragoons, 3927 light-horse, 928 artillery, or gunners, 1410 pioneers, 250 sappers, and 406 miners: what is the whole number of men? Ans. 70995.

OF SUBTRACTION.

Subtraction teaches to find how much one number exceeds another, called their *difference*, or the *remainder*, by taking the less from the greater. The method of doing which is as follows:

Place the less number under the greater, in the same manner as in Addition, that is, units under units, tens under tens, and so on; and draw a line below them.—Begin at the right hand, and take each figure in the lower line, or number, from the figure above it, setting down the remainder below it.—But if the figure in the lower line be greater than that above it, first borrow, or add, 10 to

In like manner, the same property may be shown to belong to the number 3: but the preference is usually given to the number 9, on account of its being more convenient in practice. A similar property belongs to the number 11.

Now, from the demonstration above given, the reason of the rule itself is evident: for the excess of 9s in two or more numbers being taken separately, and the excess of 9s taken also out of the sum of the former excesses, it is plain that this last excess must be equal to the excess of 9s contained in the total sum of all these numbers; all the parts taken together being equal to the whole.—This rule was first given by Dr. Wallis in his Arithmetic, published in the year 1657.

* The whole body of foot soldiers is denoted by the word *Infantry*; and all those that charge on horseback by the word *Cavalry*.—Some authors conjecture that the term infantry is derived from a certain Infanta of Spain, who, finding that the army commanded by the king her father had been defeated by the Moors, assembled a body of the people together on foot, with which she engaged and totally routed the enemy. In honour of this event, and to distinguish the foot soldiers, who were not before held in much estimation, they received the name of Infantry, from her own title of Infanta.

the upper one, and then take the lower figure from that sum, setting down the remainder, and carrying 1, for what was borrowed, to the next lower figure, with which proceed as before; and so on till the whole is finished.

TO PROVE SUBTRACTION.

Add the remainder to the less number, or that which is just above it; and if the sum be equal to the greater or uppermost number, the work is right *.

EXAMPLES.

	1.	2.	3.
From	5386427	5386427	1234567
Take	2164315	4258792	702973
Rem.	3222112	1127635	531594
Proof	5386427	5386427	1234567

4. From 5331806 take 5073918. Ans. 257888.
5. From 7020974 take 2766809. Ans. 4254165.
6. From 8503402 take 574271. Ans. 7929131.
7. Sir Isaac Newton was born in the year 1642, and he died in 1727: how old was he at the time of his decease? Ans. 85 years.
8. Homer was born 2568 years ago, and Christ 1835 years ago: then how long before Christ was the birth of Homer? Ans. 733 years.
9. Noah's flood happened about the year of the world 1656, and the birth of Christ about the year 4000: then how long was the flood before Christ? Ans. 2344 years.
10. The Arabian or Indian method of notation was first known in England about the year 1150: then how long is it since to this present year 1840? Ans. 690 years.
11. Gunpowder was invented in the year 1330: how long was that before the invention of printing, which was in 1441? Ans. 111 years.
12. The mariner's compass was invented in Europe in the year 1302: how long was that before the discovery of America by Columbus, which happened in 1492? Ans. 190 years.

OF MULTIPLICATION.

MULTIPLICATION is a compendious method of Addition, teaching how to find the amount of any given number when repeated a certain number of times; as, 4 times 6, which is 24.

The number to be multiplied, or repeated, is called the *Multiplicand.*—The number you multiply by, or the number of repetitions, is the *Multiplier.*—And the number found, being the total amount, is called the *Product.* Also, both the multiplier and multiplicand are, in general, named the *Terms* or *Factors.*

* The reason of this method of proof is evident; for if the difference of two numbers be added to the less, it must manifestly make up a sum equal to the greater.

Before proceeding to any operations in this rule, it is necessary to commit thoroughly to memory the following Table, of all the products of the first 12 numbers, commonly called the Multiplication Table, or sometimes the Table of Pythagoras, from its alleged inventor.

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

To multiply any Given Number by a Single Figure, or by any Number not exceeding 12.

* Set the multiplier under the units figure or right hand place of the multiplicand, and draw a line below it. Then, beginning at the right hand, multiply every figure in this by the multiplier. Count how many tens there are in the product of every single figure, and set down the remainder directly under the figure that is multiplied; and if nothing remains, set down a cipher. Carry as many units or ones as there are tens counted, to the product of the next figures; and proceed in the same manner till the whole is finished.

EXAMPLE.

Multiply 9876543210, the Multiplicand.
By - - - - 2, the Multiplier.

19753086420, the Product.

---

* The reason of this rule is the same as for the process in Addition, in which 1 is carried for every 10, to the next place, gradually as the several products are produced one after another, instead of setting them all down below each other, as in the annexed example.

$$\begin{array}{rcr} 5678 & & \\ 4 & & \\ \hline 32 & = & 8 \times 4 \\ 280 & = & 70 \times 4 \\ 2400 & = & 600 \times 4 \\ 20000 & = & 5000 \times 4 \\ \hline 22712 & = & 5678 \times 4 \end{array}$$

To multiply by a Number consisting of Several Figures.

* Set the multiplier below the multiplicand, placing them as in Addition, namely, units under units, tens under tens, &c. drawing a line below it. Multiply the whole of the multiplicand by each figure of the multiplier, as in the last article; setting down a line of products for each figure in the multiplier, so as that the first figure of each line may stand straight under the figure multiplying by. Add all the lines of products together, in the order in which they stand, and their sum will be the answer or whole product required. It will, of course, be always best to take that number as the multiplier which has the fewest effective figures.

TO PROVE MULTIPLICATION.

THERE are three different ways of proving multiplication, which are as below:

First Method. Make the multiplicand and multiplier change places, and multiply the latter by the former in the same manner as before. Then if the product found in this way be the same as the former, the number is right.

Second Method. † Cast all the 9s out of the sum of the figures in each of the two factors, as in Addition, and set down the remainders. Multiply these two remainders together, and cast the 9s out of the product, as also out of the whole product or answer of the question, reserving the remainders of these last two, which remainders must be equal when the work is right.

Note. It is common to set the four remainders within the four angular spaces of a cross, as in the example below.

Third Method. Multiplication is also very naturally proved by Division; for the product divided by either of the factors, will evidently give the other. But this cannot be practised till the rule of division is learned.

Or thus:—

Having placed the multiplier under the multiplicand as in the previous rule, multiply by the *left-hand* figure, setting down the product as if that figure were

---

* After having found the product of the multiplicand by the first figure of the multiplier, as in the former case, the multiplier is supposed to be divided into parts, and the product is found for the second figure in the same manner: but as this figure stands in the place of tens, the product must be ten times its simple value; and therefore the first figure of this product must be set in the place of tens; or, which is the same thing, directly under the figure multiplying by. And proceeding in this manner separately with all the figures of the multiplier, it is evident that we shall multiply all the parts of the multiplicand by all the parts of the multiplier, or the whole of the multiplicand by the whole of the multiplier: therefore these several products being added together, will be equal to the whole required product; as in the example annexed.

1234567		the multiplicand.
4567		
8641969	=	7 times the mult.
74074020	=	60 times ditto.
617283500	=	500 times ditto.
4938268000	=	4000 times ditto.
5638267489	=	4567 times ditto.

† This method of proof is derived from the peculiar property of the number 9, mentioned in the proof of Addition, and the reason for the one includes that of the other. Another more ample demonstration of this rule may, however, be as follows:—Let P and Q denote the number of 9s in the factors to be multiplied, and a and b what remain; then $9P + a$ and $9Q + b$ will be the numbers themselves, and their product is $(9P \times 9Q) + (9P \times b) + (9Q \times a) + (a \times b)$; but the first three of these products are each a precise number of 9s, because their factors, either one or both, are so: these therefore being cast away, there remains only $a \times b$; and if the 9s also be cast out of this, the excess is the excess of 9s in the total product: but a and b are the excesses in the factors themselves, and $a \times b$ is their product; therefore the rule is true. This mode of proof, however, is not an entire check against the errors that might arise from a transposition of figures, or other compensation of errors.

the only multiplier. Proceed to the next figure of the multiplier, putting the first figure one place to the right of the right-hand figure of the last product, in the line below: proceed to the next, carrying out the first figure one place more to the right; and so on till all the partial products are made. Add up as in the last rule *.

EXAMPLE.

Mult. 3542	Proof	or Mult. 3542
by 6196	2	by 6196
21252	5 4	21252
31878	2	3542
3542		31878
21252		21252
21946232 = Product.		21946232 = Product.

OTHER EXAMPLES.

Multiply 123456789 by 3.	Ans. 370370367.
Multiply 123456789 by 4.	Ans. 493827156.
Multiply 123456789 by 5.	Ans. 617283945.
Multiply 123456789 by 6.	Ans. 740740734.
Multiply 823456789 by 7.	Ans. 5764197523.
Multiply 123456789 by 8.	Ans. 987654312.
Multiply 123456789 by 9.	Ans. 1111111101.
Multiply 123456789 by 11.	Ans. 1358024679.
Multiply 123456789 by 12.	Ans. 1481481468.
Multiply 302914603 by 16.	Ans. 4846633648.
Multiply 273580961 by 23.	Ans. 6292362103.
Multiply 402097316 by 195.	Ans. 78408976620.
Multiply 82164973 by 3027.	Ans. 248713373271.
Multiply 7564900 by 579.	Ans. 4380077100.
Multiply 8496427 by 874359.	Ans. 7428927415293.
Multiply 2760325 by 37072.	Ans. 102330768400.

CONTRACTIONS IN MULTIPLICATION.

I. *When there are Ciphers in the Factors.*

If the ciphers be at the right-hand of the numbers; multiply the other figures only, and annex as many ciphers to the right-hand of the whole product, as are in both the factors. When the ciphers are in the middle parts of the multiplier; neglect them as before, only taking care to place the first figure of every line of products exactly under the figure by which you are multiplying.

EXAMPLES.

1.	2.
Mult. 9001635	Mult. 390720400
by - 70100	by - 406000
9001635	23443224
63011445	15628816
631014613500 Product.	158632482400000 Product.

* This is the eastern mode of putting down the work; and is evidently productive of the same final result, only that the progressive partial multiplications are taken in an inverse order.

3. Multiply 81503600 by 7030. Ans. 572970308000.
4. Multiply 9030100 by 2100. Ans. 18963210000.
5. Multiply 8057069 by 70050. Ans. 564397683450.

II. *When the Multiplier is the product of two or more Numbers in the Table; then*

* Multiply by each of those parts successively, instead of the whole number at once.

EXAMPLES.

1. Multiply 51307298 by 56, or 7 times 8.

```
 51307298
        7
---------
359151086
        8
---------
2873208688
```

2. Multiply 31704592 by 36. Ans. 1141365312.
3. Multiply 29753804 by 72. Ans. 2142273888.
4. Multiply 7128368 by 96. Ans. 684323328.
5. Multiply 160430800 by 108. Ans. 17326526400.
6. Multiply 61835720 by 1320. Ans. 81623150400.
7. There was an army composed of 104 † battalions, each consisting of 500 men; what was the number of men contained in the whole? Ans. 52000.
8. A convoy of ammunition ‡ bread, consisting of 250 waggons, and each waggon containing 320 loaves, having been intercepted and taken by the enemy, what is the number of loaves lost? Ans. 80000.

OF DIVISION.

Division is a kind of compendious method of Subtraction, teaching to find how often one number is contained in another, or may be taken from it, which is the same thing.

The number to be divided, is called the *Dividend*. The number to divide by, is the *Divisor*: and the number of times the dividend contains the divisor is called the *Quotient*. Sometimes there is a *Remainder* left, after the division is finished.

The usual manner of placing the terms, is, the dividend in the middle, having the divisor on the left hand, and the quotient on the right, each separated by a curve line; as, to divide 12 by 4, the quotient is 3,

---

The chief advantage of this process is, that it assimilates with the method employed in contracted decimals, in the extractions of roots in duodecimals, and in Algebra.

* The reason of this rule is obvious; for any number multiplied by the component parts of another, must give the same product as if it were multiplied by that number at once. Thus, in the 1st example, 7 times the product of 8 by the given number, make 56 times the same number, as plainly as 7 times 8 make 56.

† A battalion is a body of foot, consisting of 500, or 600, or 700 men, more or less.

‡ The ammunition bread is that which is provided for, and distributed to, the soldiers; the usual allowance being a loaf of 6 pounds to every soldier, once in 4 days.

Dividend,
Divisor 4) 12 (3 Quotient; showing that the number 4 is 3 times contained in 12, or may be 3 times subtracted out of it, as in the margin.

* *Rule.* Having placed the divisor before the dividend, as above directed, find how often the divisor is contained in as many figures of the dividend as are just necessary, and place the number on the right in the quotient. Multiply the divisor by this number, and set the product under the figures of the dividend before mentioned.—Subtract this product from that part of the dividend under which it stands, and bring down the next figure of the dividend, or more if necessary, to join on the right of the remainder.—Divide this number, so increased, in the same manner as before; and so on, till all the figures are brought down and used.

12	
4	subtr.
—	
8	
4	subtr.
—	
4	
4	subtr.
—	
0	
—	

Note. If it be necessary to bring down more figures than one to any remainder, in order to make it as large as the divisor, or larger, a cipher must be set in the quotient for every figure so brought down more than one.

TO PROVE DIVISION.

† Multiply the quotient by the divisor; to this product add the remainder, if there be any; then the sum will be equal to the dividend, when the work is right.

---

* In this way the dividend is resolved into parts, and by trial is found how often the divisor is contained in each of those parts, one after another, arranging the several figures of the quotient one after another, into one number.

When there is no remainder to a division, the quotient is the whole and perfect answer to the question. But when there is a remainder, it goes so much towards another time, as it approaches to the divisor: so, if the remainder be half the divisor, it will go the half of a time more; if the fourth part of the divisor, it will go one-fourth of a time more; and so on. Therefore to complete the quotient, set the remainder at the end of it, above a small line, and the divisor below it, thus forming a fractional part of the whole quotient.

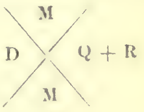
† This method of proof is plain enough: for since the quotient is the number of times the dividend contains the divisor, the quotient multiplied by the divisor must evidently be equal to the dividend.

There are several other methods sometimes used for proving Division, some of the most useful of which are as follow:

Second Method. Subtract the remainder from the dividend, and divide what is left by the quotient; so shall the new quotient from this last division be equal to the former divisor, when the work is right.

Third Method. Add together the remainder and all the products of the several quotient figures by the divisor, according to the order in which they stand in the work; and the sum will be equal to the dividend, when the work is right.

Fourth Method, by casting out the nines. Make a cross as in multiplication, and cast out the nines from the divisor and quotient, and place the respective remainders, instead of D and Q respectively. Cast the nines also out of the remainder, and annex it to Q by the sign plus, at R. Multiply D by Q, and add in the number R; and from this also cast out the nines. Place the result at M: and if this last number be the same as that left after casting out the nines from the dividend, the work is probably correct.



EXAMPLES.

```
         1.        Quot.
   3 )  1234567  (411522
        12         mult. 3
        ---      --------
          3       1234566
          3         add 1
         ---     --------
          4       1234567 Proof.
          3      --------
         ---
          15
          15
          ---
           6
           6
          ---
           7
           6
          ---
        Rem. 1
          ---
```

```
          1.          Quot.
   37 )  12345678  (333666
         111            37
         ---        -------
          124       2335662
          111       1000998
          ---        rem. 36
           135      --------
           111      12345678 Proof.
           ---      --------
            246
            222
            ---
             247
             222
             ---
              258
              222
              ---
         Rem. 36
              ---
```

3. Divide 73146085 by 4. Ans. $18286521\frac{1}{4}$.

4. Divide 5317986027 by 7. Ans. $759712289\frac{4}{7}$.

5. Divide 570196382 by 12. Ans. $47516365\frac{2}{12}$.

6. Divide 74638105 by 37. Ans. $2017246\frac{3}{37}$.

7. Divide 137896254 by 97. Ans $1421610\frac{84}{97}$.

8. Divide 35821649 by 764. Ans. $46886\frac{745}{764}$.

9. Divide 72091365 by 5201. Ans. $13861\frac{304}{5201}$.

10. Divide 4637064283 by 57606. Ans. $80496\frac{11707}{57606}$.

11. Suppose 471 men are formed into ranks of 3 deep, what is the number in each rank. Ans. 157.

12. A party, at the distance of 378 miles from the head-quarters, receive orders to join their corps in 18 days: what number of miles must they march each day to obey their orders. Ans. 21.

13. The annual revenue of a nobleman being 37960*l.*; how much per day is that equivalent to, there being 365 days in the year? Ans. 104*l.*

CONTRACTIONS IN DIVISION.

There are certain contractions in Division, by which the operation in particular cases may be performed more concisely: as follows:

I. *Division by any Small Number*, not greater than 12, may be expeditiously performed, by multiplying and subtracting mentally, omitting to set down the work except only the quotient immediately below the dividend.

EXAMPLES.

3) 56103961	4) 52619675	5) 1379192
Quot. $18701320\frac{1}{3}$		
6) 38672940	7) 81396627	8) 23718920
9) 43981962	11) 57614230	12) 27980373

II. * *When Ciphers are annexed to the Divisors;* cut off those ciphers from it, and cut off the same number of figures from the right-hand of the dividend; then divide with the remaining figures, as usual. And if there be any thing remaining after this division, place the figures cut off from the dividend to the right of it, and the whole will be the true remainder; otherwise, the figures cut off only will be the remainder.

EXAMPLES.

1. Divide 3704196 by 20.

```
2,0) 370419 6
     --------
Quot. 185209 16/20
```

2. Divide 31086901 by 7100.

```
71,00) 310869,01 ( 4378 3101/7100.
       284
       ---
        268
        213
        ---
         556
         497
         ---
          599
          568
          ---
           31
```

3. Divide 7330964 by 23000. Ans. $320\frac{20964}{23000}$.

4. Divide 2304109 by 5800. Ans. $397\frac{1509}{5800}$.

III. *When the Divisor is the exact Product of two or more of the Numbers not greater than* 12; divide by one of the factors of the divisor, putting down the remainder to the right of the quotient, but separated by the mark (; then this quotient by the next of the factors, setting down the remainder to the right of the quotient, as in the former case; then this quotient by the next factor, and so on till all the factors have been used. The final quotient is the integer part of the quotient required.

* This method serves to avoid a needless repetition of ciphers, which would occur in the common way. And the truth of the principle on which it is founded, is evident; for, cutting off the same number of ciphers, or figures, from each, is the same as dividing each of them by 10, or 100, or 1000, &c. according to the number of ciphers cut off; and it is evident, that as often as the whole divisor is contained in the whole dividend, so often must any part of the former be contained in a like part of the latter.

To find the fractional part, proceed thus:

Write the several remainders in a horizontal line from right to left, beginning at the left hand with the last; then write the several factors in the same manner to the right of these, but separated by a curve, (. Multiply the first remainder by the first divisor, and to the product add the second remainder, (this can be done mentally in all cases to which this method of division applies), the sum of which is to be placed under the second remainder: multiply this sum by the next divisor, and add the product to the third remainder, putting the sum under the third remainder: multiply this sum by the next divisor, and so on till the last sum falls under the last remainder. This will be the entire remainder which would result from dividing the dividend by the entire divisor *.

EXAMPLES.

Ex. 1. Divide 3672965 by $2 \times 3 \times 4 \times 5 \times 6$.

2	3672965	
3	1836482	1
4	612160	2
5	153040	0
6	30608	0
	5101	2

The work of finding the remainder is placed below, and below the several successive numbers is given their composition. The resulting remainder is 245, and this combined, as before, with the quotient and divisor gives for the entire quotient $5101 \frac{245}{720}$.

		Remainders.			Divisors.
2	0	0	2	1	(5, 4, 3, 2.
	10	40	122	245	
	$5 \times 2 + 0$	$4 \times 10 + 0$	$3 \times 40 + 2$	$2 \times 122 + 1$	

* The proof of the truth of this rule may be given as follows; and the example worked will show the nature of the notation employed.

Let the several remainders (reckoned *backwards*) be $r_1, r_2, r_3, \ldots$ and the divisors which gave them be $d_1, d_2, d_3, \ldots$ Then the preceding fractions being all to be divided by the successive divisors (they forming parts of the numbers successively divided) we have

$$\frac{r_1}{d_1} + \frac{r_2}{d_1 d_2} + \frac{r_3}{d_1 d_2 d_3} + \frac{r_4}{d_1 d_2 d_3 d_4} + \frac{r_5}{d_1 d_2 d_3 d_4 d_5} + \ldots$$

Reducing these to a common denominator, we have

$$\frac{r_1 d_2 d_3 d_4 d_5 \ldots + r_2 d_3 d_4 d_5 \ldots + r_3 d_4 d_5 \ldots + r_4 d_5 \ldots + r_5 \ldots + \ldots}{d_1 d_2 d_3 d_4 d_5 \ldots}$$

where the continuing dots express that the multiplication and addition are to be carried to the extent of embracing all the terms. We may suppose them to be five, as in the work written down, since the process is the same however many there may be, and the steps are continuous. Then this reduction may be gradually effected thus:

$$\begin{array}{l}
r_1 \quad + r_2 \quad + r_3 \quad + r_4 \quad + r_5 \\
\underline{d_2} \\
r_1 d_2 + r_2 \\
\underline{\quad d_3 \quad} \\
r_1 d_2 d_3 + r_2 d_3 + r_3 \\
\underline{\quad d_4 \quad} \\
r_1 d_2 d_3 d_4 + r_2 d_3 d_4 + r_3 d_4 + r_4 \\
\underline{\quad d_5 \quad} \\
r_1 d_2 d_3 d_4 d_5 + r_2 d_3 d_4 d_5 + r_3 d_4 d_5 + r_4 d_5 + r_5
\end{array}$$

&c. &c. &c.

2. Divide 7014596 by 72. Ans. $97424\frac{68}{72}$.
3. Divide 5130652 by 132. Ans. $38868\frac{76}{132}$.
4. Divide 83016572 by 240. Ans. $345902\frac{92}{240}$.

IV. *Common Division may be performed more concisely*, by omitting the several products, and setting down only the remainders; namely, multiply the divisor by the quotient figures as before, and, without setting down the product, subtract each figure of it from the dividend, as it is produced; always remembering to carry as many to the next figure as were borrowed before. This is not, however, to be recommended till considerable practice has conferred on the pupil the power of carrying on two processes at once; namely, multiplication and subtraction.

EXAMPLES.

1. Divide 3104679 by 833.

```
833 ) 3104679 ( 3727 88/833.
        6056
         2257
          5919
            88
```

2. Divide 79165238 by 238. Ans. $332627\frac{12}{238}$.
3. Divide 29137062 by 5317. Ans. $5479\frac{5219}{5317}$.
4. Divide 62015735 by 7803. Ans. $7947\frac{5294}{7803}$.

OF REDUCTION.

Reduction is the changing of numbers from one name or denomination to another, without altering their value. This is chiefly concerned in reducing money, weights, and measures.

When the numbers are to be reduced from a higher name to a lower, it is called *Reduction descending*; but when contrariwise, from a lower name to a higher, it is *Reduction ascending*.

Before we proceed to the rules and questions of Reduction, it will be proper to set down the usual tables of money, weights, and measures, which are as follow.

OF MONEY, WEIGHTS, AND MEASURES.

TABLES OF MONEY *.

2 Farthings	=	1 Halfpenny	½
4 Farthings	=	1 Penny	*d*
12 Pence	=	1 Shilling	*s*
20 Shillings	=	1 Pound	£

qrs		*d*				
4	=	1		*s*		
48	=	12	=	1		£
960	=	240	=	20	=	1

---

* £ denotes pounds, *s* shillings, and *d* denotes pence.
¼ denotes 1 farthing, or one quarter of a penny.
½ denotes a halfpenny, or the half of a penny.
¾ denotes 3 farthings, or three quarters of a penny.

[The

PENCE TABLE.			
d.		*s.*	*d.*
20	are	1	8
30	—	2	6
40	—	3	4
50	—	4	2
60	—	5	0
70	—	5	10
80	—	6	8
90	—	7	6
100	—	8	4
110	—	9	2
120	—	10	0

SHILLINGS TABLE.		
s.		*d.*
1	are	12
2	—	24
3	—	36
4	—	48
5	—	60
6	—	72
7	—	84
8	—	96
9	—	108
10	—	120
11	—	132

The full weight and value of the English gold and silver coin, both old and new, are subjoined.

GOLD.	VALUE.			WEIGHT.		SILVER.	VALUE.		OLD WT.		NEW WT.	
	£	*s.*	*d.*	*dwt.*	*gr.*		*s.*	*d.*	*dwt.*	*gr.*	*dwt.*	*gr.*
Guinea	1	1	0	5	$9\frac{1}{2}$	A Crown ...	5	0	19	$8\frac{1}{2}$	18	$4\frac{4}{11}$
Half do.	0	10	6	2	$16\frac{3}{4}$	Half-crown	2	6	9	$16\frac{1}{4}$	9	$2\frac{2}{11}$
Third do. ...	0	7	0	1	$19\frac{1}{6}$	Shilling.....	1	0	3	21	3	$15\frac{3}{11}$
Double Sov.	2	0	0	10	$6\frac{6}{11}$	Sixpence ...	0	6	1	$22\frac{1}{4}$	1	$19\frac{7}{11}$
Sovereign ...	1	0	0	5	$3\frac{3}{11}$							
Half do. ...	0	10	0	2	$13\frac{7}{11}$							

The usual value of gold is nearly *4l.* an ounce, or *2d.* a grain: and that of silver is nearly *5s.* an ounce. Also the value of any quantity of gold, was to the value of the same weight of standard silver, as $15\frac{9}{124}$ to 1, in the old coin; but in the new coin they are $14\frac{2}{7}$ to 1.

Pure gold, free from mixture with other metals, usually called fine gold, is of so pure a nature, that it will endure the fire without wasting, though it be kept continually melted. But silver, not having the purity of gold, will not endure the fire like it: yet fine silver will waste but a very little by being in the fire any moderate time; whereas copper, tin, lead, &c. will not only waste, but may be calcined, or burnt to a powder.

Both gold and silver, in their purity, are so soft and flexible, (like new lead, &c.) that they are not so useful, either in coin or otherwise (except to beat into leaf gold or silver), as when they are alloyed, or mixed and hardened with copper or brass. And though most nations differ, more or less, in the quantity of such alloy, as well as in the same place at different times, yet in England the standard for gold and silver coin has been for a long time as follows: viz. That 22 parts of fine gold, and 2 parts of copper, being melted together, shall be esteemed the true standard for gold coin: And that 11 ounces and 2 pennyweights of fine silver, and 18 pennyweights of copper, being melted together, be esteemed the true standard for silver coin, called Sterling silver.

In the old coin the pound of sterling gold was coined into $42\frac{1}{2}$ guineas, of 21 shillings each, of which the pound of sterling silver was divided into 62. The new coin is also of the same quality or degree of fineness with that of the old sterling gold and silver above described, but divided into pieces of other names or values; viz. the pound of the silver into 66 shillings, of course each shilling is the 66th part of a pound; and 20 pounds of the gold into $934\frac{1}{2}$ pieces called sovereigns, or the pound weight into $46\frac{29}{40}$ sovereigns, each equal to 20 of the new shillings. So that the weight of the sovereign is $46\frac{29}{40}$ths of a pound, which is equal to $5\frac{85}{623}$ pennyweights, or equal to 5 dwt. $3\frac{3}{11}$ gr. very nearly, as stated in the preceding tables. And multiples and parts of the sovereign and shilling in their several proportions.

WEIGHTS AND MEASURES,

Agreeably to the Act of Uniformity, which took effect 1st January, 1826.

The term MEASURE is the most comprehensive of the two, and it is distinguished into six kinds, viz.:—

Measure of ....
1. Length.
2. Surface.
3. Solidity, or Capacity.
4. Force of Gravity, or what is commonly called Weight.
5. Angles.
6. Time.

The several denominations of these Measures have reference to certain standards, which are entirely arbitrary, and consequently vary among different nations. In this kingdom

The standard of...
- Length is a Yard.
- Surface is a Square Yard, the $\frac{1}{4840}$ of an Acre.
- Solidity is a Cubic Yard.
- Capacity is a Gallon.
- Weight is a Pound.

The Standards of Angular Measure, and of Time, are the same in all European and most other countries.

1. MEASURE OF LENGTH.

12	Inches	=	1 Foot.
3	Feet	=	1 Yard.
$5\frac{1}{2}$	Yards	=	1 Rod, or Pole.
40	Poles	=	1 Furlong.
8	Furlongs	=	1 Mile.
$69\frac{1}{15}$	Miles	=	1 Degree of a Great Circle of the Earth.

An Inch is the smallest lineal measure to which a name is given; but subdivisions are used for many purposes. Among mechanics, the Inch is commonly divided into *eighths.* By the officers of the revenue, and by scientific persons, it is divided into *tenths*, *hundredths*, &c. Formerly it was made to consist of 12 parts, called *lines*, but these have properly fallen into disuse.

Particular Measures of Length.

A Nail	=	$2\frac{1}{4}$ Inches,	Used for measuring cloth of all kinds.
Quarter	=	4 Nails,	
Yard	=	4 Quarters,	
Ell	=	5 Quarters,	
Hand	=	4 Inches,	Used for the height of horses.
Fathom	=	6 Feet,	Used in measuring depths.
Link	=	7·92 Inc.	Used in Land Measure to facilitate computation of the content, 10 square chains being equal to an acre.
Chain	=	100 Links,	

2. MEASURE OF SURFACE.

144	Square Inches	=	1 Square Foot.
9	Square Feet	=	1 Square Yard.
$30\frac{1}{4}$	Square Yards	=	1 Perch, or Rod.
40	Perches	=	1 Rood.
4	Roods	=	1 Acre.
640	Acres	=	1 Square Mile.

3. MEASURES OF SOLIDITY AND CAPACITY.

DIVISION I. SOLIDITY.

1728 Cubic Inches	=	1 Cubic Foot.
27 Cubic Feet	=	1 Cubic Yard.

Division II.

Imperial Measure of Capacity for all liquids, and for all dry goods, except such as are comprised in the third Division.

4 Gills	=	1 Pint	=	34·65925		Cubic Inches.
2 Pints	=	1 Quart	=	69·3185		
4 Quarts	=	1 Gallon	=	277·274		
2 Gall.	=	1 Peck	=	554·548		
8 Gall.	=	1 Bushel	=	2218·192		
8 Bushels	=	1 Quarter	=	10·26936		Cubic Feet.
5 Qrs.	=	1 Load	=	51·34681		

The four last denominations are used for dry goods only. For liquids, several denominations have been heretofore adopted, viz. :—For Beer, the Firkin of 9 gallons, the Kilderkin of 18, the Barrel of 36, the Hogshead of 54, and the Butt of 108 gallons. These will probably continue to be used in practice. For Wine and Spirits, there are the Anker, Runlet, Tierce, Hogshead, Puncheon, Pipe, Butt, and Tun; but these may be considered rather as the names of the casks in which such commodities are imported, than as expressing any definite number of gallons. It is the practice to gauge all such vessels, and to charge them according to their actual content.

Flour is sold, nominally, by measure, but actually by weight, reckoned at 7 lb. Avoirdupois to a gallon.

Division III.

Imperial measure of Capacity for coals, culm, lime, fish, potatoes, fruit, and other goods, commonly sold by *heaped measure :*

2 Gallons	=	1 Peck	=	704	Cubic Inches, nearly.
8 Gallons	=	1 Bushel	=	$2815\frac{1}{2}$	
3 Bushels	=	1 Sack	=	$4\frac{8}{9}$	Cubic Feet, nearly.
12 Sacks	=	1 Chald.	=	$58\frac{2}{3}$	

The goods are to be heaped up in the form of a cone, to a height above the rim of the measure of at least $\frac{3}{4}$ of its depth. The outside diameter of Measures used for heaped goods are to be at least double the depth ; consequently, not less than the following dimensions :—

Bushel	$19\frac{1}{2}$ inches.	Peck	$12\frac{1}{4}$ inches.
Half-bushel	$15\frac{1}{2}$	Gallons	$9\frac{3}{4}$ ......

Half-gallon, $7\frac{3}{4}$ inches.

The Imperial Measures described in the second and third Divisions were established by Act 5 Geo. IV. c. 74. Before that time there were four different measures of capacity used in England. 1. For wine, spirits, cider, oils, milk, &c. ; this was one-sixth less than the Imperial Measure. 2. For malt liquor, this was $\frac{1}{55}$ part greater than the Imperial Measure. 3. For corn, and all other dry goods not heaped, this was $\frac{1}{33}$ part less than the Imperial Measure. 4. For coals, which did not differ sensibly from the Imperial measure.

The Imperial Gallon contains exactly 10 lbs. Avoirdupois of pure water ; consequently the pint will hold $1\frac{1}{4}$ lb., and the bushel 80 lbs.

4. MEASURE OF WEIGHT.

Division I. Avoirdupois Weight.

$27\frac{11}{32}$ Grains	=	1 Dram	=	$27\frac{11}{32}$	Grains.
16 Drams	=	1 Ounce	=	$437\frac{1}{2}$	
16 Ounces	=	1 Pound (lb.)	=	7000	
28 Pounds	=	1 Quarter (qr.)			
4 Quarters	=	1 Hundredweight (cwt.)			
20 Cwt.	=	1 Ton.			

This weight is used in almost all commercial transactions, and in the common dealings of life.

Particular weights belonging to this division:—

					cwt.	qr.	lb.	
8 Pounds	=	1 Stone						Used for Meat.
14 Pounds	=	1 Stone	=		0	0	14	Used in the Wool Trade.
2 Stone	=	1 Tod	=		0	1	0	
$6\frac{1}{2}$ Tod	=	1 Wey	=		1	2	14	
2 Weys	=	1 Sack	=		3	1	0	
12 Sacks	=	1 Last	=		39	0	0	

Division II. Troy Weight.

24 Grains	=	1 Pennyweight	=	24	Grains.
20 Pennyweights	=	1 Ounce	=	480	
12 Ounces	=	1 Pound	=	5760	

These are the denominations of Troy Weight when used for weighing gold, silver, and precious stones (except diamonds). But Troy Weight is also used by Apothecaries in compounding medicines, and by them the ounce is divided into 8 drams, and the dram into 3 scruples, so that the latter is equal to 20 grains.

For scientific purposes the grain only is used; and sets of weights are constructed in decimal progression from 10,000 grains downwards to $\frac{1}{100}$ of a grain.

By comparing the number of grains in the Avoirdupois and Troy pound and ounce respectively, it appears that the Troy pound is less than the Avoirdupois, in the proportion of 144 to 175; but the Troy ounce is greater than the Avoirdupois, in the proportion of 192 to 175.

		oz	dwts	grs	
1 lb Avoirdupois	=	14	11	$15\frac{1}{2}$	Troy.
1 oz	=	0	18	$5\frac{1}{2}$	......
1 dr	=	0	1	$3\frac{1}{2}$	......

The *carat*, used for weighing diamonds, is $3\frac{1}{5}$ grains. The term, however, when used to express the fineness of gold, has a relative meaning only. Every mass of alloyed gold is supposed to be divided into 24 equal parts: thus the standard for coin is 22 carats fine; that is, it consists of 22 parts of pure gold and 2 parts of alloy. What is called the *new standard*, used for watch cases, &c. is 18 carats fine.

5. ANGULAR MEASURE; or, DIVISIONS OF THE CIRCLE.

60 Seconds	= 1 Minute.
60 Minutes	= 1 Degree.
30 Degrees	= 1 Sign.
90 Degrees	= 1 Quadrant.
360 Degrees	= 1 Circumference.

Formerly, the subdivisions were carried on by sixties; thus, the second was divided into 60 thirds, the third into 60 fourths, &c. At present, the second is more generally divided decimally into 10ths, 100ths, &c. The degree is frequently so divided.

6. MEASURE OF TIME.

60 Seconds	= 1 Minute.
60 Minutes	= 1 Hour.
24 Hours	= 1 Day.
7 Days	= 1 Week.
28 Days	= 1 Lunar Month.
28, 29, 30, or 31 Days	= 1 Calendar Month.
12 Calendar Months	= 1 Year.
365 Days	= 1 Common Year.
366 Days	= 1 Leap Year.
$365\frac{1}{4}$ Days	= 1 Julian Year.
365 Days, 5 Hours, 48 M. $45\frac{1}{2}$ Seconds	= 1 Solar Year.

In 400 years, 97 are leap years, and 303 common.

The same remark, as in the case of angular measure, applies to the mode of subdividing the second of time.

COMPARISON OF MEASURES.

The *old ale gallon* contained 282 cubic inches.
The *old wine gallon* contained 231 cubic inches.
The *old Winchester bushel* contained 2150$\frac{2}{5}$ cubic inches.
The *imperial gallon* contains 277·274 cubic inches.
The *corn bushel*, eight times the above.

Hence, with respect to Ale, Wine, and Corn, it will be expedient to possess a

TABLE OF FACTORS,

For converting old measures into new, and the contrary.

	By decimals.			By vulgar fractions nearly.		
	Corn Measure.	Wine Measure.	Ale Measure.	Corn Measure.	Wine Measure.	Ale Measure.
To convert old measures to new.	·96943	·83311	1·01704	$\frac{31}{32}$	[illegible]	$\frac{59}{58}$
To convert new measures to old.	1·03153	1·20032	·98324	$\frac{32}{31}$	$\frac{6}{5}$	$\frac{58}{59}$

N.B. For reducing the *prices*, these numbers must all be reversed.

SIZES OF DRAWING-PAPER.

Wove antique	4 ft 4 ×	2 ft 7
Double elephant	3 ft 4 ×	2 ft 2
Atlas	2 ft 9 ×	2 ft 2
Columbier	2 ft 9$\frac{3}{4}$ ×	1 ft 11
Elephant	2 ft 3$\frac{3}{4}$ ×	1 ft 10$\frac{1}{4}$
Imperial	2 ft 5 ×	1 ft 9$\frac{1}{4}$
Super royal	2 ft 3 ×	1 ft 7
Royal	2 ft 0 ×	1 ft 7
Medium	1 ft 10 ×	1 ft 6

MISCELLANEOUS INFORMATION.

1 Aum of hock contains	36	gallons.
1 Barrel, imperial measure	9981·864	cubic inches.
1 Barrel, anchovies	30	pounds.
soap	256	pounds.
herrings	32	gallons.
salmon or eels	42	gallons.
1 Bushel of coal	88	pounds.
flour	56	pounds.
1 Butt of sherry	130	gallons.
1 Chaldron of coals, with ingrain	104809·572	cubic inches.
without ingrain	99818·64	cubic inches.
at Newcastle, is	53	cwt.
8 Chaldrons of coals at Newcastle are equal to	15$\frac{1}{2}$	London chaldrons.
1 Clove of wool	7	pounds.
1 Firkin of butter	56	pounds.
soap	64	pounds.
soap	8	gallons.

1 Fodder of lead, at Stockton	22	cwt.
Newcastle	21	cwt.
London	19½	cwt.
1 Gross	12	dozen.
1 Great gross	12	gross.
1 Hand	4	inches.
1 Hogshead of claret	58	gallons.
tent	63	gallons.
1 Hundred of salt	7	lasts.
1 Keg of sturgeon	4	or 8 gallons.
1 Last of salt	18	barrels.
gunpowder	24	barrels.
beer	12	barrels.
potash	12	barrels.
cod-fish	12	barrels.
herrings	12	barrels.
meal	12	barrels.
soap	12	barrels.
pitch and tar	12	barrels.
flax	17	cwt.
feathers	17	cwt.
wool	4368	pounds.
1 Pack of wool	240	pounds.
1 Palm	3	inches.
1 Pipe of Madeira	110	gallons.
Cape Madeira	110	gallons.
Teneriffe	120	gallons.
Bucellas	140	gallons.
Barcelona	120	gallons.
Vidonia	120	gallons.
Mountain	120	gallons.
Port	138	gallons.
Lisbon	140	gallons.
1 Pole, Woodland	18	feet.
Plantation	21	feet.
Cheshire	24	feet.
1 Sack of wool	364	pounds.
1 Seam of glass	124	pounds.
1 Span	9	inches.
1 Stone of meat	8	pounds.
fish	8	pounds.
(horseman's weight)	14	pounds.
glass	5	pounds.
wool	14	pounds.
1 Tun of vegetable oil	236	gallons.
animal oil	252	gallons.
1 Tod of wool	28	pounds.
1 Wey of Cheese, in Suffolk	256	pounds.
in Essex	336	pounds.
1 Wey of wool	182	pounds.
1 Ton or load of rough timber	40	cubic feet.
hewn	50	cubic feet.
40 Chaldrons of coal at Newcastle	106	tons.
at London	55	tons.

DIGGING.

24 Cubic feet of sand, or 18 cubic feet of earth, or 17 cubic feet of clay, make 1 ton.

1 Yard cube of solid gravel or earth contains 18 heaped bushels before digging, and 27 heaped bushels when dug.

27 Heaped bushels make 1 load.

FRENCH AND ENGLISH WEIGHTS AND MEASURES COMPARED.

The following is a comparative Table of the Weights and Measures of England and France, which was published by the Royal and Central Society of Agriculture of Paris, in the Annuaire for 1829, and founded on a Report made by Mr. Mathieu, to the Royal Academy of Sciences of France, on the bill passed the 17th of May, 1824, relative to the Weights and Measures termed "Imperial," which are now used in Great Britain.

MEASURES OF LENGTH.

ENGLISH.	FRENCH.
1 Inch (1-36th of a yard)	2·539954 centimetres.
1 Foot (1-3d of a yard)	3·0479449 decimetres.
Yard imperial	0·91438348 metre.
Fathom (2 yards)	1·82876696 metre.
Pole, or perch, (5½ yards)	5·02911 metres.
Furlong (220 yards)	201·16437 metres.
Mile (1760 yards)	1609·3149 metres.

FRENCH.	ENGLISH.
1 Millimetre	0·03937 inch.
1 Centimetre	0·393708 inch.
1 Decimetre	3·937079 inches.
1 Metre	39·37079 inches. 3·2808992 feet. 1·093633 yard.
Myriametre	6·21382 miles.

SQUARE MEASURES.

ENGLISH.	FRENCH.
1 Yard square	0·836097 metre square.
1 Rod (square perch)	25·291939 metres square.
1 Rood (1210 yards square)	10·116775 ares.
1 Acre (4840 yards square)	0·404671 hectares.

FRENCH.	ENGLISH.
1 Metre square	1·196033 yard square.
1 Are	0·098845 rood.
1 Hectare	2·473614 acres.

SOLID MEASURES.

ENGLISH.	FRENCH.
1 Pint (1-8th of a gallon)	0·567932 litre.
1 Quart (1-4th of a gallon)	1·135864 litre.
1 Gallon imperial	4·54345794 litres.
1 Peck (2 gallons)	9·0869159 litres.
1 Bushel (8 gallons)	36·347664 litres.
1 Sack (3 bushels)	1·09043 hectolitre.
1 Quarter (8 bushels)	2·907813 hectolitre
1 Chaldron (12 sacks)	13·08516 hectolitres.

FRENCH.	ENGLISH.
1 Litre	1·760773 pint. 0·2200967 gallon.
1 Decalitre	2·2009667 gallons.
1 Hectolitre	22·009667 gallons.

WEIGHTS.

ENGLISH TROY.	FRENCH.
1 Grain (1-24th of a pennyweight)	0·06477 gramme.
1 Pennyweight (1-20th of an ounce)	1·55456 gramme.
1 Ounce (1-12th of a pound troy)	31·0913 grammes.
1 Pound troy imperial	0·3730956 kilogramme.

ENGLISH AVOIRDUPOIS.	FRENCH.
1 Drachm (1-16th of an ounce)	1·7712 gramme.
1 Ounce (1-16th of a pound)	28·3384 grammes.
1 Pound avoirdupois imperial	0·4534148 kilogramme.
1 Hundredweight (112 pounds)	50·78246 kilogrammes.
1 Ton (20 hundredweight)	1015·649 kilogrammes.
FRENCH.	**ENGLISH.**
1 Gramme	15·438 grains troy. 0·643 pennyweight. 0·03216 ounce troy.
1 Kilogramme	2·68027 pounds troy. 2·20548 pounds avoirdupois.

RULES FOR REDUCTION.

I. *When the Numbers are to be reduced from a Higher Denomination to a Lower :*

Multiply the number in the highest denomination by as many of the next lower as make an integer, or 1, in that higher ; to this product add the number, if any, which was in this lower denomination before, and set down the amount.

Reduce this amount in like manner, by multiplying it by as many of the next lower as make an integer of this, taking in the odd parts of this lower, as before. And so proceed through all the denominations to the lowest ; so shall the number last found be the value of all the numbers which are in the higher denominations, taken together *.

EXAMPLE.

1. In 1234*l* 15*s* 7*d*, how many farthings ?

	£	s	d
	1234	15	7
	20		
	24695	Shillings.	
	12		
	296347	Pence.	
	4		
Answer	1185388	Farthings.	

II. *When the Numbers are to be reduced from a Lower Denomination to a Higher :*

Divide the given number by as many of that denomination as make 1 of the next higher, and set down what remains, as well as the quotient.

Divide the quotient by as many of this denomination as make one of the next higher ; setting down the new quotient, and remainder, as before.

Proceed in the same manner through all the denominations to the highest ; and the quotient last found, together with the several remainders, if any, will be of the same value as the first number proposed.

---

* The reason of this rule is very evident ; for pounds are brought into shillings by multiplying them by 20 ; shillings into pence, by multiplying them by 12 ; and pence into farthings, by multiplying by 4 ; and the reverse of this rule by division. And the like, it is evident, will be true in the reduction of numbers consisting of any denominations whatever.

EXAMPLES.

2. Reduce 1185388 farthings into pounds, shillings, and pence.

4	1185388
12	296347*d*
20	2469 5*s* 7*d*
Answer	1234*l* 15*s* 7*d*

3. Reduce 24*l* to farthings. Ans. 23040.
4. Reduce 337587 farthings to pounds, &c. Ans. 351*l* 13*s* 0¾*d*.
5. How many farthings are in 36 guineas? Ans. 36288.
6. In 36288 farthings how many guineas? Ans. 36.
7. In 59lb 13 dwts 5 gr. how many grains? Ans. 340157.
8. In 8012131 grains how many pounds, &c.
Ans. 1390 lb 11 oz 18 dwt 19 gr.
9. In 35 tons 17 cwt 1 qr 23 lb 7 oz 13 dr how many drams? Ans. 20571005.
10. How many barley-corns will reach round the earth, supposing it to be 25000 miles? Ans. 4752000000.
11. How many seconds are in a solar year, or 365 days 5 hrs 48 min 45½ sec? Ans. 31556925½.
12. In a lunar month, or 29 days 12 hrs 44 min 3 sec, how many seconds? Ans. 2551443.

COMPOUND ADDITION.

COMPOUND ADDITION serves to add or collect several numbers of different denominations into one sum.

RULE. Place the numbers so, that those of the same denomination may stand directly under each other, and draw a line below them. Add up the figures in the lowest denomination, and find, by Reduction, how many units, or ones, of the next higher denomination are contained in their sum. Set down the remainder below its proper column, and carry those units or ones to the next denomination, which add up in the same manner as before. Proceed thus through all the denominations, to the highest, whose sum, together with the several remainders, will give the answer sought.

The method of proof is the same as in Simple Addition.

EXAMPLES OF MONEY.

1.			2.			3.			4.		
£	*s*	*d*	£	*s*	*d*	£	*s*	*d*	£	*s*	*d*
7	13	3	14	7	5	15	17	10	53	14	8
3	5	10½	8	19	2¼	3	14	6	5	10	2¾
6	18	7	7	8	1½	23	6	2¾	93	11	6
0	2	5¾	21	2	9	14	9	4½	7	5	0
4	0	3	7	16	8½	15	6	4	13	2	5
17	15	4½	0	4	3	6	12	9¾	0	18	7
39	15	9¾									
32	2	6¾									
39	15	9¾									

5.			6.			7.			8.		
£	s	d	£	s	d	£	s	d	£	s	d
14	0	$7\frac{1}{4}$	37	15	8	61	3	$2\frac{1}{2}$	472	15	3
8	15	3	14	12	$9\frac{3}{4}$	7	16	8	9	2	$2\frac{1}{2}$
62	4	7	17	14	9	29	13	$10\frac{3}{4}$	27	12	$6\frac{1}{4}$
4	17	8	23	10	$9\frac{1}{4}$	12	16	2	370	16	$2\frac{1}{2}$
23	0	$4\frac{3}{4}$	8	6	0	0	7	$5\frac{1}{4}$	13	7	4
6	6	7	14	0	$5\frac{1}{2}$	24	13	0	6	10	$5\frac{1}{4}$
91	0	$10\frac{1}{4}$	54	2	$7\frac{1}{2}$	5	0	$10\frac{3}{4}$	30	0	$11\frac{3}{4}$

EXAM. 9. A nobleman, going out of town, is informed by his steward, that his butcher's bill comes to 197*l* 13*s* $7\frac{1}{2}d$; his baker's bill to 59*l* 5*s* $2\frac{3}{4}d$; his brewer's to 85*l*; his wine-merchant's to 103*l* 13*s*; to his corn-chandler is due 75*l* 3*d*; to his tallow-chandler and cheesemonger, 27*l* 15*s* $11\frac{1}{4}d$; and to his tailor 55*l* 3*s* $5\frac{3}{4}d$; also for rent, servants' wages, and other charges, 127*l* 3*s*: Now supposing he would take 100*l* with him, to defray his charges on the road, for what sum must he send to his banker? Ans. 830*l* 14*s* $6\frac{1}{4}d$.

10. The strength of a regiment of foot, of 10 companies, and the amount of their subsistence *, for a month of 30 days, according to the annexed Table, are required:

Numb.	Rank.	Subsistence for a Month.		
1	Colonel	£ 27	0	0
1	Lieutenant-Colonel ..	19	10	0
1	Major.............	17	5	0
7	Captains	78	15	0
11	Lieutenants	57	15	0
9	Ensigns	40	10	0
1	Chaplain	7	10	0
1	Adjutant	4	10	0
1	Quarter-Master.. ...	5	5	0
1	Surgeon............	4	10	0
1	Surgeon's Mate......	4	10	0
30	Serjeants	45	0	0
30	Corporals	30	0	0
20	Drummers...........	20	0	0
2	Fifes	2	0	0
390	Private Men	292	10	0
507	Total ..	£ 656	10	0

* Subsistence Money is the Money paid to the soldiers weekly; which is short of their full pay, because their clothes, accoutrements, &c. are to be accounted for. It is likewise the money advanced to officers till their accounts are made up, which is commonly once a year, when they are paid their arrears. The following Table shows the full pay and subsistence of each rank on the English establishment.

COMMISSIONED OFFICERS' REGIMENTAL PAY. FIXED IN 1824.

	Life Guards.		Horse Guards.		Foot Guards.		Dr.Gds. & Drs.	R. Wag. Train.	Foot.	R. Staff Corps.	Royal Artillery.		Royal Eng.	Royal Marines.	Militia and Fencib.
	Subsistence per Diem nett.	Gross Pay and Allowance per Diem as borne on the Establishment.	Subsistence per Diem nett.	Gross Pay and Allowance per Diem as borne on the Establishment.	Subsistence per Diem nett.	Gross Pay and Allowance per Diem as borne on the Establishment.					Marching and Invalid Battalion.	Horse Brigade.			
olonel Commandant	....	....	....	....	....	....	....	....	....	....	2 14 9½	3 0 0	2 14 9½	2 13 0	....
olonel	1 7 0	1 16 0	0 11 0	2 1 0	1 10 0	1 19 0	1 12 10	....	1 2 6	1 16 10	1 6 3	1 12 0	1 6 3	1 2 6	1 2 6
eutenant-Colonel	1 3 3	1 11 0	1 2 6	1 9 6	1 1 6	1 8 6	1 3 0	1 12 10	0 17 0	1 3 0	0 18 1	1 7 1	{ 16 1 18 1 }	} 17 0	0 15 11
ajor	0 19 6	1 6 0	1 1 6	1 7 0	0 18 6	1 4 6	0 19 3	0*18 0	0 16 0	0 19 3	0 16 11	1 2 11	....	0 16 0	0 14 1
aptain	0 12 0	0 16 0	0 16 6	1 1 6	0 12 6	0 16 6	0 14 7	0*12 6	0 10 6	0 14 7	0 11 1	0 16 1	0 11 1	0 10 6	0 10 6
itto with Brevet of Major, or superior rank	....	....	....	....	....	....	....	0*14 6	0 12 6	....	0 13 1	0 18 1	0 13 1	0 12 6	....
eutenant	0 8 3	0 11 0	0 11 6	0 15 0	0 6 0	0 7 10	0 9 0	0 *8 6	0 6 6	0 9 0	0 6 10	0 9 10	0 6 10	0 6 6	0 §6 0
itto above 7 years standing	....	....	....	....	....	....	....	0 *9 6	0 7 6	....	0 7 10	0 10 10	0 7 10	0 7 6	....
ornet, Ensign, & 2d Lieut.	0 7 3	4 8 6	0 11 0	0 14 0	0 4 6	0 5 10	0 8 0	0 *7 3	0 5 3	0 8 0	0 5 7	....	0 5 7	0 5 3	0 §5 3
aymaster	....	....	....	....	....	....	0 15 0	0 15 0	0 15 0	0 15 0	....	....	....	....	0‡15 0
djutant	0 13 0	0 13 0	0 10 0	0 10 0	0 10 0	0 10 0	0 10 0	0 10 0	0 8 6	0 10 0	0 8 +6	0 10 6	0 6 0	0 8 6	0 8 6
uarter Master	0 4 9	0 6 0	0 6 6	0 8 6	0 6 6	0 6 6	0 *8 0	....	0 6 6	0 6 6	0 7 10	0 10 10	0 6 0	0 6 6	0 6 6
urgeon Major	....	....	....	....	0 16 8	1 0 0	....	....	....	....	....	....	....	....	....
attalion Surgeon	....	....	....	....	0 10 0	0 12 0	....	....	....	....	....	....	....	....	....
urgeon	0 9 0	0 12 0	0 9 0	0 12 0	....	....	0 11 4	0 11 4	0 11 4	0 11 4	0 11 4	....	....	....	0 11 4
ssistant Surgeon	0 8 6	0 8 6	0 8 6	0 8 6	0 7 6	0 7 6	0 8 6	....	0 7 6	0 7 6	0 7 6	....	....	....	....
urgeon's Mate	....	....	....	....	....	....	....	....	....	....	....	....	....	....	0 §5 0
eterinary Surgeon	0 8 0	0 8 0	0 8 0	0 8 0	....	....	0 8 0	....	....	....	....	....	....	....	....

The difference between the Subsistence and Gross Pay of the Officers of these Regiments, after deducting Poundage, Hospital, and Agency, is paid as "*Arrears.*"

* These rates include 2s a day for a horse. + If 2d Captain, 12s 8d.
‡ Including pay as a Subaltern.
§ If holding another Commission, Lieutenants of Militia receive only 4s 8d; Ensigns, 3s 8d; and Surgeons' Mates, 3s 6d.

MEM.—Regimental Surgeons of the Line, those of the Royal Artillery, and Veterinary Surgeons, after certain periods of Service, receive the following Rates of Pay, viz.—

Surgeons of the Line and Royal Artillery.	s d
After 7 years' service	14 1 per diem.
— 20 ditto	18 10 ditto.

Veterinary Surgeons.	s
After 3 years' service	10 per diem.
— 10 ditto	12 ditto
— 20 ditto	15 ditto

EXAMPLES OF WEIGHTS, MEASURES, &c.

TROY WEIGHT.

1. lb	oz	dwt	2. oz	dwt	gr
17	3	15	37	9	3
7	9	4	9	5	3
0	10	7	8	12	12
9	5	0	17	7	8
176	2	17	5	9	0
23	11	12	3	0	19

APOTHECARIES' WEIGHT.

3. lb	oz	dr	sc	4. oz	dr	sc	gr
3	5	7	2	3	5	1	17
13	7	3	0	7	3	2	5
19	10	6	2	16	7	0	12
0	9	1	2	7	3	2	9
36	3	5	0	4	1	2	18
5	8	6	1	36	4	1	14

AVOIRDUPOIS WEIGHT.

5. lb	oz	dr	6. cwt	qr	lb
17	10	13	15	2	15
5	14	8	6	3	24
12	9	18	9	1	14
27	1	6	9	1	17
0	4	0	10	2	6
6	14	10	3	0	3

LONG MEASURE.

7. mls	fur	pls	8. yds	ft	in
29	3	14	127	1	5
19	6	29	12	2	9
7	0	24	10	0	10
9	1	37	54	1	11
7	0	3	5	2	7
4	5	9	23	0	5

CLOTH MEASURE.

9. yds	qr	nls	10. el en	qrs	nls
26	3	1	270	1	0
13	1	2	57	4	3
9	1	2	18	1	2
217	0	3	0	3	2
9	1	0	10	1	0
55	3	1	4	4	1

LAND MEASURE.

11. ac	ro	p	12. ac	ro	p
225	3	37	19	0	16
16	1	25	270	3	29
7	2	18	6	3	13
4	2	9	23	0	34
42	1	19	7	2	16
7	0	6	75	0	23

COMPOUND SUBTRACTION.

Compound Subtraction shows how to find the difference between any two numbers of different denominations. To perform which, observe the following Rule.

* Place the less number below the greater, so that the parts of the same denomination may stand directly under each other; and draw a line below them. Begin at the right hand, and subtract each number or part in the lower

* The reason of this rule will easily appear from what has been said in Simple Subtraction; for the borrowing depends on the same principle, and is only different as the numbers to be subtracted are of different denominations.

line, from the one just above it, and set the remainder straight below it: but if any number in the lower line be greater than that above it, add as many to the upper number as make 1 of the next higher denomination; then take the lower number from the upper one thus increased, and set down the remainder. Carry the unit borrowed to the next number in the lower line; after which subtract this number from the one above it, as before; and so proceed till the whole is finished. Then the several remainders, taken together, will be the whole difference sought.

The method of proof is the same as in Simple Subtraction.

EXAMPLES OF MONEY.

	1.			2.			3.			4.		
	£	s	d	£	s	d	£	s	d	£	s	d
From	79	17	8¾	103	3	2½	81	10	11	254	12	10
Take	35	12	4¼	71	12	5¾	29	13	3¼	37	9	4¾
Rem.	44	5	4½	31	10	8¾						
Proof	79	17	8¾	103	3	2½						

5. What is the difference between 73*l* 5¼*d* and 19*l* 13*s* 10*d*?
Ans. 53*l* 6*s* 7¼*d*.

6. A lends to B 100*l*; how much is B in debt after A has taken goods of him to the amount of 73*l* 12*s* 4¾*d*? Ans. 26*l* 7*s* 7¼*d*.

7. Suppose that my rent for half a year is 20*l* 12*s*, and that I have laid out for the land-tax 14*s* 6*d*, and for several repairs 1*l* 3*s* 3¼*d*; what have I to pay of my half-year's rent? Ans. 18*l* 14*s* 2¾*d*.

8. A trader failing, owes to A 35*l* 7*s* 6*d*, to B 91*l* 13*s* 0½*d*, to C 53*l* 7¼*d*, to D 87*l* 5*s*, and to E 111*l* 3*s* 5¾*d*. When this happened, he had by him in cash, 23*l* 7*s* 5*d*, in wares 53*l* 11*s* 10¼*d*, in household furniture 63*l* 17*s* 7¾*d*, and in recoverable book-debts 25*l* 7*s* 5*d*. What will his creditors lose by him, supposing these things delivered to them? Ans. 212*l* 5*s* 3½*d*.

EXAMPLES OF WEIGHTS, MEASURES, &c.

TROY WEIGHT. APOTHECARIES' WEIGHT.

	1.				2.				3.				
	lb	oz	dwt	gr	lb	oz	dwt	gr	lb	oz	dr	scr	gr
From	9	2	12	10	7	10	4	17	73	4	7	0	14
Take	5	4	6	17	3	7	16	12	29	5	3	1	19
Rem.													
P of													

AVOIRDUPOIS WEIGHT. LONG MEASURE.

	4.			5.			6.			7.		
	c	qrs	lb	lb	oz	dr	m	fu	pl	yd	ft	in
From	5	0	17	71	5	9	14	3	17	95	0	4
Take	2	3	10	17	9	13	7	6	11	71	2	9
Rem.												
Proof												

CLOTH MEASURE. / LAND MEASURE.

	8.			9.			10.			11.		
	yd	qr	nl	yd	qr	nl	ac	ro	p	ac	ro	p
From	17	2	1	9	0	2	71	1	14	57	1	16
Take	9	0	2	7	2	1	17	2	8	22	3	29
Rem.												
Proof												

DRY MEASURE. / TIME.

	12.			13.			14.			15.		
	la	qr	bu	bu	gal	pt	mo	we	da	ds	hrs	min
From	9	4	7	13	7	1	71	2	5	114	17	26
Take	6	3	5	9	2	7	17	1	6	72	10	37
Rem.												
Proof												

20. The line of defence in a certain polygon being 236 yards, and that part of it which is terminated by the curtain and shoulder being 146 yards 1 foot 4 inches; what then was the length of the face of the bastion?

Ans. 89 yds 1ft 8 inches.

COMPOUND MULTIPLICATION.

Compound Multiplication shows how to find the amount of any given number of different denominations repeated a certain proposed number of times; which is performed by the following rule.

Set the multiplier under the lowest denomination of the multiplicand, and draw a line below it. Multiply the number in the lowest denomination by the multiplier, and find how many units of the next higher denomination are contained in the product, setting down what remains. In like manner, multiply the number in the next denomination, and to the product carry or add the units, before found, and find how many units of the next higher denomination are in this amount, which carry in like manner to the next product, setting down the overplus. Proceed thus to the highest denomination proposed: so shall the last product, with the several remainders, taken as one compound number, be the whole amount required.

EXAMPLES OF MONEY.

1. To find the amount of 8lb of tea, at 5*s* 8½*d* per lb.

	s	*d*	
	5	8½	
		8	
£2	5	8	Answer.

		£	*s*	*d*
2. 4 lb of tea at 7*s* 8*d* per lb.	Ans.	1	10	8
3. 6 lb of butter, at 9½*d* per lb.	Ans.	0	4	9

	£	s	d
4. 7 lb of tobacco, at 1*s* 8½*d* per lb.	Ans. 0	11	11½
5. 8 stone of beef, at 2*s* 7½*d* per stone.	Ans. 1	1	0
6. 10 cwt of cheese, at 2*l* 17*s* 10*d* per cwt.	Ans. 28	18	4
7. 12 cwt of sugar, at 3*l* 7*s* 4*d* per cwt.	Ans. 40	8	0

CONTRACTIONS.

I. If the multiplier exceed 12, multiply successively by its component parts, instead of the whole number at once.

EXAMPLES.

1. 15 cwt of cheese, at 17*s* 6*d* per cwt.

£	s	d
0	17	6
		3
2	12	6
		5
£13	2	6 Answer.

	£	s	d
2. 20 cwt of hops, at 4*l* 7*s* 2*d* per cwt.	Ans. 87	3	4
3. 24 tons of hay, at 3*l* 7*s* 6*d* per ton.	Ans. 81	0	0
4. 45 ells of cloth, at 1*s* 6*d* per ell.	Ans. 3	7	6
5. 63 gallons of oil, at 2*s* 3*d* per gallon.	Ans. 7	1	9
6. 70 barrels of ale, at 1*l* 4*s* per barrel.	Ans. 84	0	0
7. 84 quarters of oats, at 1*l* 12*s* 8*d* per qr.	Ans. 137	4	0
8. 96 quarters of barley, at 1*l* 3*s* 4*d* per qr.	Ans. 112	0	0
9. 120 days' wages, at 5*s* 9*d* per day.	Ans. 34	10	0
10. 144 reams of paper, at 13*s* 4*d* per ream.	Ans. 96	0	0

II. If the multiplier cannot be exactly produced by the multiplication of simple numbers, take the nearest number to it, either greater or less, which can be so produced, and multiply by its parts, as before. Then multiply the given multiplicand by the difference between this assumed number and the multiplier, and add the product to that before found, when the assumed number is less than the multiplier, but subtract the same when it is greater.

EXAMPLES.

1. 26 yards of cloth, at 3*s* 0¾*d* per yard.

£	s	d
0	3	0¾
		5
0	15	3¾
		5
3	16	6¾
	3	0¾ add.
£3	19	7½ Answer.

	£	s	d
2. 29 quarters of corn, at 2*l* 5*s* 3¼*d* per qr.	Ans. 65	12	10¼
3. 53 loads of hay, at 3*l* 15*s* 2*d* per load.	Ans. 199	3	10
4. 79 bushels of wheat, at 11*s* 5¾*d* per bushel.	Ans. 45	6	10¼
5. 97 casks of beer, at 12*s* 2*d* per cask.	Ans. 59	0	2
6. 114 stone of meat, at 15*s* 3¾*d* per stone.	Ans. 87	5	7½

EXAMPLES OF WEIGHTS AND MEASURES.

7.

lb	oz	dwt	gr
28	7	14	10
			5

8.

lb	oz	dr	sc	gr
2	6	3	2	10
				8

9.

cwt	qr	lb	oz
29	2	16	14
			12

10.

mls	fu	pls	yds
22	5	29	3½
			4

11.

yds	qrs	na
126	3	1
		7

12.

ac	ro	po
28	3	27
		9

13.

we	qr	bu	pe	gal
24	2	5	3	1
				6

14.

mo	we	da	ho	min
172	3	5	16	49
				10

COMPOUND DIVISION.

Compound Division teaches how to divide a number of several denominations by any given number, or into any number of equal parts. It is performed as follows:—

Place the divisor on the left of the dividend, as in simple division. Begin at the left hand, and divide the number of the highest denomination by the divisor, setting down the quotient in its proper place. If there be any remainder after this division, reduce it to the next lower denomination, which add to the number, if any, belonging to that denomination, and divide the sum by the divisor. Set down again this quotient, reduce its remainder to the next lower denomination again, and so on through all the denominations to the last.

EXAMPLES OF MONEY.

1. Divide 237*l* 8*s* 6*d* by 2.

	£	*s*	*d*	
2)	237	8	6	
	£118	14	3	the Quotient.

	£	*s*	*d*			£	*s*	*d*
2. Divide	432	12	1¾	by 3.	Ans.	144	4	0½
3. Divide	507	3	5	by 4.	Ans.	126	15	10¼
4. Divide	632	7	6½	by 5.	Ans.	126	9	6
5. Divide	690	14	3¼	by 6.	Ans.	115	2	4½
6. Divide	705	10	2	by 7.	Ans.	100	15	8¾

	£	s	d			£	s	d
7. Divide	760	5	6	by 8.	Ans.	95	0	8¼
8. Divide	761	5	7¾	by 9.	Ans.	84	11	8¾
9. Divide	829	17	10	by 10.	Ans.	82	19	9¼
10. Divide	937	8	8¾	by 11.	Ans.	85	4	5
11. Divide	1145	11	4¼	by 12.	Ans.	95	9	3¼

CONTRACTIONS.

I. If the divisor exceed 12, find what simple numbers, multiplied together, will produce it, and divide by them separately, as in simple division, as below.

EXAMPLES.

1. What is cheese per cwt if 16 cwt cost 25*l* 14*s* 8*d*?

	£	s	d
4	25	14	8
4	6	8	8
	£1	12	2 the Answer.

2. If 20 cwt of tobacco come to 150*l* 6*s* 8*d*, what is that per cwt? — £ *s* *d* Ans. 7 10 4
3. Divide 98*l* 8*s* by 36. — Ans. 2 14 8
4. Divide 71*l* 13*s* 10*d* by 56. — Ans. 1 5 7¼
5. Divide 44*l* 4*s* by 96. — Ans. 0 9 2½
6. At 31*l* 10*s* per cwt, how much per lb? — Ans. 0 5 7½

II. If the divisor cannot be produced by the multiplication of small numbers, divide by the whole divisor at once, after the manner of long division, as follows:—

EXAMPLES.

1. Divide 59*l* 6*s* 3¾*d* by 19.

```
     £  s  d    £  s  d
19) 59  6  3¾  (3  2  5¼ Ans.
    57
    —
     2
    20
    —
    46 (2
    38
    —
     8
    12
    —
    99 (5
    95
    —
     4
     4
    —
    19 (1
```

	£	s	d.			£	s	d
2. Divide	39	14	5¼	by 57.	Ans.	0	13	11¼
3. Divide	125	4	9	by 43	Ans.	2	18	3
4. Divide	542	7	10	by 97.	Ans	5	11	10
5. Divide	123	11	2½	by 127.	Ans.	0	19	5½

EXAMPLES OF WEIGHTS AND MEASURES.

1. Divide 17 lb 9 oz 0 dwts 2 gr by 7.	Ans. 2 lb 6 oz 8 dwts 14 gr.
2. Divide 17 lb 5 oz 2 dr 1 scr 4 gr by 12.	Ans 1 lb 5 oz 3 dr 1 sc 12 gr.
3. Divide 178 cwt 3 qrs 14 lb by 53.	Ans. 3 cwt 1 qr 14 lb.
4. Divide 144 mi 4 fur 20 po 1 yd 2 ft by 39.	Ans. 3 mi 5 fur 26 po 2 ft 8 in.
5. Divide 534 yds 2 qrs 2 na by 47.	Ans. 11 yds 1 qr 2 na.
6. Divide 77 ac 1 ro 33 po by 51.	Ans. 1 ac 2 ro 3 po.
7. Divide 206 mo 4 da by 26.	Ans. 7 mo 3 we 5 ds.

THE GOLDEN RULE, OR RULE OF THREE.

THE Rule of Three enables us to find a fourth proportional to three numbers given: for which reason it is sometimes called the Rule of Proportion. It is called the Rule of Three, because three terms or numbers are given, to find a fourth. And because of its great and extensive usefulness, it was often called, by early writers on Arithmetic, the Golden Rule. This Rule is usually by practical men considered as of two kinds, namely, Direct and Inverse. The distinction, however, as well as the manner of stating, though retained here for practical purposes, does not well accord with the principles of proportion; as will be shown farther on.

The Rule of Three Direct is that in which more requires more, or less requires less. As in this: if 3 men dig 21 yards of trench in a certain time, how much will 6 men dig in the same time? Here more requires more, that is, 6 men, which are more than 3 men, will also perform more work in the same time. Or when it is thus: if 6 men dig 42 yards, how much will 3 men dig in the same time? Here, then, less requires less, or 3 men will perform proportionably less work than 6 men in the same time. In both these cases, then, the Rule, or the Proportion, is Direct; and the stating must be

thus, as $3 : 21 :: 6 : 42$, or as $3 : 6 :: 21 : 42$.
And, as $6 : 42 :: 3 : 21$, or as $6 : 3 :: 42 : 21$.

But the Rule of Three Inverse, is when more requires less, or less requires more. As in this: if 3 men dig a certain quantity of trench in 14 hours, in how many hours will 6 men dig the like quantity? Here it is evident that 6 men, being more than 3, will perform an equal quantity of work in less time, or fewer hours. Or thus: if 6 men perform a certain quantity of work in 7 hours, in how many hours will 3 men perform the same? Here less requires more, for 3 men will take more hours than 6 to perform the same work. In both these cases, then, the Rule, or the Proportion, is Inverse; and the stating must be

thus, as $6 : 14 :: 3 : 7$, or as $6 : 3 :: 14 : 7$.
And, as $3 : 7 :: 6 : 14$, or as $3 : 6 :: 7 : 14$.

And in all these statings, the fourth term is found, by multiplying the 2d and 3d terms together, and dividing the product by the 1st term.

Of the three given numbers: two of them contain the supposition, and the third a demand. And for stating and working questions of these kinds, observe the following general Rule:

STATE the question, by setting down in a straight line the three given numbers, in the following manner, viz. so that the 2d term be that number of supposition which is of the same kind that the answer or 4th term is to be; making the other number of supposition the 1st term, and the demanding number the 3d term,

when the question is in direct proportion ; but contrariwise, the other number of supposition the 3d term, and the demanding number the 1st term, when the question has inverse proportion *.

Then, in both cases, multiply the 2d and 3d terms together, and divide the product by the first, which will give the answer, or 4th term sought, viz. of the same denomination as the second term.

Note I. If the first and third terms consist of different denominations, reduce them both to the same ; and if the second term be a compound number, it is mostly convenient to reduce it to the lowest denomination mentioned. If, after division, there be any remainder, reduce it to the next lower denomination, and divide by the same divisor as before, and the quotient will be of this last denomination. Proceed in the same manner with all the remainders, till they be reduced to the lowest denomination which the second admits of, and the several quotients taken together will be the answer required.

Note II. The reason for the foregoing rules will appear when we come to treat of the nature of proportions. Sometimes two or more statings are necessary, which may always be known from the nature of the question : but in this case it falls under compound proportion, and may be more easily worked by the rule for that case.

Note III. When the first term is divisible by any number which also divides the second or third, we may so divide them, using the quotients instead of the original terms. This will often diminish the labour of the calculation considerably.

EXAMPLES.

1. If 8 yards of cloth cost 1*l* 4*s* what will 96 yards cost ?

```
   yds  £  s    yds  £  s
As  8 : 1  4 :: 96 : 14  8
        20
       ----
        24
        96
       ----
       144
       216
       ----
    8| 2304
       ----
   20| 28|8s
       ----
  £ 14  8 Answer.
```

Or, in accordance with note iii :—

```
   yd  £  s    yds  £  s
As 1 : 1  4 :: 12 : 14  8
          12
        -----
     £ 14  8
```

Or again,

```
   yd  s    yds   £  s
As 1 : 3 :: 96 : 14  8
                3
             ------
           20)28|8
             ------
             £14  8
```

* If we adhere to the rigid geometrical principles of ratio, as Mr. Bonnycastle has done, we should put the term which is of the same kind with the answer in the third place instead of the second. It is not, however, with concrete but with abstract numbers that we work ; and, hence, though the relations of things show us the relations of the numbers by which they are represented, still we may conceive a ratio between the numbers whilst the things themselves are dissimilar. Such a restriction was necessary in the geometry of the Greeks, but is not at all implied, and is therefore not necessary, in the arithmetic of symbols.

It should be added, that this rule is of very great European antiquity, and it has been universally given in this form : though the application of Jones's Rule (see Compound Proportion) is certainly more simple, and upon the whole more easily applied. As, however, the very form of stating the Rule of Three has been almost universally adopted in writing proportions, and it has acquired so strong a hold upon the language, habits, and practice of mankind, it has not been considered desirable to alter it here.

Ex. 2. An engineer having raised 100 yards of a certain work in 24 days with 5 men; how many men must he employ to finish a like quantity of work in 15 days?

	ds	men		ds		men
As	15	: 5	: :	24	:	8 Ans.
				5		
			15)	120	(8 Answer.	
				120		

3. What will 72 yards of cloth cost, at the rate of 9 yards for 5*l* 12*s*?
Ans. 44*l* 16*s*.

4. A person's annual income being 146*l*; how much is that per day?
Ans. 8*s*.

5. If 3 paces or common steps of a certain person be equal to 2 yards, how many yards will 160 of his paces make? Ans. 106 yds 2 ft.

6. What length must be cut off a board, that is 9 inches broad, to make a square foot, or as much as 12 inches in length and 12 in breadth contains?
Ans. 16 inches.

7. If 750 men require 22500 rations of bread for a month, how many rations will a garrison of 1200 men require? Ans. 36000.

8. If 7 cwt 1 qr of sugar cost 26*l* 10*s* 4*d*; what will be the price of 43 cwt 2 qrs? Ans. 159*l* 2*s*.

9. The clothing of a regiment of foot of 750 men amounting to 2831*l* 5*s*; what will the clothing of a body of 3500 men amount to? Ans. 13212*l* 10*s*.

10. How many yards of matting, that is 3 ft. broad, will cover a floor that is 27 feet long and 20 feet broad? Ans. 60 yards.

11. What is the value of six bushels of coals, at the rate of 1*l* 14*s* 6*d* the chaldron? Ans. 5*s* 9*d*.

12. If 6352 stones of 3 feet long complete a certain quantity of walling; how many stones of 2 feet long will raise a like quantity? Ans. 9528.

13. What must be given for a piece of silver weighing 73 lb 5 oz 15 dwts, at the rate of 5*s* 9*d* per ounce? Ans. 253*l* 10*s* $0\frac{3}{4}$*d*.

14. A garrison of 536 men having provision for 12 months; how long will those provisions last, if the garrison be increased to 1124 men?
Ans. 174 $\frac{64}{1124}$ days.

15. What will be the tax upon 763*l* 15*s*, at the rate of 3*s* 6*d* per pound sterling? Ans. 133*l* 13*s* $1\frac{1}{2}$*d*.

16. A certain work being raised in 12 days, by working 4 hours each day; how long would it have been in raising by working 6 hours per day?
Ans. 8 days.

17. What quantity of corn can I buy for 90 guineas, at the rate of 6*s* the bushel? Ans. 39 qrs 3 bushels.

18. A person, failing in trade, owes in all 977*l*; at which time he has, in money, goods, and recoverable debts, 420*l* 6*s* $3\frac{1}{4}$*d*; now supposing these things delivered to his creditors, how much will they get per pound? Ans. 8*s* $7\frac{1}{4}$*d*.

19. A plain of a certain extent having supplied a body of 3000 horse with forage for 18 days; then how many days would the same plain have supplied a body of 2000 horse? Ans. 27 days.

20. Suppose a gentleman's income is 600 guineas a year, and that he spends 25*s* 6*d* per day, one day with another; how much will he have saved at the year's end? Ans. 164*l* 12*s* 6*d*.

21. What cost 30 pieces of lead, each weighing 1 cwt 12 lb, at the rate of 16*s* 4*d* the cwt? Ans. 27*l* 2*s* 6*d*.

22. The governor of a besieged place having provision for 54 days, at the rate of $1\frac{1}{2}$lb of bread; but being desirous to prolong the siege to 80 days, in expectation of succour, in that case what must the ration of bread be? Ans. $1\frac{1}{80}$lb.

23. At half-a-guinea per week, how long can I be boarded for 20 pounds? Ans. $38\frac{2}{21}$ wks.

24. How much will 75 chaldrons 7 bushels of coals come to, at the rate of 1*l* 13*s* 6*d* per chaldron? Ans. 125*l* 19*s* $0\frac{1}{6}$*d*.

25. If the penny loaf weigh 8 ounces when the bushel of wheat cost 7*s* 3*d*, what ought the penny loaf to weigh when the wheat is at 8*s* 4*d*? Ans. 6 oz $15\frac{36}{100}$ dr.

26. What rent will 173 acres 2 roods 14 poles of land yield, at the rate of 1*l* 7*s* 8*d* per acre? Ans. 240*l* 2*s* $7\frac{1}{20}$*d*.

27. To how much amount 73 pieces of lead each weighing 1 cwt 3 qrs 7 lb, at 10*l* 4*s* per fother of $19\frac{1}{2}$ cwt? Ans. 69*l* 4*s* 2*d* $1\frac{18}{73}$q.

28. How many yards of stuff, of 3 qrs wide, will line a cloak that is $1\frac{3}{4}$ yards in length and $3\frac{1}{2}$ yards wide? Ans. 8 yds. 0 qrs. $2\frac{2}{3}$ nl.

29. If 5 yards of cloth cost 14*s* 2*d*, what must be given for 9 pieces, containing each 21 yards 1 quarter? Ans. 27*l* 1*s* $10\frac{1}{2}$*d*

30. If a gentleman's estate be worth 2107*l* 12*s* a year; what may he spend per day, to save 500*l* in the year? Ans 4*l* 8*s* $1\frac{19}{365}$*d*.

31. Wanting just an acre of land cut off from a piece which is $13\frac{1}{2}$ poles in breadth, what length must the piece be? Ans. 11 po 4 yds 2 ft $0\frac{18}{27}$ in.

32. At 7*s* $9\frac{1}{2}$*d* per yard, what is the value of a piece of cloth containing 53 ells English 1 qr. Ans. 25*l* 18*s* $1\frac{3}{4}$*d*.

33. If the carriage of 5 cwt 14 lb for 96 miles be 1*l* 12*s* 6*d*; how far may I have 3 cwt 1 qr carried for the same money? Ans. 151 m 3 fur $3\frac{1}{13}$ pol.

34. Bought a silver tankard, weighing 1 lb 7 oz 14 dwts; what did it cost me at 6*s* 4*d* the ounce? Ans. 6*l* 4*s* $9\frac{1}{5}$*d*.

35. What is the half year's rent of 547 acres of land, at 15*s* 6*d* the acre? Ans. 211*l* 19*s* 3*d*.

36. A wall that is to be built to the height of 36 feet, was raised 9 feet high by 16 men in 6 days; then how many men must be employed to finish the wall in 4 days, at the same rate of working? Ans. 72 men.

37. What will be the charge of keeping 20 horses for a year, at the rate of $14\frac{1}{2}$*d* per day for each horse? Ans. 441*l* 0*s* 10*d*.

38. If 18 ells of stuff that is $\frac{3}{4}$ yard wide, cost 39*s* 6*d*; what will 50 ells, of the same quality, cost, being yard wide? Ans. 7*l* 6*s* $3\frac{5}{9}$*d*.

39. How many yards of paper that is 30 inches wide, will hang a room that is 20 yards in circuit and 9 feet high. Ans. 72 yards.

40. If a gentleman's estate be worth 384*l* 16*s* a year, and the land tax be assessed at 2*s* $9\frac{1}{2}$*d* per pound, what is his net annual income? Ans. 331*l* 1*s* $9\frac{1}{2}$*d*.

41. The circumference of the earth is about 25000 miles; at what rate per hour is a person at the middle of its surface carried round, one whole rotation being made in 23 hours 56 minutes? Ans. $1044\frac{816}{1436}$ miles.

42. If a person drink 20 bottles of wine per month, when it cost 8*s* a gallon, how many bottles per month may he drink, without increasing the expense, when wine costs 10*s* the gallon? Ans. 16 bottles.

43. What cost 43 qrs 5 bushels of corn, at 1*l* 8*s* 6*d* the quarter? Ans. 62*l* 3*s* $3\frac{3}{4}$*d*.

44. How many yards of canvass that is ell wide will line 50 yards of say that is 3 quarters wide? Ans. 30 yards.

45. If an ounce of gold cost 4 guineas, what is the value of a grain? Ans. $2\frac{1}{10}d$.

46. If 3 cwt of tea cost 40*l* 12*s*; at how much a pound must it be retailed, to gain 10*l* by the whole? Ans. $3\frac{4}{336}s$.

COMPOUND PROPORTION.

Compound Proportion is a rule by means of which the student may resolve such questions as require two or more statings in simple proportion.

The general rule for questions of this kind may be exhibited in the following precepts, viz.

1. Set down the terms that express the *conditions* of the question in one line.
2. Under each conditional term, set its corresponding one, in another line, putting the letter Q in the (otherwise) blank place of the term required.
3. Multiply the *effective terms* of one line, and the *objective terms* of the other line, continually, and take the result for a dividend.
4. Multiply the remaining terms continually, and let the product be a divisor.
5. The quotient of this division will be Q, the term required *.

Note. By *effective terms* are here meant whatever necessarily and jointly produce any effect; as the cause and the time; length, breadth, and depth; buyer and his money; things carried, and their distance, &c. all necessarily inseparable in producing their several effects. In short, the *causes* of the effect.

By *objective terms*, those which express the effect itself.

Thus, if the number of men, the time of the siege, and the daily rations, be the *effective terms* in producing the consumption of the quantity of food in the garrison; then, in reference to the same problem, the quantity of food constitutes the *objective term*.

In a question where a term is only understood, and not expressed, that term may always be expressed by unity.

A quotient is represented by the dividend put above a line, and the divisor put below it.

EXAMPLES.

1. How many men can complete a trench of 135 yards long in 8 days, when 6 men can dig 54 yards of the same trench in 6 days?

men	days	yds
16	6	54
Q	8	135

Here 16 men and 6 days are the effective terms of the first line, and 135 yards e objective term of the other. Therefore, by the rule,

$$Q=\frac{16\times6\times135}{8\times54}=\frac{2\times135}{9}=30,$$

the number of men required.

* This rule, which is as applicable to *Simple* as to *Compound* Proportion, was given, in 1706, by *W. Jones, Esq.* F.R.S., the father of the late *Sir W. Jones.*

ANOTHER QUESTION.

If a garrison of 3600 men have bread for 35 days, at 24 oz each day; how much a day may be allowed to 4800 men, each for 45 days, that the same quantity of bread may serve?

men	oz	days	bread
3600	24	35	1
4800	Q	45	1

$$Q=\frac{3600\times24\times35}{4800\times45}=14 \text{ oz } \textit{per diem}.$$

AN EXAMPLE IN SIMPLE PROPORTION.

If 14 yards of cloth cost 21*l*; how many yards may be bought for 73*l* 10*s*?

yd	£	yds
1	21	14
1	$73\frac{1}{2}$	Q

$$Q=\frac{73\frac{1}{2}\times14}{21}=\tfrac{2}{3}\text{ of }73\tfrac{1}{2}=49 \text{ yards, Answer.}$$

2. If 100*l* in one year gain 5*l* interest; what will be the interest of 750*l* for 7 years? Ans. 262*l* 10*s*.
3. If a family of 8 persons expend 200*l* in 9 months; how much will serve a family of 18 people 12 months? Ans. 600*l*.
4. If 27*s* be the wages of 4 men for 7 days; what will be the wages of 14 men for 10 days? Ans. 6*l* 15*s*.
5. If a footman travel 130 miles in 3 days, when the days are 12 hours long; in how many days, of 10 hours each, may he travel 360 miles? Ans. $9\frac{63}{65}$ days.
6. If 120 bushels of corn can serve 14 horses 56 days; how many days will 94 bushels serve 6 horses? Ans. $102\frac{16}{45}$ days.
7. If 3000 lbs of beef serve 340 men 15 days; how many lbs will serve 120 men for 25 days? Ans. 1764 lb $11\frac{5}{17}$oz.
8. If a barrel of beer be sufficient to last a family of 8 persons 12 days; how many barrels will be drunk by 16 persons in the space of a year? Ans. $60\frac{5}{6}$ barrels.
9. If 180 men, in 6 days, of 10 hours each, can dig a trench 200 yards long, 3 wide, and 2 deep; in how many days of 8 hours long, will 100 men dig a trench of 360 yards long, 4 wide, and 3 deep? Ans. $48\frac{3}{5}$ days.

OF VULGAR FRACTIONS.

A FRACTION, or broken number, is an expression of a part, or some parts, of something considered as a whole.

It is denoted by two numbers, placed one below the other, with a line between them:

Thus, $\left.\begin{matrix}3 \text{ numerator}\\ \overline{4} \text{ denominator}\end{matrix}\right\}$, which is named 3-fourths.

The Denominator, or number placed below the line, shows how many equal parts the whole quantity is divided into; and it represents the Divisor in Division. And the Numerator, or number set above the line, shows how many of these parts are expressed by the fraction: being the remainder after division. Also, both these numbers are in general named the Terms of the Fraction.

Fractions are either Proper, Improper, Simple, Compound, Mixed, or Complex.

A Proper Fraction, is when the numerator is less than the denominator; as, $\frac{1}{2}$, or $\frac{2}{3}$, or $\frac{3}{5}$.

An Improper Fraction, is when the numerator is equal to, or exceeds, the denominator; as $\frac{3}{3}$, or $\frac{5}{4}$, or $\frac{7}{5}$. In these cases the fraction is called *improper*, because it is equal to or exceeds unity.

A Simple Fraction, is a single expression, denoting any number of parts of the integer; as, $\frac{2}{5}$, or $\frac{3}{2}$.

A Compound Fraction, is the fraction of a fraction, or two or more fractions connected with the word *of* between them; as $\frac{1}{2}$ of $\frac{2}{3}$, or $\frac{3}{5}$ of $\frac{5}{6}$ of 3.

A Mixed Number, is composed of a whole number and a fraction together; as, $3\frac{1}{4}$, or $12\frac{4}{5}$.

A Complex Fraction, is one that has a fraction or a mixed number for its numerator, or its denominator, or both;

as, $\dfrac{\frac{1}{2}}{3}$, or $\dfrac{2}{\frac{3}{4}}$, or $\dfrac{\frac{2}{3}}{\frac{4}{5}}$, or $\dfrac{3\frac{1}{3}}{4}$.

A whole or integer number may be expressed like a fraction, by writing 1 below it, as a denominator; so 3 is $\frac{3}{1}$, or 4 is $\frac{4}{1}$.

A fraction denotes division; and its value is equal to the quotient obtained by dividing the numerator by the denominator: so $\frac{12}{4}$ is equal to 3, and $\frac{21}{5}$ is equal to $4\frac{1}{5}$.

Hence then, if the numerator be less than the denominator, the value of the fraction is less than 1. But if the numerator be the same as the denominator the fraction is just equal to 1. And if the numerator be greater than the denominator, the fraction is greater than 1.

REDUCTION OF VULGAR FRACTIONS.

Reduction of Vulgar Fractions, is the bringing them out of one form or denomination into another; commonly to prepare them for the operations of Addition, Subtraction, &c.; of which there are several cases.

PROBLEM.

To find the greatest common measure of two or more numbers.

The common measure of two or more numbers, is that number which will divide them all without remainder: so, 3 is a common measure of 18 and 24; the quotient of the former being 6, and of the latter 8. And the greatest number that will do this is the greatest common measure: so 6 is the greatest common measure of 18 and 24; the quotient of the former being 3, and of the latter 4, which will not both divide further.

RULES.

If there be two numbers only, divide the greater by the less; then divide the divisor by the remainder; and so on, dividing always the last divisor by the last remainder, till nothing remains; so shall the last divisor of all be the greatest common measure sought.

When there are more than two numbers, find the greatest common measure of two of them, as before; then do the same for that common measure and

another of the numbers; and so on, through all the numbers; so will the greatest common measure last found be the answer.

If it happen that the common measure thus found is 1; then the numbers are said to be incommensurable, or not to have any common measure, or they are said to be prime to each other *.

Ex. 1. Find the greatest common measure of 3852 and 762896.

```
3852) 762896 (198_1
      3852
      -----
      37769
      34668
      -----
       31016
       30816
       -----
        200) 3852 (19_2
             200
             ----
             1852
             1800
             ----
              52) 200 (3_3
                  156
                  ---
                  44) 52 (1_4
                      44
                      --
                       8) 44 (5_5
                          40
                          --
                           4) 8 (2_6
                              8
```

But the mode of putting down the work may be more compactly done as below †.

```
19_2 | 3852 | 762896 | 198_1
     |  200 |   3852 |
     | ---- | ------ |
     | 1852 |  37769 |
     | 1800 |  34668 |
---- | ---- | ------ |
1_4  |   52 |  31016 |
     |   44 |  30816 |
---- | ---- | ------ | ----
2_6  |    8 |    200 | 3_3
     |    8 |    156 |
---- | ---- |   ---- | ----
     |    0 |     44 | 5_5
     |      |     40 |
     |      |    --- | ---
     |      |      4 |
```

And 4 the last divisor is the greatest common measure.

* It is not absolutely necessary that our products should be less than the dividend. All that the principle requires is, that we should take the *difference between the dividend and the nearest multiple of the divisor*. The method given in the rule is that most usually employed: though when the next higher multiple would be nearer to the dividend than the next lower, the actual work is considerably lessened by the adoption of the higher multiple. Thus in the example in the text, had we taken the quotient 4 in the third division, it will be obvious that one division would have been saved.

For a proof of this rule see the corresponding subject in algebra.

† The several quotients in both processes are numbered by *subscribed* figures, as 1_4 or 2_6 showing that 1 is the 4th quotient, and 2 is the 6th. In the new method the remainder is considered and treated as the divisor of the previous quotient, without being placed (after the first step) in the *usual* place of the divisor in common operations. This can occasion no difficulty in any case, as the divisor is not more removed from the place of the successive products than in the old method.

One advantage is, that it saves the repetition of the writing of the dividend figures. A more important one (and which is of great practical convenience in the corresponding algebraical operation) is the compactness of the work, and the small space it occupies. Both considerations concur in recommending it to general adoption.

The 2nd, 4th, &c. quotients are placed on the left of the work; the 1st, 3d, &c. on the right. It will conduce to ready re-examination of the work to draw the horizontal lines above each final remainder *through* the side lines, as in the example.

Ex. 2. To find the greatest common measure of 1908, 936, and 630.

26_2	936	1908	2_1
	72	1872	
	216	36	
	216		

2_2	36	630	17
	36	36	
	0	270	
		252	
		18	

And 36 is the g. c. m of 1908 and 936; and 18 the g. c. m. of 36 and 630 is the g. c. m. of all the three numbers 1908, 936, and 630.

3. What is the greatest common measure of 246 and 372? Ans. 6.

4. What is the greatest common measure of 324, 612, and 1032? Ans. 12.

CASE I.

To abbreviate or reduce fractions to their lowest terms.

* Divide the terms of the given fraction by any number that will divide them without a remainder; then divide these quotients again in the same manner; and so on, till it appears that there is no number greater than 1 which will divide them; then the fraction will be in its lowest terms.

---

* That dividing both the terms of the fraction by the same number, whatever it be, will give another fraction equal to the former, is evident. And when these divisions are performed as often as can be done, or when the common divisor is the greatest possible, the terms of the resulting fraction must be the least possible.

Note. 1. Any number ending with an even number, or a cipher, is divisible, or can be divided, by 2.

2. Any number ending with 5, or 0, is divisible by 5.

3. If the right-hand place of any number be 0, the whole is divisible by 10; if there be two ciphers, it is divisible by 100; if three ciphers, by 1000: and so on; which is only cutting off those ciphers.

4. If the two right-hand figures of any number be divisible by 4, the whole is divisible by 4. And if the three right-hand figures be divisible by 8, the whole is divisible by 8. And so on.

5. If the sum of the digits in any number be divisible by 3, or by 9, the whole is divisible by 3, or by 9.

6. If the right-hand digit be even, and the sum of all the digits be divisible by 6, then the whole is divisible by 6.

7. A number is divisible by 11, when the sum of the 1st, 3d, 5th, &c. or of all the odd places, is equal to the sum of the 2d, 4th, 6th, &c. or of all the even places of digits.

8. If a number cannot be divided by some quantity less than the square root of the same, that number is a prime, or cannot be divided by any number whatever.

9. All prime numbers, except 2 and 5, have either 1, 3, 7, or 9, in the place of units; and all other numbers are composite, or can be divided. It is not, however, to be inferred that all numbers which end in 1, 3, 7, 9, are prime numbers. No method, indeed, is yet known by which prime numbers can be either immediately calculated, or assigned, or detected. The best *practical* method for numbers not very high, is the *sieve* of Eratosthenes (κοσκινον), an account of which may be seen in the *Phil. Trans.* by *Dr. Horsley*, and in most works on the theory of numbers.

10. When numbers, with the sign of addition or subtraction between them, are to be divided by any number, then each of those numbers must be divided by it. Thus, $\frac{10+8-4}{2}=5+4-2=7$.

11. But if the numbers have the sign of multiplication between them, only one of them must be divided. Thus, $\frac{10\times8\times3}{6\times2}=\frac{10\times4\times3}{6\times1}=\frac{10\times4\times1}{2\times1}=\frac{10\times2\times1}{1\times1}=\frac{20}{1}=20$.

Or, Divide both the terms of the fraction by their greatest common measure at once, and the quotients will be the terms of the fraction required, of the same value as at first.

EXAMPLES.

1. Reduce $\frac{216}{288}$ to its least terms.

$\frac{216}{288} = \frac{72}{96} = \frac{36}{48} = \frac{12}{16} = \frac{6}{8} = \frac{3}{4}$, the Answer.

Or thus:

```
216) 288 (1
     216
     ---
      72) 216 (3
          216
          ---
```

Therefore 72 is the greatest common measure; and 72) $\frac{216}{288} = \frac{3}{4}$ the Answer, the same as before.

2. Reduce $\frac{195}{780}$ to its lowest terms. Ans. $\frac{1}{4}$.
3. Reduce $\frac{136}{204}$ to its lowest terms. Ans. $\frac{2}{3}$.
4. Reduce $\frac{525}{630}$ to its lowest terms. Ans. $\frac{5}{6}$.

CASE II.

To reduce a mixed number to its equivalent improper fraction.

* Multiply the integer or whole number by the denominator of the fraction, and to the product add the numerator; then set that sum above the denominator for the fraction required.

EXAMPLES.

1. Reduce $23\frac{2}{5}$ to a fraction.

```
 23
  5
---
115
  2
---
117=numerator,
```

and the fraction is $\frac{117}{5}$.

Or the work may be written thus, when the denominator is capable of being used at once :—

$$\frac{(23 \times 5) + 2}{5} = \frac{117}{5} \text{ the Answer.}$$

2. Reduce $12\frac{7}{9}$ to a fraction. Ans. $\frac{115}{9}$.
3. Reduce $14\frac{7}{10}$ to a fraction. Ans. $\frac{147}{10}$.
4. Reduce $183\,\frac{5}{21}$ to a fraction. Ans. $\frac{3848}{21}$.

CASE III.

To reduce an improper fraction to its equivalent whole or mixed number.

† Divide the numerator by the denominator, and the quotient will be the whole or mixed number sought.

EXAMPLES.

1. Reduce $\frac{12}{3}$ to its equivalent number.

Here $\frac{12}{3}$ or $12 \div 3 = 4$, the Answer.

---

* This is no more than first multiplying a quantity by some number, and then dividing the result back again by the same: which it is evident does not alter the value; for any fraction represents a division of the numerator by the denominator.

† This rule is evidently the reverse of the former; and the reason of it is manifest from the nature of common division.

2. Reduce $\frac{15}{7}$ to its equivalent number.
   Here $\frac{15}{7}$ or $15 \div 7 = 2\frac{1}{7}$, the Answer.
3. Reduce $\frac{749}{17}$ to its equivalent number.

Thus 17) 749 ($44\frac{1}{17}$
              68
              ——
               69
               68
               ——
                1

So that $\frac{749}{17} = 44\frac{1}{17}$, the Answer.

4. Reduce $\frac{56}{7}$ to its equivalent number. Ans. 8.
5. Reduce $\frac{1362}{25}$ to its equivalent number. Ans. $54\frac{12}{25}$.
6. Reduce $\frac{2918}{17}$ to its equivalent number. Ans. $171\frac{11}{17}$.

CASE IV.

To reduce a whole number to an equivalent fraction, having a given denominator.

* Multiply the whole number by the given denominator; then set the product over the said denominator, and it will form the fraction required.

EXAMPLES.

1. Reduce 9 to a fraction whose denominator shall be 7.
   Here $9 \times 7 = 63$: then $\frac{63}{7}$ is the Answer;
   For $\frac{63}{7} = 63 \div 7 = 9$, the Proof.
2. Reduce 12 to a fraction whose denominator shall be 13. Ans. $\frac{156}{13}$.
3. Reduce 27 to a fraction whose denominator shall be 11. Ans. $\frac{297}{11}$.

CASE V.

To reduce a compound fraction to an equivalent simple one.

† Multiply all the numerators together for a numerator, and all the denominators together for a denominator, and they will form the simple fraction sought.

When part of the compound fraction is a whole or mixed number, it must first be reduced to a fraction by one of the former cases.

And, when it can be done, any two terms of the fraction may be divided by the same number, and the quotients used instead of them. Or, when there are terms that are common, they may be omitted, or cancelled.

---

* Multiplication and Division being here equally used, the result must be the same as the quantity first proposed.

† The truth of this rule may be shown as follows: Let the compound fraction be $\frac{2}{3}$ of $\frac{5}{7}$. Now $\frac{1}{3}$ of $\frac{5}{7}$ is $\frac{5}{7} \div 3$, which is $\frac{5}{21}$; consequently $\frac{2}{3}$ of $\frac{5}{7}$ will be $\frac{5}{21} \times 2$ or $\frac{10}{21}$; that is, the numerators are multiplied together, and also the denominators, as in the Rule. When the compound fraction consists of more than two single ones; having first reduced two of them as above, then the resulting fraction and a third will be the same as a compound fraction of two parts; and so on to the last of all.

EXAMPLES.

1. Reduce $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ to a simple fraction.

Here $\frac{1 \times 2 \times 3}{2 \times 3 \times 4} = \frac{6}{24} = \frac{1}{4}$, the Answer.

Or, $\frac{1 \times \cancel{2} \times \cancel{3}}{\cancel{2} \times \cancel{3} \times 4} = \frac{1}{4}$ by cancelling 2 and 3.

2. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{10}{11}$ to a simple fraction.

Here $\frac{2 \times 3 \times 10}{3 \times 5 \times 11} = \frac{60}{165} = \frac{12}{33} = \frac{4}{11}$ the Answer.

Or, $\frac{2 \times \cancel{3} \times \overset{2}{\cancel{10}}}{\cancel{3} \times \cancel{5} \times 11} = \frac{4}{11}$ the same as before, by cancelling the 3 and dividing both terms by 5.

3. Reduce $\frac{3}{7}$ of $\frac{4}{5}$ to a simple fraction. Ans $\frac{12}{35}$.
4. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{5}{9}$ to a simple fraction. Ans. $\frac{2}{9}$.
5. Reduce $\frac{2}{5}$ of $\frac{5}{8}$ of $3\frac{1}{2}$ to a simple fraction. Ans. $\frac{7}{8}$.
6. Reduce $\frac{2}{7}$ of $\frac{5}{8}$ of $\frac{7}{2}$ of 4 to a simple fraction. Ans. $\frac{5}{2}$.
7. Reduce $2\frac{2}{5}$ of $\frac{5}{6}$ to a fraction. Ans. $\frac{2}{1}$ or 2.

CASE VI.

To reduce fractions of different denominators to equivalent fractions having a common denominator.

* MULTIPLY each numerator by all the denominators except its own, for the new numerators: and multiply all the denominators together for a common denominator.

Note. It is evident, that in this and several other operations, when any of the proposed quantities are integers, or mixed numbers, or compound fractions, they must first be reduced, by their proper rules, to the form of simple fractions.

---

* This is evidently no more than multiplying each numerator and its denominator by the same quantity, and consequently the value of the fraction is not altered.

It is in many cases not only useful, but easy, to reduce fractions to their *least* common denominator.

The rule is this :

Find the least common multiple of all the denominators, and it will be the common denominator required.

Then divide the common denominator by the denominator of each fraction, and multiply the quotient by the numerator—the several products will be the numerators ; which are to be placed respectively over the common denominator for the answer.

To find the *least common multiple* proceed thus :

Divide by any numbers that will divide two or more of the given numbers without a remainder, and set the quotients, together with the undivided numbers, in a line beneath.

Divide the second line, as before, and so on, until there are no two numbers, beginning with the lowest numbers, and only *primes* need be used, that can be divided ; then the continued product of the divisors, quotients, and undivided numbers, will give the multiple required

Example

EXAMPLES.

1. Reduce $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$, to a common denominator.

$1 \times 3 \times 4 = 12$ the new numerator for $\frac{1}{2}$.
$2 \times 2 \times 4 = 16$ ditto $\frac{2}{3}$.
$3 \times 2 \times 3 = 18$ ditto $\frac{3}{4}$.
$2 \times 3 \times 4 = 24$ the common denominator.

Therefore the equivalent fractions are $\frac{12}{24}$, $\frac{16}{24}$, and $\frac{18}{24}$.

Or the whole operation of multiplying may often be performed mentally, only setting down the results and given fractions thus, $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, = \frac{12}{24}, \frac{16}{24}, \frac{18}{24}, = \frac{6}{12}, \frac{8}{12}, \frac{9}{12}$, by abbreviation.

2. Reduce $\frac{2}{7}$ and $\frac{5}{9}$ to fractions of a common denominator. Ans. $\frac{18}{63}$, $\frac{35}{63}$.
3. Reduce $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{3}{4}$ to a common denominator. Ans. $\frac{40}{60}$, $\frac{36}{60}$, $\frac{45}{60}$.
4. Reduce $\frac{5}{6}$, $2\frac{3}{5}$, and 4 to a common denominator. Ans. $\frac{25}{30}$, $\frac{78}{30}$, $\frac{120}{30}$.

Note. 1. When the denominators of two given fractions have a common measure, let them be divided by it; then multiply the terms of each given fraction by the quotient arising from the other's denominator.

Ex. $\underset{5}{\frac{3}{25}}$ and $\underset{7}{\frac{4}{35}} = \frac{21}{175}$ and $\frac{20}{175}$, by multiplying the former by 7 and the latter by 5.

2. When the less denominator of two fractions exactly divides the greater, multiply the terms of that which has the less denominator by the quotient.

Ex. $\underset{2}{\frac{3}{7}}$ and $\frac{5}{14} = \frac{6}{14}$ and $\frac{5}{14}$, by multiplying the terms of the former by 2.

3. When more than two fractions are proposed, it is sometimes convenient, first to reduce two of them to a common denominator; then these and a third; and so on till they be all reduced to their least common denominator.

Ex. $\frac{2}{3}$ and $\frac{3}{4}$ and $\frac{7}{8} = \frac{2}{3}$ and $\frac{6}{8}$ and $\frac{7}{8} = \frac{16}{24}$ and $\frac{18}{24}$ and $\frac{21}{24}$.

CASE VII.

To reduce complex fractions to single ones.

Reduce the two parts both to simple fractions; then multiply the numerator of each by the denominator of the other; which is in fact only increasing each part by equal multiplications, which makes no difference in the value of the whole.

$$\text{So, } \frac{\frac{5}{2}}{3} = \frac{5}{6}. \quad \text{And } \frac{2\frac{1}{3}}{4} = \frac{7}{12}. \quad \text{Also } \frac{3\frac{2}{5}}{4\frac{1}{2}} = \frac{\frac{17}{5}}{\frac{9}{2}} = \frac{17}{5} \times \frac{2}{9} = \frac{34}{45}.$$

Example. Reduce $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{7}{12}$, $\frac{8}{15}$, and $\frac{11}{20}$ to fractions having the least common denominator.

3	3	4	5	12	15	20
4	1	4	5	4	5	20
5	1	1	5	1	5	5
	1	1	1	1	1	1

Therefore we have $3 \times 4 \times 5 = 60 =$ least common denominator.

$$\left.\begin{array}{l} \text{Then } 60 \div 3 = 20, \text{ and } 20 \times 2 = 40 \\ 60 \div 4 = 15, \text{ and } 15 \times 3 = 45 \\ 60 \div 5 = 12, \text{ and } 12 \times 4 = 48 \\ 60 \div 12 = 5, \text{ and } 5 \times 7 = 35 \\ 60 \div 15 = 4, \text{ and } 4 \times 8 = 32 \\ 60 \div 20 = 3, \text{ and } 3 \times 11 = 33 \end{array}\right\} \text{new numerators.}$$

Hence $\frac{40}{60}$, $\frac{45}{60}$, $\frac{48}{60}$, $\frac{35}{60}$, $\frac{32}{60}$, $\frac{33}{60}$ are the fractions required. It is of great importance that the student should be made familiar with this rule, both on account of the facility which it gives in actual reductions, and especially in the reductions that occur in algebraic fractions and equations.

CASE VIII.

To find the value of a fraction in parts of the integer.

MULTIPLY the integer by the numerator, and divide the product by the denominator, by compound multiplication and division, if the integer be a compound quantity.

Or, if it be a single integer, multiply the numerator by the parts in the next inferior denomination, and divide the product by the denominator. Then, if any thing remains, multiply it by the parts in the next inferior denomination, and divide by the denominator, as before; and so on as far as necessary; so shall the quotients, placed in order, be the value of the fraction required *.

EXAMPLES.

1. What is the $\frac{4}{5}$ of $2l$ $6s$?

By the former part of the rule,

$$\begin{array}{r|l} & 2l\ 6s \\ & \quad 4 \\ \hline 5 & 9\ \ 4 \\ \hline & \pounds\ 1\ \ 16\ \ 9\tfrac{1}{2}\ \tfrac{2}{5} \end{array}$$

2. What is the value of $\frac{2}{3}$ of $1l$?

By the 2d part of the rule,

$$\begin{array}{r|l} 3 & 2l \\ \hline & \pounds\ 0\ \ 13\ \ 4 \end{array}$$

3. Find the value of $\frac{3}{8}$ of a pound sterling. Ans. 7*s* 6*d*.
4. What is the value of $\frac{2}{9}$ of a guinea? Ans. 4*s* 8*d*.
5. What is the value of $\frac{3}{4}$ of a half-crown? Ans. 1*s* 10½*d*.
6. What is the value of $\frac{2}{5}$ of 4*s* 10*d*? Ans. 1*s* 11⅕*d*.
7. What is the value of $\frac{4}{5}$ lb troy? Ans. 9 oz 12 dwts.
8. What is the value of $\frac{5}{16}$ of a cwt? Ans. 1 qr 7 lb.
9. What is the value of $\frac{7}{8}$ of an acre? Ans. 3 ro 20 po.
10. What is the value of $\frac{3}{10}$ of a day? Ans. 7 hrs 12 min.

CASE IX.

To reduce a fraction from one denomination to another.

† CONSIDER how many of the less denomination make one of the greater; then multiply the numerator by that number, if the reduction be to a less name, but multiply the denominator, if to a greater.

EXAMPLES.

1. Reduce $\frac{2}{9}$ of a pound to the fraction of a penny.

$\frac{2}{9} \times \frac{20}{1} \times \frac{12}{1} = \frac{480}{9} = \frac{160}{3}$, the answer.

2. Reduce $\frac{5}{7}$ of a penny to the fraction of a pound.

$\frac{5}{7} \times \frac{1}{12} \times \frac{1}{20} = \frac{1}{336}$, the answer.

---

* The numerator of a fraction being considered as a remainder, in division, and the denominator as a divisor, this rule is of the same nature as compound division, or the valuation of remainders in the rule of three, before explained.

† This is the same as the rule of reduction in whole numbers from one denomination to another.

3. Reduce $\frac{2}{15}l$ to the fraction of a penny. Ans. $\frac{32}{1}d$.
4. Reduce $\frac{2}{5}q$ to the fraction of a pound. Ans. $\frac{1}{2400}$.
5. Reduce $\frac{2}{7}$ cwt to the fraction of a lb. Ans. $\frac{32}{1}$.
6. Reduce $\frac{3}{5}$ dwt to the fraction of a lb troy. Ans. $\frac{1}{400}$.
7. Reduce $\frac{3}{8}$ crown to the fraction of a guinea. Ans. $\frac{5}{56}$.
8. Reduce $\frac{5}{6}$ half-crown to the fraction of a shilling. Ans. $\frac{25}{12}$.
9. Reduce 2*s* 6*d* to the fraction of a £. Ans. $\frac{1}{8}$.
10. Reduce 17*s* 7*d* $3\frac{3}{5}q$ to the fraction of a £. Ans. $\frac{2119}{2400}$.

ADDITION OF VULGAR FRACTIONS.

If the fractions have a common denominator; add all the numerators together, then place the sum over the common denominator, and that will be the sum of the fractions required.

* If the proposed fractions have not a common denominator, they must be reduced to one. Also compound fractions must be reduced to simple ones, and fractions of different denominations to those of the same denomination. Then add the numerators, as before. As to the mixed numbers, they may either be reduced to improper fractions, and so added with the others; or else the fractional parts only added, and the integers united afterwards.

EXAMPLES.

1. To add $\frac{3}{5}$ and $\frac{4}{5}$ together.
   Here $\frac{3}{5} + \frac{4}{5} = \frac{7}{5} = 1\frac{2}{5}$, the answer.

---

* Before fractions are reduced to a common denominator, they are quite dissimilar, as much as shillings and pence are, and therefore cannot be incorporated with one another any more than these can. But when they are reduced to a common denominator, and made parts of the same thing, their sum, or difference, may then be as properly expressed by the sum or difference of the numerators, as the sum or difference of any two quantities whatever, by the sum or difference of their individuals. Whence the reason of the rule is manifest, both for addition and subtraction.

Note 1. When several fractions are to be collected, it is commonly best first to add two of them together that most easily reduce to a common denominator; then add their sum and a third, and so on.

Note 2. Taking any two fractions whatever, $\frac{7}{11}$ and $\frac{35}{55}$, for example, after reducing them to a common denominator, we judge whether they are equal or unequal, by observing whether the products 35×11, and 7×55, which constitute the new numerators, are equal or unequal. If, therefore, we have two equal products $35 \times 11 = 7 \times 55$, we may compose from them two equal fractions, as $\frac{35}{55} = \frac{7}{11}$, or $\frac{35}{7} = \frac{55}{11}$.

If, then, we take two equal fractions, such as $\frac{7}{11}$ and $\frac{35}{55}$, we shall have $35 \times 11 = 7 \times 55$; taking from each of these 7×11, there will remain $(35 - 7) \times 11 = (55 - 11) \times 7$, whence we have $\frac{35 - 7}{55 - 11} = \frac{7}{11}$, or $\frac{28}{44} = \frac{7}{11}$.

In like manner, if the terms of $\frac{7}{11}$ were respectively *added* to those of $\frac{35}{55}$, we should have $\frac{35 + 7}{55 + 11} = \frac{42}{66} = \frac{7}{11}$.

Or, generally, if $\frac{a}{b} = \frac{c}{d}$, it may in a similar way be shown, that $\frac{a \pm c}{b \pm d} = \frac{a}{b} = \frac{c}{d}$.

Hence, *when two fractions are of equal value, the fraction formed by taking the sum or the difference of their numerators respectively, and of their denominators respectively, is a fraction equal in value to each of the original fractions.* This proposition will be found useful in the doctrine of proportions.

2. To add $\frac{3}{5}$ and $\frac{5}{6}$ together.

$\frac{3}{5} + \frac{5}{6} = \frac{18}{30} + \frac{25}{30} = \frac{43}{30} = 1\frac{13}{30}$, the answer.

3. To add $\frac{5}{8}$ and $7\frac{1}{2}$ and $\frac{1}{3}$ of $\frac{3}{4}$ together.

$\frac{5}{8} + 7\frac{1}{2} + \frac{1}{3}$ of $\frac{3}{4} = \frac{5}{8} + \frac{15}{2} + \frac{1}{4} = \frac{5}{8} + \frac{60}{8} + \frac{2}{8} = \frac{67}{8} = 8\frac{3}{8}$.

4. To add $\frac{3}{7}$ and $\frac{6}{7}$ together. Ans. $1\frac{2}{7}$.
5. To add $\frac{3}{4}$ and $\frac{5}{9}$ together. Ans. $1\frac{11}{36}$.
6. Add $\frac{2}{7}$ and $\frac{5}{14}$ together. Ans. $\frac{9}{14}$.
7. What is the sum of $\frac{2}{3}$ and $\frac{3}{5}$ and $\frac{5}{7}$? Ans. $1\frac{103}{105}$.
8. What is the sum of $\frac{5}{9}$ and $\frac{3}{5}$ and $2\frac{1}{6}$? Ans. $3\frac{29}{90}$.
9. What is the sum of $\frac{3}{5}$ and $\frac{4}{5}$ of $\frac{1}{3}$, and $9\frac{3}{20}$? Ans. $10\frac{1}{60}$.
10. What is the sum of $\frac{2}{3}$ of a pound and $\frac{5}{9}$ of a shilling?

Ans. $\frac{125}{9}s$ or $13s\ 10d\ 2\frac{2}{3}q$.

11. What is the sum of $\frac{3}{5}$ of a shilling and $\frac{4}{15}$ of a penny? Ans. $\frac{112}{15}d$ or $7d\ 1\frac{13}{15}q$.
12. What is the sum of $\frac{1}{7}$ of a pound, and $\frac{2}{9}$ of a shilling, and $\frac{5}{12}$ of a penny?

Ans. $\frac{3139}{1008}s$ or $3s\ 1d\ 1\frac{10}{21}q$.

SUBTRACTION OF VULGAR FRACTIONS.

Prepare the fractions the same as for addition, when necessary; then subtract the one numerator from the other, and set the remainder over the common denominator, for the difference of the fractions sought.

EXAMPLES.

1. To find the difference between $\frac{5}{6}$ and $\frac{1}{6}$. Here $\frac{5}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$, the answer.
2. To find the difference between $\frac{3}{4}$ and $\frac{5}{9}$. $\frac{3}{4} - \frac{5}{9} = \frac{27}{36} = \frac{20}{36} = \frac{7}{36}$, the answer.
3. What is the difference between $\frac{5}{12}$ and $\frac{7}{12}$? Ans. $\frac{1}{6}$.
4. What is the difference between $\frac{3}{13}$ and $\frac{4}{39}$? Ans. $\frac{5}{39}$.
5. What is the difference between $\frac{5}{12}$ and $\frac{7}{13}$? Ans. $\frac{19}{156}$.
6. What is the difference between $5\frac{3}{8}$ and $\frac{2}{7}$ of $4\frac{1}{6}$? Ans. $4\frac{31}{168}$.
7. What is the difference between $\frac{5}{9}$ of a pound, and $\frac{2}{3}$ of $\frac{3}{4}$ of a shilling?

Ans $\frac{191}{18}s$ or $10s\ 7d\ 1\frac{1}{3}q$.

8. What is the difference between $\frac{2}{7}$ of $5\frac{1}{6}$ of a pound, and $\frac{3}{5}$ of a shilling?

Ans. $\frac{3037}{2100}l$ or $1l\ 8s\ 11\frac{3}{35}d$.

MULTIPLICATION OF VULGAR FRACTIONS.

* Reduce mixed numbers, if there be any, to equivalent fractions; then multiply all the numerators together for a numerator, and all the denominators together for a denominator, which will give the product required.

---

* Multiplication of any thing by a fraction, implies the taking some part or parts of the thing; it may therefore be truly expressed by a compound fraction; which is resolved by multiplying together the numerators and the denominators.

Note. A fraction is best multiplied by an integer, by dividing the denominator by it; but if it will not exactly divide, then multiply the numerator by it.

EXAMPLES.

1. Required the product of $\frac{3}{4}$ and $\frac{2}{9}$.
   Here $\frac{3}{4} \times \frac{2}{9} = \frac{6}{36} = \frac{1}{6}$, the answer.
   Or, $\frac{3}{4} \times \frac{2}{9} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.
2. Required the continued product of $\frac{2}{3}$, $3\frac{1}{4}$, 5, and $\frac{3}{4}$ of $\frac{3}{5}$.
   Here $\frac{\cancel{2}}{\cancel{3}} \times \frac{13}{4} \times \frac{\cancel{5}}{1} \times \frac{\cancel{3}}{4} \times \frac{3}{\cancel{5}} = \frac{13 \times 3}{4 \times 2} = \frac{39}{8} = 4\frac{7}{8}$, Answer.
3. Required the product of $\frac{2}{7}$ and $\frac{5}{8}$. Ans. $\frac{5}{28}$.
4. Required the product of $\frac{4}{13}$ and $\frac{5}{24}$. Ans. $\frac{1}{18}$.
5. Required the product of $\frac{3}{7}$, $\frac{4}{9}$, and $\frac{14}{15}$. Ans. $\frac{8}{45}$.
6. Required the product of $\frac{1}{2}$, $\frac{2}{3}$, and 3. Ans. 1.
7. Required the product of $\frac{7}{9}$, $\frac{3}{5}$, and $4\frac{5}{14}$. Ans. $2\frac{1}{30}$.
8. Required the product of $\frac{5}{6}$, and $\frac{2}{3}$ of $\frac{6}{7}$. Ans. $\frac{10}{21}$.
9. Required the product of 6, and $\frac{2}{3}$ of 5. Ans. 20.
10. Required the product of $\frac{2}{9}$ of $\frac{3}{5}$, and $\frac{5}{8}$ of $3\frac{2}{7}$. Ans. $\frac{23}{84}$.
11. Required the product of $3\frac{2}{7}$ and $4\frac{14}{33}$. Ans. $14\frac{124}{231}$.
12. Required the product of 5, $\frac{2}{3}$, $\frac{2}{7}$ of $\frac{3}{5}$, and $4\frac{1}{6}$. Ans. $2\frac{8}{21}$.

DIVISION OF VULGAR FRACTIONS.

* Prepare the fractions as before in multiplication: then divide the numerator by the numerator, and the denominator by the denominator, if they will exactly divide: but if not, invert the terms of the divisor, and multiply the dividend by it, as in multiplication.

EXAMPLES.

1. Divide $\frac{25}{9}$ by $\frac{5}{3}$. Here $\frac{25}{9} \div \frac{5}{3} = \frac{5}{3} = 1\frac{2}{3}$, by the first method.
2. Divide $\frac{5}{9}$ by $\frac{2}{15}$. Here $\frac{5}{9} \div \frac{2}{15} = \frac{5}{9} \times \frac{15}{2} = \frac{5}{3} \times \frac{5}{2} = \frac{25}{6} = 4\frac{1}{6}$.
3. It is required to divide $\frac{16}{25}$ by $\frac{4}{5}$. Ans. $\frac{4}{5}$.
4. It is required to divide $\frac{7}{16}$ by $\frac{3}{4}$. Ans. $\frac{7}{12}$.
5. It is required to divide $\frac{14}{9}$ by $\frac{7}{6}$. Ans. $1\frac{1}{3}$,
6. It is required to divide $\frac{5}{6}$ by $\frac{15}{7}$. Ans. $\frac{7}{18}$.
7. It is required to divide $\frac{12}{35}$ by $\frac{3}{5}$. Ans. $\frac{4}{7}$.
8. It is required to divide $\frac{2}{7}$ by $\frac{3}{5}$. Ans. $\frac{10}{21}$.
9. It is required to divide $\frac{9}{16}$ by 3. Ans. $\frac{3}{16}$.
10. It is required to divide $\frac{3}{5}$ by 2. Ans. $\frac{3}{10}$.
11. It is required to divide $7\frac{1}{3}$ by $9\frac{5}{9}$. Ans. $\frac{33}{43}$.
12. It is required to divide $\frac{2}{3}$ of $\frac{1}{3}$ by $\frac{5}{7}$ of $7\frac{8}{5}$. Ans. $\frac{7}{171}$.

RULE OF THREE IN VULGAR FRACTIONS.

Make the necessary preparations as before directed (p. 36, 37); then multiply continually together the second and third terms, and the first with its parts inverted as in division, for the answer †.

---

* Division being the reverse of multiplication, the reason of the rule is evident.

Note. A fraction is best divided by an integer, by dividing the numerator by it; but if it will not exactly divide, then multiply the denominator by it.

† This is only multiplying the 2d and 3d terms together, and dividing the product by the first, as in the rule of three in whole numbers.

EXAMPLES.

1. If $\frac{3}{8}$ of a yard of velvet cost $\frac{2}{5}$ of a pound sterling; what will $\frac{5}{16}$ of a yard cost?

$$\frac{3}{8} : \frac{2}{5} :: \frac{5}{16} : \frac{\cancel{8}}{3} \times \frac{\cancel{2}}{\cancel{5}} \times \frac{\cancel{5}}{\cancel{16}} = \tfrac{1}{3}l = 6s\ 8d, \text{ Answer.}$$

2. What will $3\frac{3}{8}$ oz of silver cost, at $6s$ $4d$ an ounce? Ans. $1l$ $1s$ $4\frac{1}{2}d$.

3. If $\frac{3}{16}$ of a ship be worth $273l$ $2s$ $6d$; what are $\frac{5}{32}$ of her worth? Ans. $227l$ $12s$ $1d$.

4. What is the purchase of $1230l$ bank-stock, at $108\frac{3}{8}$ per cent? Ans. $1336l$ $1s$ $9d$.

5. What is the interest of $273l$ $15s$ for a year, at $3\frac{1}{4}$ per cent? Ans. $8l$ $17s$ $11\frac{1}{4}d$.

6. If $\frac{1}{8}$ of a ship be worth $73l$ $1s$ $3d$; what part of her is worth $250l$ $10s$? Ans. $\frac{3}{7}$.

7. What length must be cut off a board that is $7\frac{3}{4}$ inches broad, to contain a square foot, or as much as another piece of 12 inches long and 12 broad? Ans. $18\frac{18}{31}$ inches.

8. What quantity of shalloon that is $\frac{3}{4}$ of a yard wide, will line $9\frac{1}{2}$ yards of cloth, that is $2\frac{1}{2}$ yards wide? Ans. $31\frac{2}{3}$ yds.

9. If the penny loaf weigh $6\frac{9}{10}$ oz. when the price of wheat is $5s$ the bushel; what ought it to weigh when the wheat is $8s$ $6d$ the bushel? Ans. $4\frac{1}{17}$ oz.

10. How much in length, of a piece of land that is $11\frac{11}{12}$ poles broad, will make an acre of land? Ans. $13\frac{61}{143}$ poles.

11. If a courier perform a certain journey in $35\frac{1}{2}$ days, travelling $13\frac{5}{8}$ hours a day; how long would he be in performing the same, travelling only $11\frac{9}{10}$ hours a day? Ans: $40\frac{615}{952}$ days.

12. A regiment of soldiers, consisting of 976 men, are to be new clothed; each coat to contain $2\frac{1}{2}$ yards of cloth that is $1\frac{5}{8}$ yard wide, and lined with shalloon $\frac{7}{8}$ yard wide; how many yards of shalloon will line them? Ans. 4351 yds 1 qr $2\frac{6}{7}$ nails.

Scholium. A rule for operations of this nature, where the first term is unity, was long in great use under the name of Practice, and was broken down into a variety of separate cases adapted to the peculiar circumstances of each question. It was doubtless owing to the apparent complexity produced by the number of cases that it was generally considered very difficult of acquisition. and has now fallen into very general disuse. It is, however, an exceedingly useful process for daily purposes, and is, in fact, of very easy acquirement; but though the present Editor intended to give a page or two on the subject here, he is compelled to omit it for want of sufficient disposable space. A very systematic view of it is given by Mr. Rutherford in his edition of Gray's Arithmetic, and to that the student is referred *.

---

* The Editor takes this opportunity also to remark, that the very best work with which he is acquainted on Arithmetic for *commercial purposes*, and apart from ulterior views, is one bearing the title of "The Quadrantal System of Arithmetic, by Daniel Harrison." The scientific reader, too, will find some articles worthy of his attention in the work; though from its object it does not take a strictly scientific form.

DECIMAL FRACTIONS.

A Decimal Fraction is that which has for its denominator an unit (1), with as many ciphers annexed as the numerator has places; and it is usually expressed by setting down the numerator only, with a point before it, on the left hand. Thus $\frac{4}{10}$ is ·4, and $\frac{24}{100}$ is ·24, and $\frac{74}{1000}$ is ·074, and $\frac{124}{100000}$ is ·00124; where ciphers are prefixed to make up as many places as there are ciphers in the denominator, when there is a deficiency in the figures. Thus, the understood denominator of a decimal is always either *ten*, or some power of *ten;* whence its name.

A mixed number is made up of a whole number with some decimal fraction, the one being separated from the other by a point. Thus, 3·25 is the same as $3\frac{25}{100}$, or $\frac{325}{100}$.

Ciphers on the right-hand of decimals make no alteration in their value; for ·4, or ·40, or ·400 are decimals having all the same value, each being $= \frac{4}{10}$, or $\frac{2}{5}$. But when they are placed on the left-hand, they decrease the value in a tenfold proportion: Thus, ·4 is $\frac{4}{10}$, or 4 tenths; but ·04 is only $\frac{4}{100}$, or 4 hundredths, and ·004 is only $\frac{4}{1000}$, or four thousandths.

In decimals as well as in whole numbers, the values of the places increase towards the left-hand, and decrease towards the right, both in the same tenfold proportion; as in the following scale or table of notation.

	millions	hundred thousands	ten thousands	thousands	hundreds	tens	units	tenth parts	hundredth parts	thousandth parts	ten thousandth parts	hundred thousandth parts	millionth parts	
. . . .	3	3	3	3	3	3	3 ·	3	3	3	3	3	3	. . . .

ADDITION OF DECIMALS.

Set the numbers under each other according to the value of their places, as in whole numbers; in which state the decimal separating points will stand all exactly under each other. Then, beginning at the right-hand, add up all the columns of numbers as in integers; and point off as many places for decimals, as are in the greatest number of decimal places as any of the lines that are added; or, place the point directly below all the other points.

EXAMPLES.

1. Add together 29·0146, and 3146·5, and 2109, and ·62417, and 14·16.

29·0146
3146·5
2109·
·62417
14·16
5299·29877 = the sum.

2. What is the sum of 276, 39·213, 72014·9, 417, and 5032?

Ans. 77779·113.

3. What is the sum of 7530, 16·201, 3·0142, 957·13, 6·72119, and ·03014?

Ans. 8513·09653.

4. What is the sum of 312·09, 3·5711, 7195·6, 71·498, 9739·215, 179, and ·0027?

Ans. 17500·9768.

SUBTRACTION OF DECIMALS.

Place the numbers under each other according to the value of their places, as in the last rule. Then, beginning at the right-hand, subtract as in whole numbers, and point off the decimals as in addition.

EXAMPLES.

1. Find the difference between 91·73 and 2·138.

```
91·73
 2·138
------
89·592 = the difference.
```

2. Find the difference between 1·9185 and 2·73. Ans. 0·8115.
3. Subtract 4·90142 from 214·81. Ans. 209·90858.
4. Find the difference between 2714 and ·916. Ans. 2713·084.

MULTIPLICATION OF DECIMALS.

* Place the factors, and multiply them together the same as if they were whole numbers. Then point off in the product just as many places of decimals as there are decimals in both the factors. But if there be not so many figures in the product, then supply the defect by *prefixing* ciphers.

EXAMPLES.

1. Multiply ·321096 by ·2465	Or, thus, see p. 12.
·321096	·321096
·2465	·2465
1605480	642192
1926576	1284384
1284384	1926576
642192	1605480
·0791501640 = the product	= ·0791501640

* The rule will be evident from this example:—Let it be required to multiply ·12 by ·361; these numbers are equivalent to $\frac{12}{100}$ and $\frac{361}{1000}$; the product of which is $\frac{4332}{100000}$ = ·04332, by the nature of Notation, which consists of as many places as there are ciphers, that is, of as many places as there are in both numbers. And in like manner we reason for any other numbers. As a general investigation, however, let the one factor have m decimal places and the other n; and let all the figures of the first number, taken as integers, be expressed by M, and all those of the other by N. Then the actual numbers are $\frac{M}{10^m}$ and $\frac{N}{10^n}$. Whence, their product is $\frac{MN}{10^{m+n}}$: that is, there are $m + n$ decimals in the quotient.

2. Multiply 79·347 by 23·15. Ans. 1836·88305.
3. Multiply ·63478 by ·8204. Ans. ·520773512.
4. Multiply ·385746 by ·00464. Ans. ·00178986144.

CONTRACTION I.

To multiply decimals by 1 *with any number of ciphers, as by* 10, *or* 100, *or* 1000.

This is done by only removing the decimal point so many places farther to the right-hand, as there are ciphers in the multiplier; and subjoining ciphers if need be.

1. The product of 51·3 and 1000 is 51300.
2. The product of 2·714 and 100 is
3. The product of ·916 and 1000 is
4. The product of 21·31 and 10000 is

CONTRACTION II.

To contract the operation so as to retain only as many decimals in the product as may be thought necessary, when the product would naturally contain several more places.

Remove the decimal point of the multiplier (if necessary) until the left-hand figure is an integer in the unit's place; and so many places as you have moved the decimal point in the multiplier to the *left* or to the *right*, remove, on the contrary, the decimal point in the multiplicand to the *right* or to the *left*. Then, place the multiplier under the multiplicand in the usual way; and begin to multiply by the left-hand figure of the multiplier, retaining in the product only so many decimals as you wish to have at last. Then, multiply by the remaining figures in the multiplier one by one, from the left towards the right; as you proceed, set each product one figure more to the left-hand; and, of course, leave out one figure more to the right-hand in each successive multiplication. The sum of these successive lines of products will give the general product required. It will always be better to calculate one place of decimals more than are required by the question. See the subsequent example and remarks.

In multiplying be very careful to increase the first right-hand figure retained in each line by what would be *carried on* from the figures omitted, in this manner: viz. add 1 if the preceding number fall between 5 and 14, 2 from 15 to 24, 3 from 25 to 34, 4 from 35 to 44, and so on. This process will usually make the general product true to the last place of decimals.

EXAMPLES.

1. Multiply 2·714986 by 924·1035, so as to retain only 4 places of decimals in the product.

This is evidently the same as to multiply 271·4986 by 9·241035; where the decimal point in the multiplicand is moved 2 places to the *right*-hand, and that in the multiplier 2 to the *left*.

Common method.	*Eastern method.*	*Contracted method.*
2·714986	2·714986	271·4986
924·1035	924·1035	9·241035
13 574930	24434874	2443487 4
81 44958	5429972	542997 2
2714 986	10859944	108599 4
108599 44	2714986	2715 0
542997 2	8144958	81 5
24434874	13574930	13 6
2508·9280 650510	2508·9280650510	2508·9280 7

By a comparison of the common with the eastern method, it will appear upon inspection that the difference is only in the *arrangement of the work*: and by a comparison of the eastern with the contracted method, it will be seen that the only difference is to leave out that portion of the multiplication which does not contribute to the figures within the limits prescribed for the contraction.

In this contraction a correction column is kept to the right of the vertical line, which, in fact, is computing one decimal place more than was required, in order to insure accuracy in the required number of places. It is always desirable to do this, as otherwise the last figure cannot be depended on; and the more so as such a correction column must always be kept in almost the only place where the method is of constant occurrence; viz. in the solution of equations, and its subordinate class of operations, the extraction of roots.

Here, in the contracted way, we have multiplied first by the left-hand figure, 9; then by the 2, omitting the product of 2 × 6, but regarding the 1 carried on; then by the 4, omitting the product of 4 × 86, but regarding the 3 carried on. The rest of the process is, in like manner, conformable to the rule; and it is much easier than the usual method of contracted multiplication by inverting the multiplier.

2. Multiply 480·14936 by 2·72416, retaining only four decimals in the product.

3. Multiply 2490·3048 by ·573286, retaining only five decimals in the product.

4. Multiply 325·701428 by ·7218393, retaining only three decimals in the product.

DIVISION OF DECIMALS.

Divide as in whole numbers; and point off in the quotient as many places for decimals as the decimal places in the dividend exceed those in the divisor *.

* The reason of this rule is evident; for, since the divisor multiplied by the quotient gives the dividend, therefore the number of decimal places in the dividend is equal to those in the divisor and quotient, taken together, by the nature of multiplication; and consequently the quotient itself must contain as many as the dividend exceeds the divisor.

The investigation may, as in the last case, be here given in a general form.

Let $\frac{M}{10^m}$ and $\frac{N}{10^n}$ be the divisor and dividend respectively, which designates

$$\frac{N}{10^n} \div \frac{M}{10^m} = \frac{10^m N}{10^n M} = \frac{10^{m-n} N}{M} = \frac{N}{10^{n-m} M}$$

When m is greater than n, there will be a removal of the decimal point $m-n$ places to the right, or, in other words, if the division be complete, $m-n$ ciphers must be added; but when n is greater than m, there will be $n-m$ decimal places.

Another way to know the place for the decimal point is this: *The first figure of the quotient must be made to occupy the same place of integers or decimals, as that figure of the dividend which stands over the unit's figure of the first product.*

When the places of the quotient are not so many as the rule requires, the defect is to be supplied by prefixing ciphers.

When there happens to be a remainder after the division; or when the decimal places in the divisor are more than those in the dividend; then ciphers may be annexed to the dividend, and the quotient carried on as far as required.

EXAMPLES.

1.

```
178 ) ·48520998 ( ·00272589
          1292
            460
            1049
              1599
                1758
                  156
```

2.

```
·2639 ) 27·00000 ( 102·3114
           6100
             8220
               3030
                 3910
                 12710
                   2154
```

3. Divide 123·70536 by 54·25 Ans. 2·2802.
4. Divide 12 by ·7854. Ans. 15·278.
5. Divide 4195·68 by 100. Ans. 41·9568.
6. Divide ·8297592 by ·153. Ans. 5·4232.

CONTRACTION I.

When the divisor is an integer, with any number of ciphers annexed; cut off those ciphers, and remove the decimal point in the dividend as many places farther to the left as there are ciphers cut off, prefixing ciphers, if need be; then proceed as before.

EXAMPLES.

1. Divide 45·5 by 2100.

```
21·00 ) ·455 ( ·0216 ....
          35
          140
           14
```

Or thus:—

```
3 | ·455
  ---------
7 | ·151666 ....
  ---------
    ·021666 ....
```

2. Divide 41020 by 32000.
3. Divide 953 by 21600.
4. Divide 61 by 79000.

CONTRACTION II.

Hence, if the divisor be one with ciphers, as 10, 100, or 1000; then the quotient will be found by merely moving the decimal point in the dividend so many places farther to the left, as the divisor has ciphers; prefixing ciphers, if need be.

EXAMPLES.

So, 217·3 ÷ 100 = 2·173
And 5·16 ÷ 100 =
And 419 ÷ 10 =
And ·21 ÷ 1000 =

CONTRACTION III.

When there are many figures in the divisor; or when only a certain number of decimals are necessary to be retained in the quotient; then take only as many

figures of the divisor as will be equal to the number of figures, both integers and decimals, to be in the quotient, and find how many times they may be contained in the first figures of the dividend, as usual.

Let each remainder be a new dividend; and for every such dividend, leave out one figure more on the right-hand side of the divisor: remembering to carry for the increase of the figures cut off, as in the second contraction in multiplication.

Note. When there are not so many figures in the divisor as are required to be in the quotient, begin the operation with all the figures, and continue it as usual till the number of figures in the divisor be equal to those remaining to be found in the quotient; after which begin the contraction.

EXAMPLES.

1. Divide 2508·92806 by 92·41035, so as to have only four decimals in the quotient, in which case the quotient will contain six figures.

1. *Common method.*	2. *Contracted method.*
92·41035) 2508·928 06 (27·1498	92·4103 5) 2508·928 0,6 (27·1498
1848207 0	1848207 0
660721 06	660721 1
646872 45	646872 5
13848 610	13848 6
9241 035	9241 0
4607 5750	4607 6
3696 4140	3696 4
911 16100	911 2
831 69315	831 7
79 467850	79 5
73 928280	73 9
5 539570	5 6

In this operation, as in multiplication, a correction column to the right of the vertical line should be kept. The method itself is obviously only a rejection of the figures which do not contribute to the result within the prescribed limits.

2. Divide 4109·2351 by 230·409, so that the quotient may contain only four decimals. Ans. 17·8345.

3. Divide 37·10438 by 5713·96, that the quotient may contain only five decimals. Ans. ·00649.

4. Divide 913·08 by 2137·2, that the quotient may contain only three decimals.

REDUCTION OF DECIMALS.

CASE I.

To reduce a vulgar fraction to its equivalent decimal.

Divide the numerator by the denominator, as in division of decimals, annexing ciphers to the numerator, as far as necessary; so shall the quotient be the decimal required *.

* It will frequently happen (indeed always when the fraction in its lowest terms has in its denominator any factors besides 2 and 5, or powers and products of these,) that the division will

EXAMPLES.

1. **Reduce $\frac{7}{24}$ to a decimal.**

$$24 = 4 \times 6. \quad \text{Then } 4\,|\,7\cdot$$
$$6\,|\,1\cdot750000$$
$$\cdot291666\ \ldots\ldots$$

---

never terminate. In this case, it is commonly sufficient to carry it to some specific number of quotient places, and neglect the remaining ones as of *comparatively* no value. If, however, the question be one in which the *accurate result instead of the approximation* is required, it will be necessary to work by vulgar fractions instead of decimals. In this case any other decimal quantities that have presented themselves amongst the data of the question can be readily thrown into the usual fractional form for greater facility of combination with the fraction above mentioned.

A rapid and elegant method of throwing a vulgar fraction, whose denominator is a prime number, into a decimal consisting of a great number of figures, is given by Mr. *Colson*, in page 162 of *Sir Isaac Newton's Fluxions*. It will be readily understood from the following example :—

Let $\frac{1}{29}$ be the fraction which is to be converted into an equivalent decimal.

Then, by dividing in the common way till the remainder becomes a single figure, we shall have $\frac{1}{29} = \cdot03448\frac{8}{29}$ for the complete quotient, and this equation being multiplied by the numerator 8, will give $\frac{8}{29} = \cdot27584\frac{64}{29}$, or rather $\frac{8}{29} = \cdot27586\frac{6}{29}$: and if this be substituted instead of the fraction in the first equation, it will make $\frac{1}{29} = \cdot0344827586\frac{6}{29}$. Again, let this equation be multiplied by six, and it will give $\frac{6}{29} = \cdot2068965517\frac{7}{29}$; and then by substituting as before

$$\tfrac{1}{29} = \cdot03448275862068965517\tfrac{7}{29};$$

and so on, as far as may be thought proper; each fresh multiplication doubling the number of figures in the decimal value of the fraction.

In the present instance the decimal *circulates* in a complete period of 23 figures, i. e. one less than the denominator of the fraction. This, again, may be divided into equal periods, each of 14 figures, as below :

$$\cdot03448275862068$$
$$\cdot96551724137931$$

in which it will be found that each figure with the figure vertically below it makes 9; $0 + 9 = 9$; $3 + 6 = 9$; and so on. This circulate also comprehends all the separate values of $\frac{2}{29}$, $\frac{3}{29}$, $\frac{4}{29}$…… in corresponding circulates of 28 figures, only each beginning in a distinct place, easily ascertainable. Thus, $\frac{2}{29} = \cdot06896$…… beginning at the 12th place of the primitive circulate. $\frac{3}{29} = \cdot103448$…… beginning at the 28th place. So that, in fact, this circle includes 28 complete circles.

The property of circulation of some number of the figures in periods is not one peculiar to this or any other of the *interminable decimals*, but belongs to them all. Sometimes the period is composed of a single figure, as $\cdot333$…… which is the *decimal expansion* of $\frac{1}{3}$; sometimes of a considerable number, as in the example above given from *Colson*. It is, however, never composed of a number of places so great as is expressed by the denominator : as the expansion of $\frac{1}{3}$ is composed of *one* term, and *one* is less than the denominator, three ; and the period of the expansion of $\frac{1}{29}$ is composed of 28 places, and less therefore than the denominator 29. Sometimes the circulating periods are composed of a half, or a fourth of the number of places that would be expressed by subtracting one from the denominator.

Whenever in the division a remainder occurs that has occurred before, (the number brought down from the dividend being necessarily the same in both cases, viz. 0,) then the same quotient figure will again occur, and the same remainders will occur after both ; and hence the same quotient figures, and the same remainders, will continually succeed, till the first mentioned remainder again occurs. A continual circulation of the same process will thence take place without end.

The same circumstances would obviously take place if, instead of bringing down ciphers, as above mentioned, we brought down any other figure from the dividend, or even the figures of any circulating dividend.

That there must be a less number of figures in the circulating period, than is expressed by the denominator, appears from this. The only remainders that can exist are all less than the denominator, and being integers, their number is less than is expressed by it : and hence the number of steps in the division that takes place without falling upon the same remainder must

2. Reduce $\frac{1}{4}$, and $\frac{1}{2}$, and $\frac{3}{4}$, to decimals. Ans. ·25, and ·5, and ·75.
3. Reduce $\frac{5}{8}$ to a decimal. Ans. ·625.
4. Reduce $\frac{3}{25}$ to a decimal. Ans. ·12.
5. Reduce $\frac{6}{192}$ to a decimal. Ans. ·03125.
6. Reduce $\frac{550}{3842}$ to a decimal. Ans. ·143154 ....

CASE II.

To find the value of a decimal in terms of the inferior denominations.

Multiply the decimal by the number of parts in the next lower denomination; and cut off as many places for a remainder, to the right-hand, as there are places in the given decimal.

Multiply that remainder by the parts in the next lower denomination again, cutting off for another remainder as before.

Proceed in the same manner through all the parts of the integer; then the several denominations separated on the left-hand will make up the answer.

Note. This operation is the same as reduction descending in whole numbers.

EXAMPLES.

1. Required to find the value of ·775 pounds sterling.

$$\begin{array}{r} \cdot775 \\ 20 \\ \hline s\ 15\cdot500 \\ 12 \\ \hline d\ 6\cdot000 \end{array} \quad \text{Ans. } 15s\ 6d.$$

2. What is the value of ·625 shil? Ans. $7\frac{1}{2}d$.
3. What is the value of ·8635*l*? Ans. 17*s* 3·24*d*.
4. What is the value of ·0125 lb troy? Ans. 3 dwts.
5. What is the value of ·4694 lb troy? Ans. 5 oz 12 dwts 15·744 gr.
6. What is the value of ·625 cwt? Ans. 2 qr 14lb.
7. What is the value of ·009943 miles? Ans. 17 yd 1 ft 5·98848 inc.
8. What is the value of ·6875 yd? Ans. 2 qr 3 nls.
9. What is the value of ·3375 acre? Ans. 1 rd 14 poles.
10. What is the value of ·2083 hhd of wine? Ans. 13·1229 gal.

---

at the most be one less than the number expressed by the denominator; and hence again the number of quotient figures must be at most one less than the same number. That is, the quotient is composed of such repeating circles as we have stated.

Any further examination of this subject (which, nevertheless, is a very curious and a very important one) would be incompatible with the limits of this work. The method of finding the value of such a decimal will, however, be found in the Chapter on *Geometrical Progression*: and we may refer the inquiring student also to Mr. Goodwin's *Tables of Decimal Circles*, and to the *Ladies' Diary* for 1824.

A convenient notation has been used to designate the circulating period which is that of putting a dot over the first and last figures of the period. Thus $72\cdot968\dot{6}2\dot{5}$ signifies that 625 is the circulating period. When the period is at only one place, there is but one dot required, as in the value of $\frac{1}{3}$, which is $\cdot\dot{3}$.

The reverse problem, of finding the finite fractional value of an interminable decimal, will be found in Geometrical Progression, in the Algebra.

CASE III.

To reduce integers or decimals to equivalent decimals of higher denominations.

Divide by the number of parts in the next higher denomination; continuing the operation to as many higher denominations as may be necessary, the same as in reduction ascending of whole numbers.

EXAMPLES.

1. Reduce 1 dwt to the decimal of a pound troy.

20	1 dwt
12	0·05 oz
	$0·00416\dot{6}$ lb. Ans.

2. Reduce 9*d*. to the decimal of a pound.	Ans. ·0375*l*.
3. Reduce 7 drams to the decimal of a pound avoird.	Ans. ·02734375 lb.
4. Reduce ·26*d* to the decimal of a pound.	Ans. $·00108\dot{3}36l$.
5. Reduce 2·15 lb to the decimal of a cwt.	Ans. ·019196.... cwt.
6. Reduce 24 yards to the decimal of a mile.	Ans. ·013636.... mile
7. Reduce ·056 pole to the decimal of an acre.	Ans. ·00035 acre.
8. Reduce 1·2 pint of wine to the decimal of a hhd.	Ans. ·00238.... hhd.
9. Reduce 14 minutes to the decimal of a day.	Ans. ·009722.... day.
10. Reduce .21 pint to the decimal of a peck.	Ans. ·013125 peck.
11. Reduce 28^{s} 12^{th} to the decimal of a minute.	

Note. *When there are several numbers, to be reduced all to the decimal of the highest.*

Set the given numbers directly under each other, for dividends, proceeding orderly from the lowest denomination to the highest.

Opposite to each dividend, on the left hand, set such a number for a divisor as will bring it to the next higher name; drawing a perpendicular line between all the divisors and dividends.

Begin at the uppermost, and perform all the divisions: only observing to set the quotient of each division, as decimal parts, on the right hand of the dividend next below it; so shall the last quotient be the decimal required.

EXAMPLES.

1. Reduce 17*s*. 9¾*d* to the decimal of a pound.

4	3·
12	9·75
20	17·8125
	£0·890625 Ans.

2. Reduce 19*l* 17*s* 3¼*d* to pounds.	Ans. 19·8635416*l*.
3. Reduce 15*s* 6*d* to the decimal of a pound.	Ans. ·775*l*.
4. Reduce 7½*d* to the decimal of a shilling.	Ans. ·625*s*.
5. Reduce 5 oz 12 dwts 16 gr. to lb.	Ans. ·4694 lb.

RULE OF THREE IN DECIMALS.

Prepare the terms, by reducing the vulgar fractions to decimals, and any compound numbers either to decimals of the higher denominations, or to integers of the lower, also the first and third terms to the same name: then multiply and divide as in whole numbers.

Note. Any of the convenient examples in the single or double rule of three in integers, or vulgar fractions, may be taken as proper examples to the same rules in decimals.—The following example, which is the first in vulgar fractions, is wrought out here, to show the method.

If $\frac{3}{8}$ of a yard of velvet cost $\frac{2}{3}l$, what will $\frac{5}{16}$ yd cost?

$\frac{3}{8} = {\cdot}375$

$\frac{2}{3} = {\cdot}4$

$\frac{5}{16} = {\cdot}3125$

yd		£		yd		£	s	d
·375	:	·4	∷	·3125	:	·333 .... or	6	8
				·4				

·375) ·12500 (·333333....
1250 — 20
125 — s 6·66666....
12
d 7·99999.... = 8*d*.

Ans. 6*s* 8*d*.

DUODECIMALS.

Duodecimals, or Cross Multiplication, is a rule used by workmen and artificers, in computing the contents of their works.

Dimensions are usually taken in feet, inches, and quarters; any parts smaller than these being neglected as of no consequence. And the same in multiplying them together, or computing the contents. The method is as follows:

Set down the two dimensions to be multiplied together, one under the other, so that feet may stand under feet, inches under inches, and so on.

Multiply each term in the multiplicand, beginning at the lowest, by the feet in the multiplier, and set the result of each directly under its corresponding term, observing to carry 1 for every 12, from the inches to the feet.

In like manner, multiply all the multiplicand by the inches and parts of the multiplier, and set the result of each term one place removed to the right-hand of those in the multiplicand; omitting, however, what is below parts of inches, only carrying to these the proper numbers of units from the lowest denomination.

Or, instead of multiplying by the inches, take such parts of the multiplicand as these are of a foot.

Then add the two lines together, after the manner of compound addition, carrying 1 to the feet for every 12 inches, when these come to so many.

EXAMPLES.

1. Multiply 4 f 7 inc
 by 6 4
 27 6
 1 $6\frac{1}{3}$
 Ans. 29 $0\frac{1}{3}$

2. Multiply 14 f 9 inc
 by 4 6
 59 0
 7 $4\frac{1}{2}$
 Ans. 66 $4\frac{1}{2}$

3. Multiply 4 feet 7 inches by 9 feet 6 inches. Ans. 43 f $6\frac{1}{2}$ inc
4. Multiply 12 f 5 inc by 6 f 8 inc. Ans. 82 $9\frac{1}{3}$
5. Multiply 34 f $4\frac{1}{2}$ inc by 12 f 3 inc. Ans. 421 $1\frac{1}{8}$
6. Multiply 64 f 6 inc by 8 f $9\frac{1}{4}$ inc. Ans. 565 $8\frac{5}{8}$

Note. The denomination which occupies the place of inches in these products, means not square inches, but *rectangles of an inch broad and a foot long.* Thus,

the answer to the first example is 29 sq. feet, 4 sq. inches ; to the second 66 sq
feet, 54 sq. inches.

If the resulting product be one of three dimensions, length, breadth, an
thickness, then the first denomination to the right of the feet must be multiplie
by 144, the second by 12, and these products added to the figure in the thir
place, will give cubic inches *.

---

* Though it is the *practice* to neglect all the smaller dimensions than inches or half inche
both in actual measuring among artificers, and of course in the computations which are mad
from such surveys; yet in theory, all the subordinate dimensions are reckoned in a descendin
scale of twelves, as in our common numbers, we employ a descending scale of tens: and in a
cases where the theoretical result is required, the process must be continued in the same wa
Instead of the descending denominations below the units, tenths, hundredths, thousandths, &
the terms, parts or primes, seconds, thirds, &c. descending successively below the inch, ar
employed. We are obliged, however, to keep the denominations in separate columns, o
separated by a blank space, or by dots („) in our calculations in Duodecimals, instead of placin
the numbers in each successive denomination in juxta-position as in our common notation. Bu
if there were distinct and concentrated symbols employed to designate the numbers 10 and 1
(as Φ or ϕ and Π or π, as is done by some writers on the *Theory of Numbers*,) then we migh
dispense with the extra spaces, columns, or dots, and write the results continuously. All th
advantages of the decimal notation in point of simplicity of writing would thus be gained for th
duodecimal : and it is quite obvious that the same method is applicable to any other scale o
numbers and its corresponding notation.

The great barrier, however, to any change, except in the particular instance of feet an
inches, is the *terminology*, or *names of numbers*, which could not possibly co-exist with a change o
scale. The names are of an origin so decidedly and obviously decimal, that it requires som
degree of fixed attention to ascertain how many *dozens* there are in any number specifie
decimally. All our language and all our ideas of number flow in terms of the decimal scale
and hence, however desirable it might appear in the eyes of some of the most enlightene
mathematicians to adopt the duodecimal or dozen scale, the inveterate adherence which ever
people feels to its old *language*, and the consequent (in this case) daily practice, forbids eve
the most distant hope of ever realizing such a project.

The intelligent teacher, however, may avail himself of the principle employed in duodecima
to explain to his more intelligent pupils the nature of numerical scales in general. Such pupi
are now arrived at a stage in their arithmetical studies which renders such knowledge essential.

A single specimen of the process of complete duodecimal multiplication expressed in th
common and contracted notations is here subjoined, which it is hoped will give a clear idea o
the views expressed above.

	ft	in	pr	''			
Multiply	3	10	11	8			
by	2	3	11	10			
	7	9	11	4			
		11	8	11	0		
		3	7	0	8	4	
			3	3	1	8	8
Product	9	1	6	6	10	0	8
	ft	in	pr	''	'''	iv	v

By the contracted notation.

```
     3 φ π 8
   2 3 π φ
   ---------
   3 3 1 8 8
 3 7 0 8 4
 π 8 π 0
7 9 π 4
-------------
9 1 6 6 φ 0 8
ft. in. pr. '' ''' iv. v.
```

```
3 φ π 8
2 3 π φ
---------
7 9 π 4
 π 8 π 0
  3 7 0 8 4
   3 3 1 8 8
-------------
9 1 6 6 φ 0 8
ft. in. pr. '' ''' iv. v.
```

The want of *higher* classes of twelves prevents our proceeding to the left without encounterin
the decimal notation for 10, 100, ... feet.

INVOLUTION.

INVOLUTION is the raising of Powers from any given number, as a root.

A Power is a quantity produced by multiplying any given number, called the Root, a certain number of times continually by itself. Thus,

$$2 = 2 \text{ is the root, or 1st power of } 2;$$
$$2 \times 2 = 4 \text{ is the 2d power, or square of } 2;$$
$$2 \times 2 \times 2 = 8 \text{ is the 3d power, or cube of } 2;$$
$$2 \times 2 \times 2 \times 2 = 16 \text{ is the 4th power of } 2; \text{ and so on};$$

and in this manner may be calculated the following table of the first nine powers of the first 9 numbers.

TABLE OF THE FIRST NINE POWERS OF THE FIRST NINE NUMBERS.

1st	2d	3d	4th	5th	6th	7th	8th	9th
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

The Index or Exponent of a power, is the number denoting the height or degree of that power; and it is 1 more than the number of multiplications used in producing the same. Thus 1 is the index or exponent of the 1st power or root, 2 of the 2nd power or square, 3 of the 3rd power or cube, 4 of the 4th power, or biquadrate, and so on.

Powers, that are to be raised, are usually denoted by placing the index above the root or first power.

$$\text{So } 2^2 = 4 \text{ is the 2d power of 2.}$$
$$2^3 = 8 \text{ is the 3d power of 2.}$$
$$2^4 = 16 \text{ is the 4th power of 2.}$$
$$540^4 = 85030560000 \text{ is the 4th power of 540.}$$

When two or more powers are multiplied together, their product is that power whose index is the sum of the exponents of the factors or powers multiplied. Or the multiplication of the powers answers to the addition of the indices. Thus, in the following powers of 2,

	1st	2d	3d	4th	5th	6th	7th	8th	9th	10th
	2	4	8	16	32	64	128	256	512	1024
or	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}

Here, $4 \times 4 = 16$, and $2 + 2 = 4$ its index;
and $8 \times 16 = 128$, and $3 + 4 = 7$ its index;
also $16 \times 64 = 1024$, and $4 + 6 = 10$ its index.

OTHER EXAMPLES.

1. What is the 2d power of 45? Ans. 2025.
2. What is the square of 4·16? Ans. 17·3056.
3. What is the 3d power of 3·5? Ans. 42·875.
4. What is the 5th power of ·029? Ans. ·000000020511149
5. What is the square of $\frac{2}{3}$? Ans. $\frac{4}{9}$.
6. What is the 3d power of $\frac{5}{9}$? Ans. $\frac{125}{729}$.
7. What is the 4th power of $\frac{3}{4}$? Ans. $\frac{81}{256}$.

EVOLUTION.

Evolution, or the reverse of Involution, is the extracting or finding the roots of any given powers.

The root of any number, or power, is such a number, as being multiplied into itself a certain number of times, will produce that power. Thus, 2 is the square root, or 2d root of 4, because $2^2 = 2 \times 2 = 4$; and 3 is the cube root or 3d root of 27, because $3^3 = 3 \times 3 \times 3 = 27$.

Any power of a given number or root may be found exactly, namely, by multiplying the number continually into itself. But there are many numbers of which a proposed root can never be exactly found; those numbers being themselves incapable of being produced by the involution (to the corresponding power) of any root composed of a finite number of integer or decimal places. Yet, by means of decimals, we may approximate or approach towards the root, to any assigned degree of exactness.

Those roots which only approximate are called Surd Roots; but those which can be found quite exactly, are called Rational Roots. Thus, the square root of 3 is a surd root; but the square root of 4 is a rational root, being equal to 2: also the cube root of 8 is rational, being equal to 2; but the cube root of 9 is surd or irrational.

Roots are sometimes denoted by writing the character $\sqrt{}$ before the power, with the index of the root against it. Thus, the 3d root of 20 is expressed by $\sqrt[3]{20}$; and the square root or 2d root of it is $\sqrt{20}$, the index 2 being always omitted, when only the square root is designed.

When the power is expressed by several numbers, with the sign + or − between them, a line is drawn from the top of the sign over all the parts of it: thus the third root of 45 − 12 is $\sqrt[3]{\overline{45 - 12}}$, or thus, $\sqrt[3]{}(45 - 12)$, enclosing the numbers in parentheses, which is, usually, the best way to express it.

But all roots are now often designed like powers, with fractional indices; thus the square root of 8 is $8^{\frac{1}{2}}$, the cube root of 25 is $25^{\frac{1}{3}}$, and the 4th root of 45 − 18 is $(45 - 18)^{\frac{1}{4}}$.

TO EXTRACT THE SQUARE ROOT.

* Divide the given number into periods of two figures each, by setting a point over the place of units, another over the place of hundreds, and so on,

* The reason for separating the figures of the dividend into periods or portions of two places each, is, that the square of any single figure never consists of more than two places; the square of a number of two figures, of not more than four places, and so on. So that there will be as many figures in the root as the given number contains periods so divided or parted off.

And the reason of the several steps in the operation appears from the algebraic form of the square of any number of terms, whether two or three or more. Thus

$(a + b)^2 = a^2 + 2ab + b^2 = a^2 + (2a + b)b$, the square of two terms; where it appears that a is the first term of the root, and b the second term; also a the first divisor, and the new divisor

over every second figure, both to the left hand in integers, and to the right in decimals.

Always *begin to point at the place of units;* or, if the number to be extracted be entirely decimal, put a cipher in the unit's place, and over it put the first point.

Find the greatest square in the first period on the left hand, and set its root on the right hand of the given number, after the manner of a quotient figure in Division.

Subtract the square thus found from the same period, and to the remainder annex the two figures of the next following period, for a dividend.

Double the root above mentioned for a divisor; and find how often it is contained in the said dividend, exclusive of its right-hand figure; and set that quotient figure both in the quotient and divisor.

Multiply the whole augmented divisor by this last quotient figure, and subtract the product from the said dividend, bringing down to it the next period of the given number, for a new dividend.

Repeat the same process over again, viz. find another new divisor, by doubling all the figures now found in the root; from which, and the last dividend, find the next figure of the root as before; and so on through all the periods, to the last.

Note. The best way of doubling the root, to form the new divisors, is by adding the last figure always to the last divisor, as appears in the following examples. Also, after the figures belonging to the given number are all exhausted, the operation may be continued into decimals at pleasure, by adding any number of periods of ciphers, two in each period.

EXAMPLES.

1. To find the square root of 29506624.

$$\begin{array}{r|l}
 & 2\dot{9}5\dot{0}6\dot{6}2\dot{4}\ (\ 5432 \text{ the root.} \\
 & 25 \\
\hline
104 & 450 \\
4 & 416 \\
\hline
1083 & 3466 \\
3 & 3249 \\
\hline
10862 & 21724 \\
 & 21724
\end{array}$$

---

is $2a + b$, or double the first term increased by the second. And hence the manner of extraction is thus:

$$\text{1st divisor } a)\ a^2 + 2ab + b^2\ (a + b \text{ the root.}$$
$$a^2$$
$$\begin{array}{r|l}
\text{2d divisor } 2a + b & 2ab + b^2 \\
b & 2ab + b^2
\end{array}$$

Again, for a root of three parts, a, b, c, thus:

$$(a + b + c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2 =$$

$a^2 + (2a + b)\,b + (2a + 2b + c)\,c$, the square of three terms, where a is the first term of the root, b the second, and c the third term; also a the first divisor, $2a + b$ the second, and $2a + 2b + c$ the third, each consisting of the double of the root increased by the next term of the same. In like manner $(a + b + c + d)^2 = a^2 + (2a + b)\,b + (2a + 2b + c)\,c + (2a + 2b + 2c + d)\,d$; and so on, to whatever number of terms we proceed. And the mode of extraction agrees with the rule. See farther, Case 2, of Evolution in the Algebra.

For an approximation observe that $\sqrt{a^2 + n} = a \cdot \frac{4a^2 + 3n}{4a^2 + n}$ nearly, in all cases where n is small in respect of a.

NOTE. *When the root is to be extracted to many places of figures, the work may be considerably shortened, thus :*

Having proceeded in the extraction after the common method, till there be found *half* the required number of figures in the root, or one figure more ; then for the rest, divide the last remainder by its corresponding divisor, after the manner of the third contraction in division of decimals; thus;

2. To find the root of 2 to nine places of figures.

2 (1·4142|1356 the root,

	1
24	100
4	96
281	400
1	281
2824	11900
4	11296
28282	60400
2	56564
28284	3836
....	1007 6
	160 1
	18 7
	1 8

The figures 1356, to the right of the vertical line, being obtained simply by division.

3. What is the square root of 2025 ? Ans. 45.
4. What is the square root of 17·3056 ? Ans. 4·16.
5. What is the square root of ·000729 ? Ans. ·027.
6. What is the square root of 3 ? Ans. 1·732050.
7. What is the square root of 5 ? Ans. 2·236068.
8. What is the square root of 6 ? Ans. 2·449489.
9. What is the square root of 7 ? Ans. 2·645751.
10. What is the square root of 10 ? Ans. 3·162277.
11. What is the square root of 11 ? Ans. 3·316624.
12. What is the square root of 12 ? Ans. 3·464101.

RULES FOR THE SQUARE ROOTS OF VULGAR FRACTIONS AND MIXED NUMBERS.

FIRST prepare all vulgar fractions, by reducing them to their least terms, both for this and all other roots. Then

1. Take the root of the numerator and of the denominator for the respective terms of the root required ; which is the best way if the denominator be a complete power : but if it be not, then

2. Multiply the numerator and denominator together; take the root of the product : this root being made the numerator to the denominator of the given fraction, or made the denominator to the numerator of it, will form the fractional root required.

$$\text{That is, } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{ab}}{b} = \frac{a}{\sqrt{ab}}.$$

This rule will serve, whether the root be finite or infinite : and sometimes one of these expressions will simplify the operation, sometimes another ; as will be learnt from a little experience.

3. Or reduce the vulgar fraction to a decimal, and extract its root. This is *generally* the best method in practice when the terms of the fraction are not very low primes or very simple composite numbers.

4. Mixed numbers may be either reduced to improper fractions, and extracted by the first or second rule, or the vulgar fraction may be reduced to a decimal, then joined to the integer, and the root of the whole extracted.

EXAMPLES.

1. What is the root of $\frac{25}{36}$? Ans. $\frac{5}{6}$.
2. What is the root of $\frac{27}{147}$? Ans. $\frac{3}{7}$.
3. What is the root of $\frac{9}{12}$? Ans. 0·866025.
4. What is the root of $\frac{5}{12}$? Ans. 0·645497.
5. What is the root of $17\frac{3}{8}$? Ans. 4·168333.

By means of the square root also may readily be found the 4th root, or the 8th root, or the 16th root, that is, the root of any power whose index is some power of the number 2; namely, by extracting so often the square root as is denoted by that power of 2; that is, two extractions for the 4th root, three for the 8th root, and so on.

So, to find the 4th root of the number 21035·8, extract the square root two times in succession, as follows:

```
        21035·8000    ( 145·037237 ( 12·0431407 the 4th root.
        1               1
  24 | 110            22  | 45
   4 |  96             2  | 44
 285 | 1435          2404 | 10372
   5 | 1425             4 |  9616
29003 | 108000      24083 | 75637
    3 |  87009          3 | 72249
29006    20991      24086    3388
           687                980
           107                 17
```

Ex. 2. What is the 8th root of 97·41 to four places of decimals?

Ex. 3. Find the 16th root of 15 to three decimals.

TO EXTRACT THE CUBE ROOT *.

1. Divide the page into three columns, and call them L, M, N, in order from left to right, and so that M shall be double the breadth of L, and N double the breadth of M. At the head of N place the number whose root is to be extracted;

* This method was invented by the late Mr. W. G. Horner, and forms only one single and simple application of a universal method of extracting all roots whatever; and even his method, as applied to roots, is only a particular application of his general method of solving numerical equations of all orders.

In the form here given it is rather more concise in the operation than as generally applied to equations of a higher order, and is put in this form for the use of those students who are not likely to proceed to the higher equations. It is recommended to those who intend to pursue the study of the subject to any extent, to adopt the form of work given in the equations in this volume, in preference to the present one, on account of the uniformity amongst the processes for all roots whatever.

and mark off the place for the root as the quotient is marked in division, or the root in the extraction of the square root. Thus:—

L	M	N	
3 × root already found.	3 × square of root already found.	Number whose root is to be found.	Root.

2. Divide the number into periods of three figures * each, by setting a point over the place of units, and also over every third figure from thence to the left-hand in whole numbers, and to the right in decimals. Find the nearest les cube number to the first (or left-hand) period, and having subtracted it there-from, annex the next period to the remainder. Call this the resolvend. Also set the root of the said cube in the place appropriated to the root.

3. In the column L put three times the root already found, and in M put three times its square. With this last number as a trial divisor of the resolvend omitting the two figures to the right-hand, find the next figure of the root, and annex it to the former one, and also to the number in the column L. Multiply the number now in L by the new figure of the root, and place the product under that in M, but having its figures two places more to the right than the numbers already there. Add the two numbers together, and they will form the corrected divisor.

4. Multiply the corrected divisor by the root figure, and place it under the number in N, and subtract it therefrom †.

Note. If the number last found should be greater than the number previously in N, the subtraction cannot be performed: the work dependent upon the root figure which produced it must, therefore, be cancelled, and a smaller root figure tried instead of it. If necessary, the same process must be again repeated, till a number is found which will admit of being subtracted according to the rule.

5. Write twice the last root figure under the number L, and the square of the last root figure under M. Add the two last written numbers in L together, and

---

* The reason for pointing the given number into periods of three figures each, is, because the cube of one figure never amounts to more than three places; the cube of two figures to more than six, but always more than three; the cube of three figures never to more than nine but always more than six; and so on to any extent.

For a similar reason, a given number is pointed into periods of four figures, of five figures of n figures, when the fourth, fifth, n^{th} roots are to be extracted.

† A little attention to the composition of the algebraic expression for the cube of a binomial will render the truth of this rule very obvious. For $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a$ $+ \left\{3a^2 + (3a+b) \times b\right\} b$, the last form of which is exactly that whose composition is directed in rules 3 and 4. Thus:—

L	M	N
$3a$	$3a^2$	$a^3 + 3a^2b + 3ab^2 + b^3 + B + C \ldots (a+b$
$+b$	$(3a+b) \times b$	a^3
$3a+b$	$3a^2 + (3a+b)\,b$	$+ 3a^2b + 3ab^2 + b^3$
		$\left\{3a^2 + (3a+b)\,b\right\} b$
		$+ B + C \ldots .$

when B, C, &c. stand for other quantities to be afterwards brought down for continuing the process.

the three last written numbers in M *. These numbers will be respectively triple the new root and triple its square; and we may proceed with them to find a new root figure, and complete the operation as in rules 3 and 4.

Note. After about one-third of the number of figures in the root have been obtained (or one-third of the number intended to be taken into account if the root be interminable) we may contract the work very considerably by the following method:—

Cut off one figure from M, and two from L, by vertical lines. Then with the new root figure work as in contracted multiplication, p. 57, keeping as the correction columns those immediately to the right of the vertical lines. We shall thus avoid all the work which does not contribute to the derivation of the figures of the root to the extent assigned.

In the column M, the correction of the square of the last root figure falling to the right of the correction column is not put down, as in Ex. 2.

Proceeding thus, we shall, in as many steps as we had taken before the contraction began, have cut off all the addends that would arise from L; and the process for obtaining the remaining third of the figures will be identical with that for *Contracted Division*, p. 59.

Ex. 1. Extract the cube root of 4822854.

L	M	N	
$9\cancel{7}$	27..	$\dot{4}82\dot{2}85\dot{4}4$ $(3\cancel{7}64$	
96	$\cancel{6}79$	27	
12	$\cancel{2}\cancel{3}79$	21228	= first resolvend.
1084	576	$\cancel{2}\cancel{3}6\cancel{5}\cancel{3}$	
	3276	19656	
	36	1572544	= second resolvend.
	3888	1572544	
	4336		
	393136		

Statement of the process. Here the number N being pointed as prescribed in the rule, the first period is 48, and the greatest integral cube contained in it is 27. Its root 3 is put in the root place, and the first resolvend is 21228. The trial divisor or number in M is 27, or three times the square of the root figure, and the number L is three times the root figure, or 9. The trial quotient, or integer part of $\frac{212}{27}$, is 7. Annex this to L, which gives 97, and add 7×97 to M, the figures being carried two places to the right, which gives 3379, the product of which by 7 gives 23653. This cannot be subtracted from the resolvend: and hence the whole work dependent on 7 is to be obliterated, and a lower number, 6, tried

Performing the same operations with 6 as has been just described with 7, we get 96, 3276, and 19656 for L, M, and N, the last of which subtracted gives for the next resolvend 1572544.

Adding 2×6 and 6^2 to the columns L and M respectively, taking in the 576

* That this method produces the triple of the new root in the column L, and the triple of its square in the column M, may be thus immediately shown:—

L	M	N
$\vdots$	$\vdots$	$\vdots$ $(a + b$
	$(3a + b)\,b$	
$3a + b$	$3a^2 + (3a + b)\,b$	
$2b$	$+ b^2$	
$3a + 3b$	$3a^2 + 6ab + 3b^2$	
$= 3\,(a + b)$	$= 3\,(a + b)^2$	

And we thus start from the last root $a + b$ as we did at first from a.

of the latter, we have the triple (= 108) and triple the square (= 3888) of 36 in L and M respectively.

The trial divisor 3888 is found to be contained 4 times in 15725, which is annexed to the former figures of the root making 364, and to the column L making 1084. This multiplied by 4 and added to M gives 393136; and this last multiplied by 4 gives 1572544; which being equal to the resolvend, can be subtracted. No remainder occurring, the work is hence terminated at this step, and the root is 364.

2. Find the cube root of 9 to fifteen places of decimals.

L	M	N
6 0 8	12	9(2 08008|3823|051904
1 6	4 8 6 4	8
6·2 40 08	12·4 8 6 4	1·. . . . . .
16	6 4	·998912
6·2|40 24	12·9 7 9 2	1088. . . . .
	4 9 9 2 0 6 4	1038375936512
	12·9 7 9 6 9 9 2 0 6 4	49624063488
	6 4	38940651419|7
	12 9 8 0 1 9 8 4 1 9 2	10683412068|3
	1 8 7 2 0 7	10384192682|4
	12·9 8 0 2 1 7 1 3 9 9	299219385|9
		259604919|4
	12·9 8 0 2 3 5 8 6|0 6	39614466|5
	4 9 9|2	38940738|3
	12·9 8 0 2 4 0 8 5|3	673728|2
	12·9 8 0 2 4 5 8|4|5	649012|3
	1|2	24715|9
	12·9 8 0 2 4 5 9|7	12980|2
	12·|9 8 0 2 4 6 1|0	11735|7
		11682|2
		53|5
		51|9
		1|6

The decimal points are marked in this solution, but they are unnecessary, as the arrangement of the work itself will lead to a correct disposition of the figures.

The first vertical bar in the root marks where the contraction begins to give its figures to the root, and the second where the pure division commences. The root is true in the last figure.

This example contains specimens of the case where ciphers occur in the root; and is also a pattern for the method of contraction.

Ex. 3. Extract the cube root of 571482·19.

Ex. 4. Extract the cube root of 1628·1582.

Ex. 5. Extract the cube root of 1332.

Ex. 6. Extract the cube root of 3 to six places; and also the cube root of 6: and show how the cube root of 2, of 12, and of 18, may be obtained from these.

Ex. 7. Find how many more figures are required to be written in extracting the cube root of 3 to six places of decimals, than in finding it to three places.

Ex. 8. Find the cube root of $\frac{186096\cdot5}{7698109\cdot8917}$, and subtract it from the cube root of 5.

Ex. 9. Extract the cube root of ·009009009; and thence obtain its sixth and its ninth roots to six and to four places of decimals respectively.

Ex. 10. Find the square of the cube root of 1000, and likewise of ·001: and the cube of the square root of 100 and ·01; and determine the product of these four results.

TO EXTRACT ANY ROOT WHATEVER *.

Let P be the given number, n the index of the root to be extracted, R the true root, a the nearest approximation (either greater or less than R) that has been made by trial or otherwise; and let $a^n = \text{A}$, and $\text{R} = a + x$. Then,

$$x = \text{R} - a = \frac{a\left\{\text{P} - \text{A}\right\}}{\frac{1}{2}(n-1)\,\text{P} + \frac{1}{2}(n+1)\,\text{A}} \text{ nearly.}$$

Add or subtract x, according as P is greater or less than A, which gives a new and nearer approximation to R. With this new value of a, find a new value of x, and hence a new approximation to R. Continue this series of approximations till the required degree of accuracy is obtained.

The number of figures which may be depended on in each successive value of x, is generally equal to the number of accurate places in a; so that each operation doubles the number of figures already obtained.

---

* This rule is a modification and extension of Dr. Halley's, and was first given in its present form by Dr. Hutton. See his Tracts, vol. i. p. 213.

The following is essentially Dr. Hutton's investigation:

$$\text{Let } a + x = \frac{p\,\text{P} + q\,\text{A}}{q\,\text{P} + p\,\text{A}}\cdot a; \text{ or since } \text{P} = (a + x)^n,$$

$$a + x = \frac{(p+q)\,a^n + p\,n\,a^{n-1}x + p\cdot\frac{n(n-1)}{1\cdot 2}\,a^{n-2}x^2 + \ldots}{(p+q)\,a^n + q\,n\,a^{n-1}x + q\cdot\frac{n(n-1)}{1\cdot 2}\,a^{n-2}x^2 + \ldots} \times a$$

$$= a + \frac{p-q}{p+q}\cdot n\,x + \frac{p-q}{p+q}\cdot n\left\{\frac{n-1}{2} - \frac{q\,n}{p+q}\right\}\frac{x^2}{a} + \ldots$$

Now since a is an approximation to $\sqrt[n]{\text{P}}$, x is small in comparison with it, and hence its powers above the second may be rejected, as of values too small to materially affect the calculation within the prescribed limits. Then equating the co-efficients of the homologous powers which remain, we have

$$a = a, \frac{p-q}{p+q}\cdot n = 1, \text{ and } \frac{n-1}{2} - \frac{qn}{p+q} = o.$$

From the second and third of these equations we get the same relation between p and q: viz.

$$\frac{p}{q} = \frac{n-1}{n+1}$$

Inserting this value of $\frac{p}{q}$ in the assumed fraction for $a + x$ we obtain

$$a + x = \frac{(n-1)\,\text{P} + (n+1)\,\text{A}}{(n+1)\,\text{P} + (n-1)\,\text{A}}\,a; \text{ or } x = \frac{2\,a\left\{\text{P}-\text{A}\right\}}{(n+1)\,\text{P} + (n-1)\,\text{A}}$$

which is the working formula in the text, and perhaps the best form which the correction admits of.

In another place I have given an improvement upon this approximation from a memorandum of the late Mr. W. G. Horner, but to which I cannot make more special reference here.

Note. It will always be well to reduce the index into factors as far as possible, and extract successively the several roots in the manner directed at the head of the rule. Thus the thirtieth root is the fifth root of the third root of the square root, since $5 \times 3 \times 2 = 30$. It will also be always least laborious to commence with the lowest roots, as with the second before the third, and the third before the fifth, in the case just cited.

EXAMPLE.

Extract the tenth root of 442504881·64.

Here extracting the square root of the given number, we shall have to extract the fifth root of 21035·8.

A few trials will show that the root lies between 7·3 and 7·4. Taking $a = 7·3$, we have $\text{A} = a^5 = 20730·71593$. Also $\text{P} = 21035·8$, and $n = 5$. Hence,

$$x = \frac{7·3.\ \{21035·8 - 20730·71593\}}{2.\ 21035·8 + 3.\ 20730·71593}$$

$$= 0·021360, \text{ and hence } \sqrt[5]{21035·8} = 7·321360 \text{ nearly.}$$

This result is true in the last figure; but it is seldom that the approximation proceeds so far correctly; and even here it would not have been safe to assume its correctness without verification.

OTHER EXAMPLES.

1. What is the 3d root of 2?	Ans. 1·259921.
2. What is the 3d root of 3214?	Ans. 14·75758.
3. What is the 4th root of 2?	Ans. 1·189207.
4. What is the 4th root of 97·41?	Ans. 3·1415999.
5. What is the 5th root of 2?	Ans. 1·148699.
6. What is the 6th root of 21035·8?	Ans. 5·254037.
7. What is the 6th root of 2?	Ans. 1·122462.
8. What is the 7th root of 21035·8?	Ans. 4·145392.
9. What is the 7th root of 2?	Ans. 1·104089.
10. What is the 8th root of 21035·8?	Ans. 3·470323.
11. What is the 8th root of 2?	Ans. 1·090508.
12. What is the 9th root of 21035·8?	Ans. 3·022239.
13. What is the 9th root of 2?	Ans. 1·080059.

For a simple and ingenious method of constructing tables of square and cube roots, and the reciprocals of numbers, see Dr. Hutton's *Tracts on Mathematical and Philosophical Subjects,* vol. i. Tract 24, p. 459. By means of the method there laid down, the tables at the end of the volume were computed.

A method adapted to the square root, in which the root is exhibited as a simple vulgar fraction, is also given in the same volume, which is extremely convenient for the extraction of the roots of *isolated* numbers; but where the roots of a series of consecutive numbers are required, the one above referred to is the most convenient and rapid one ever discovered. The following is the method for the square root.

Let N denote the given number, and $\frac{n}{d}$ the fraction to which its square root is approximately equal; then if n be a near integer or fractional value of the root, the fraction $\frac{n^2 + Nd^2}{2nd}$ will denote one still nearer.

Or, for continuing the process, the following is still more convenient. Let $\frac{n}{d}$ denote the last approximation: then the next is $\frac{2n^2 - 1}{2dn}$; or if intended for final conversion into the decimal scale, it is $\frac{n}{d} - \frac{1}{2dn}$.

Example.—Extract the square root of 920. Here $N = 920$, and $n = 30$, $d = 1$. Hence by the first formula $\sqrt{920} = \frac{n^2 + Nd^2}{2nd} = \frac{1820}{60} = \frac{91}{3}$. Then from the second formula, putting $\frac{n}{d} = \frac{91}{3}$, we have $\frac{2n^2 - 1}{2dn} = \frac{2 \cdot 91^2 - 1}{2 \cdot 91 \cdot 3} = \frac{16561}{546} =$ 30·33150183, which differs from the truth only by 6 in the tenth place of figures, the true value being 30·33150177.—*Hutton's Tracts*, vol. i. p. 457—549.

OF RATIOS, PROPORTIONS, AND PROGRESSIONS OF NUMBERS.

Numbers are compared to each other in two different ways: the one comparison considers the difference of the two numbers, and is named Arithmetical Relation; and the difference sometimes the Arithmetical Ratio: the other considers their quotient, which is called Geometrical Relation; and the quotient expresses the Geometrical Ratio. So, of these two numbers, 6 and 3, the difference, or arithmetical ratio is 6 — 3 or 3, but the geometrical ratio is $\frac{6}{3}$ or 2.

There must be two numbers to form a comparison: the number which is compared, being placed first, is called the Antecedent; and that to which it is compared, the Consequent. So, in the two numbers above, 6 is the antecedent, and 3 the consequent.

If two or more couplets of numbers have equal ratios, or equal differences, the equality is named Proportion, and the terms of the ratios Proportionals. So, the two couplets, 4, 2 and 8, 6, are arithmetical proportionals, because $4 - 2 = 8 - 6 = 2$; and the two couplets, 4, 2 and 6, 3, are geometrical proportions, because $\frac{4}{2} = \frac{6}{3} = 2$, the same ratio.

To denote numbers as being geometrically proportional, a colon is set between the terms of each couplet, to denote their ratio; and a double colon, or else a mark of equality, between the couplets or ratios. So, the four proportionals, 4, 2, 6, 3, are set thus, 4 : 2 :: 6 : 3, which means, that 4 is to 2, as 6 is to 3; or thus, 4 : 2 = 6 : 3, or thus, $\frac{4}{2} = \frac{6}{3}$, both which mean, that the ratio of 4 to 2, is equal to the ratio of 6 to 3 *.

* The test of equal ratios in arithmetic is that the quotients of the antecedents and consequents in the two alleged ratios is a fraction of the same value. In reference to geometry, however, the test does not involve the idea of quotients or fractions, nor indeed of numbers at all. See Def. v. book v. *Elements of Euclid*. The difficulty felt in treating proportion by means of arithmetical ideas, and of the arithmetical definition of ratio, arises from the incommensurability of numbers; but this difficulty is not encountered in Euclid's method of treating

Proportion is distinguished into Continued and Discontinued. When the difference or ratio of the consequent of one couplet, and the antecedent of the next couplet, is not the same as the common difference or ratio of the couplets, the proportion is discontinued. So, 4, 2, 8, 6, are in discontinued arithmetical proportion, because $4 - 2 = 8 - 6 = 2$, whereas $2 - 8 = - 6$: and 4, 2, 6, 3, are in discontinued geometrical proportion, because $\frac{4}{2} = \frac{6}{3} = 2$, but $\frac{2}{6} = \frac{1}{3}$, which is not the same.

But when the difference or ratio of every two succeeding terms is the same quantity, the proportion is said to be Continued, and the numbers themselves make a series of Continued Proportionals, or a progression. So, 2, 4, 6, 8, form an arithmetical progression, because $4 - 2 = 6 - 4 = 8 - 6 = 2$, all the same common difference; and 2, 4, 8, 16, a geometrical progression, because $\frac{4}{2} = \frac{8}{4} = \frac{16}{8} = 2$, all the same ratio.

When the successive terms of progression increase, or exceed each other, it is called an Ascending Progression, or Series; but when the terms decrease, it is a Descending one.

So, 0, 1, 2, 3, 4,... is an ascending arithmetical progression, but 9, 7, 5, 3, 1, ... is a descending arithmetical progression. Also 1, 2, 4, 8, 16, ... is an ascending geometrical progression, and 16, 8, 4, 2, 1, ... is a descending geometrical progression.

ARITHMETICAL PROPORTION AND PROGRESSION.

In Arithmetical Progression, the numbers or terms have all the same common difference. The first and last terms of a Progression are called the Extremes; and the other terms, lying between them, the Means. The most useful part of arithmetical proportion is contained in the following theorems:

Theorem 1. When four quantities are in arithmetical proportion, the sum of the two extremes is equal to the sum of the two means. Thus, with regard to the four, 2, 4, 6, 8, we have $2 + 8 = 4 + 6 = 10$.

Theorem 2. In any continued arithmetical progression, the sum of the two extremes is equal to the sum of any two means that are equally distant from them, or equal to double the middle term when there is an odd number of terms.

Thus, in the terms 1, 3, 5, it is $1 + 5 = 3 + 3 = 6$.

And in the series 2, 4, 6, 8, 10, 12, 14, it is $2 + 14 = 4 + 12 = 6 + 10 = 8 + 8 = 16$.

---

the subject. It is, however, necessary to remark, that quantities (for numbers belong to the predicate quantity) which fulfil the arithmetical test (or definition, as most writers term it) of proportionality, can be readily subjected to Euclid's test: but the proposition is not convertible. Hence, therefore, though we cannot build a system of proportion adapted to geometry upon the arithmetical basis, we can establish upon grounds equally valid and convincing with those of geometry, all the properties of proportional quantities as expressed by numbers, to whatever branch of pure or applied mathematics they may refer. In reading the fifth book of Euclid, the intelligent teacher will avail himself of the opportunity of doing this. From the very early stage of the student's progress, the common definition and test is necessarily employed in this place: but by no means with a view to supersede the more logical and satisfactory investigations just referred to.

THEOREM 3. The difference between the extreme terms of an arithmetical progression, is equal to the common difference of the series multiplied by one less than the number of the terms. So, of the ten terms, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, the common difference is 2, and one less than the number of terms 9; then the difference of the extremes is $20 - 2 = 18$, and $2 \times 9 = 18$ also.

Consequently the greatest term is equal to the least term added to the product of the common difference multiplied by one less than the number of terms.

THEOREM 4. The sum of all the terms of any arithmetical progression, is equal to the sum of the two extremes multiplied by the number of terms, and divided by 2; or the sum of the two extremes multiplied by the number of the terms, gives twice the sum of all the terms in the series.

This is made evident by setting the terms of the series in an inverted order, under the same series in a direct order, and adding the corresponding terms together in that order. Thus,

in the series	1,	3,	5,	7,	9,	11,	13,	15;
ditto inverted	15,	13,	11,	9,	7,	5,	3,	1;
the sums are	16 +	16 +	16 +	16 +	16 +	16 +	16 +	16,

which must be double the sum of the single series, and is equal to the sum of the extremes repeated as often as are the number of the terms.

From these theorems may readily be found any one of these five particulars; the two extremes, the number of terms, the common difference, and the sum of all the terms, when any three of them are given, as in the following problems.

PROBLEM I.

Given the extremes, and the number of terms, to find the sum of all the terms.

ADD the extremes together, multiply the sum by the number of terms, and divide by 2.

EXAMPLES.

1. The extremes being 3 and 19, and the number of terms 9; required the sum of the terms.

$$\text{Here } \frac{19+3}{2} \times 9 = \frac{22}{2} \times 9 = 11 \times 9 = 99, \text{ the answer.}$$

2. It is required to find the number of all the strokes a common clock strikes in one whole revolution of the index, or in 12 hours. Ans. 78.

3. How many strokes do the clocks of Venice strike in the compass of a day, which go continually on from 1 to 24 o'clock? Ans. 300.

4. What debt can be discharged in a year, by weekly payments, in arithmetical progression, the first payment being 1*s*, and the last or 52d payment 5*l* 3*s*? Ans. 135*l* 4*s*.

PROBLEM II.

Given the extremes, and the number of terms, to find the common difference.

SUBTRACT the less extreme from the greater, and divide the remainder by 1 less than the number of terms, for the common difference.

EXAMPLES.

1. The extremes being 3 and 19, and the number of terms 9; required the common difference.

$$\text{Here } \frac{19-3}{9-1} = \frac{16}{8} = 2, \text{ Answer.}$$

2. If the extremes be 10 and 70, and the number of terms 21; what is the common difference, and the sum of the series?

Ans., the common difference is 3, and the sum is 840.

3. A certain debt can be discharged in one year, by weekly payments in arithmetical progression, the first payment being 1*s*, and the last 5*l* 3*s*; what is the common difference of the terms? Ans. 2.

PROBLEM III.

Given one of the extremes, the common difference, and the number of terms; to find the other extreme, and the sum of the series.

Multiply the common difference by 1 less than the number of terms, and the product will be the difference of the extremes: then add the product to the less extreme to give the greater extreme, or subtract it from the greater, to give the less.

EXAMPLES.

1. Given the least term 3, the common difference 2, of an arithmetical series of 9 terms; to find the greatest term, and the sum of the series

Here $2 \times (9-1) + 3 = 19 =$ the greatest term: hence $(19+3)\frac{9}{2} = \frac{198}{2} = 99 =$ the sum of the series.

2. If the greatest term be 70, the common difference 3, and the number of terms 21, what is the least term, and the sum of the series?

Ans., the least term is 10, and the sum is 840.

3. A debt can be discharged in a year, by paying 1 shilling the first week, 3 shillings the second, and so on, always 2 shillings more every week; what is the debt, and what will the last payment be?

Ans., the last payment will be 5*l* 3*s*, and the debt is 135*l* 4*s*.

PROBLEM IV.

To find an arithmetical mean proportional between two given terms.

Add the two given extremes or terms together, and take half their sum for the arithmetical mean required.

EXAMPLE.

To find an arithmetical mean between the two numbers 4 and 14.

$$\text{Here } \frac{14+4}{2} = 9 = \text{the mean required.}$$

PROBLEM V.

To find two arithmetical means between two given extremes.

Subtract the less extreme from the greater, and divide the difference by 3, so will the quotient be the common difference; which being continually added to the less extreme, or taken from the greater, will give the means.

EXAMPLE.

To find two arithmetical means between 2 and 8.

$$\text{Here } \frac{8-2}{3} = 2 = \text{common difference.}$$

Then $2 + 2 = 4 =$ the one mean, and $4 + 2 = 6 =$ the other mean.

PROBLEM VI.

To find any number of arithmetical means between two given terms or extremes.

Subtract the less extreme from the greater, and divide the difference by 1 more than the number of means required to be found, which will give the common difference; then this being added continually to the least term, or subtracted from the greatest, will give the mean terms required.

EXAMPLE.

To find five arithmetical means between 2 and 14.

$$\text{Here } \frac{14 - 2}{6} = 2 = \text{common difference.}$$

Then by adding this common difference continually, the means are found to be 4, 6, 8, 10, 12.

See more of arithmetical progression in the Algebra.

GEOMETRICAL PROPORTION AND PROGRESSION.

* If there be taken two ratios, as those of 6 to 3, and 14 to 7, which, by what has been already said, (p. 50, 75,) may be expressed fractionally, $\frac{6}{3}$ and $\frac{14}{7}$; to judge whether they are equal or unequal, we must reduce them to a common denominator, and we shall have 6×7, and 14×3 for the two numerators. If these are equal, the fractions or ratios are equal. Therefore,

Theorem i. If four quantities be in geometrical proportion, the product of the two extremes will be equal to the product of the two means.

And hence, if the product of the two means be divided by one of the extremes, the quotient will give the other extreme. Thus, of the above numbers, if the product of the means 42 be divided by 6, the quotient 7 is the other extreme; and if 42 be divided by 7, the quotient 6 is the first extreme. This is the foundation of the practice in the *Rule of Three.*

We see, also, that if we have four numbers, 6, 3, 14, 7, such, that the products of the means and of the extremes are equal, we may hence infer the equality of the ratios $\frac{6}{3} = \frac{14}{7}$, or the existence of the proportion 6 : 3 :: 14 : 7. Hence

Theorem ii. We may always form a proportion of the factors of two equal products.

If the two means are equal, as in the terms 3, 6, 6, 12, their product becomes a square. Hence

Theorem iii. The mean proportional between two numbers is the square root of their product.

We may without destroying the accuracy of a proportion, give to its various terms all the changes which do not affect the equality of the products of the means and extremes.

---

* See the note at p. 75.

Thus, with respect to the proportion $6 : 3 :: 14 : 7$, which gives $6 \times 7 = 3 \times 14$, we may displace the extremes, or the means, an operation which is denoted by the word *Alternando*.

This will give $6 : 14 :: 3 : 7$
or $7 : 3 :: 14 : 6$
or $7 : 14 :: 3 : 6$.

Or, 2dly, we may put the extremes in the places of the means, called *Invertendo*.

Thus $3 : 6 :: 7 : 14$.

Or, 3dly, we may multiply or divide the two antecedents, or the two consequents, by the same number, when proportionality will subsist.

As $6 \times 4 : 3 :: 14 \times 4 : 7$; viz. $24 : 3 :: 56 : 7$
and $6 \div 2 : 3 :: 14 \div 2 : 7$; viz. $3 : 3 :: 7 : 7$.

Also, applying the proposition in *note* 2, *Addition of Vulgar Fractions*, to the terms of a proportion, such as $30 : 6 :: 15 : 3$, or $\frac{30}{6} = \frac{15}{3}$, we shall have

$$\frac{30 \pm 15}{6 \pm 3} = \frac{15}{3} \text{ and } \frac{30 + 15}{6 + 3} = \frac{30 - 15}{6 - 3}. \quad \text{Hence}$$

Theorem iv. The sum or the difference of the antecedents, is to that of the consequents, as any one of the antecedents is to its consequent.

Theorem v. The sum of the antecedents is to their difference, as the sum of the consequents is to their difference.

In like manner, if there be a series of equal ratios, $\frac{6}{3} = \frac{10}{5} = \frac{14}{7} = \frac{30}{15}$; we have

Theorem vi. In any series of equal ratios, the sum of the antecedents is to that of the consequents, as any one antecedent is to its consequent.

$$\frac{6 + 10 + 14 + 30}{3 + 5 + 7 + 15} = \frac{6}{3} = \frac{10}{5} = \frac{14}{7} = \frac{30}{15}. \quad \text{Therefore,}$$

Theorem vii. If two proportions are multiplied, term by term, the products will constitute a proportion.

Thus, if $30 : 15 :: 6 : 3$
and $2 : 3 :: 4 : 6$.
Then $30 \times 2 : 15 \times 3 :: 6 \times 4 : 3 \times 6$
or $60 : 45 :: 24 : 18$; or $\frac{60}{45} = \frac{24}{18}$.

Theorem viii. If four quantities are in proportion, their squares, cubes, and all other powers will be in proportion.

For this will evidently be nothing else than assuming the proportionality of the products, term by term, of two, three, or more identical proportions.

The same properties hold with regard to surd or irrational expressions.

Thus, $\sqrt{720} : \sqrt{80} :: \sqrt{567} : \sqrt{63}$
and $\sqrt{12} : \sqrt{3} :: \sqrt{4} : \sqrt{1}$.

$$\text{For } \frac{\sqrt{720}}{\sqrt{80}} = \frac{\sqrt{(9 \times 80)}}{\sqrt{80}} = \frac{3}{1}, \text{ and } \frac{\sqrt{567}}{\sqrt{63}} = \frac{\sqrt{(9 \times 63)}}{\sqrt{63}} = \frac{3}{1}$$

$$\text{and } \frac{\sqrt{12}}{\sqrt{3}} = \sqrt{\frac{12}{3}} = \frac{\sqrt{4}}{1} = \frac{2}{1}.$$

Theorem ix. The quotient of the extreme terms of a geometrical progression is equal to the common ratio of the series raised to the power denoted by 1 less than the number of the terms.

So, of the ten terms 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, the common ratio is 2, one less than the number of terms 9; then the quotient of the extremes is $\frac{1024}{2} = 512$, and $2^9 = 512$ also.

Consequently the greatest term is equal to the less term multiplied by the said power of the ratio, whose index is 1 less than the number of terms.

Theorem x. The sum of all the terms, of any geometrical progression, is

found by adding the greatest term to the difference of the extremes divided by 1 less than the ratio.

So, the sum of 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, (whose ratio is 2,) is $1024 + \frac{1024 - 2}{2 - 1} = 1024 + 1022 = 2046$.

This subject will be resumed in the algebraic part of this work. A few examples may here be added.

EXAMPLES.

1. The least of ten terms in geometrical progression being 1, and the ratio 2; what is the greatest term, and the sum of all the terms?

Ans., the greatest term is 512, and the sum 1023.

2. What debt may be discharged in a year, or 12 months, by paying 1*l* the first month, 2*l* the second, 4*l* the third, and so on, each succeeding payment being double the last; and what will the last payment be?

Ans., the debt 4095*l* and the last payment 2048*l*.

PROBLEM I.

To find one geometrical mean proportional between any two numbers.

Multiply the two numbers together, and extract the square root of the product, which will give the mean proportional sought.

EXAMPLE.

To find a geometrical mean between the two numbers 3 and 12.

$$\sqrt{(12 \times 3)} = \sqrt{36} = 6 = \text{the mean.}$$

PROBLEM II.

To find two geometrical mean proportionals between any two numbers.

Divide the greater number by the less, and extract the cube root of the quotient, which will give the common ratio of the terms. Then multiply the least given term by the ratio for the first mean, and this mean again by the ratio for the second mean; or, divide the greater of the two given terms by the ratio for the greater mean, and divide this again by the ratio for the less mean.

EXAMPLE.

To find two geometrical means between 3 and 24.

Here 3) 24 (8; its cube root 2 is the ratio.

Then $3 \times 2 = 6$, and $6 \times 2 = 12$, the two means.

Or $24 \div 2 = 12$, and $12 \div 2 = 6$, the same.

That is, the two means between 3 and 24, are 6 and 12.

PROBLEM III.

To find any number of geometrical means between two numbers.

Divide the greater number by the less, and extract such root of the quotient whose index is 1 more than the number of means required; that is, the 2d root for one mean, the 3rd root for two means, the fourth root for three means, and so on; and that root will be the common ratio of all the terms: then, with the ratio, multiply continually from the first term, or divide continually from the last or greatest term.

EXAMPLE.

To find four geometrical means between 3 and 96.

Here 3) 96 (32; the 5th root of which is 2, the ratio.

Then $3 \times 2 = 6$, and $6 \times 2 = 12$, and $12 \times 2 = 24$, and $24 \times 2 = 48$.

Or $96 \div 2 = 48$, and $48 \div 2 = 24$, and $24 \div 2 = 12$, and $12 \div 2 = 6$.

That is, 6, 12, 24, 48, are the four means between 3 and 96.

OF HARMONICAL PROPORTION.

THERE is also a third kind of proportion, called Harmonical or Musical, which being but of rare occurrence in questions purely arithmetical, a very short account of it may here suffice. It will however be again noticed both in algebra and in geometry, but especially in the latter.

Musical Proportion is when, of three numbers, the first has the same proportion to the third, as the difference between the first and second has to the difference between the second and third.

As in these three, 6, 8, 12;

where $6 : 12 :: 8 - 6 : 12 - 8$,

that is $6 : 12 :: 2 : 4$.

When four numbers are in musical proportion, then the first has the same ratio to the fourth, as the difference between the first and second has to the difference between the third and fourth.

As in these, 6, 8, 12, 18;

where $6 : 18 :: 8 - 6 : 18 - 12$,

that is $6 : 18 :: 2 : 6$.

When numbers are in musical progression, their reciprocals are in arithmetical progression; and the converse, that is, when numbers are in arithmetical progression, their reciprocals are in musical progression.

So in these musicals 6, 8, 12, their reciprocals $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{12}$, are in arithmetical progression; for $\frac{1}{6} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$; and $\frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$; that is, the sum of the extremes is equal to double the mean, which is the property of arithmeticals.

The method of finding a series of numbers in musical proportion and progression is best expressed by algebraic methods and symbols.

FELLOWSHIP OR PARTNERSHIP.

FELLOWSHIP is the rule by which any sum or quantity may be divided into any number of parts which shall be in any given proportion to one another.

By this rule are adjusted the gains or losses or charges of partners in company; or the effects of bankrupts, or legacies in case of a deficiency of assets or effects; or the shares of prizes; or the numbers of men to form certain detachments; or the division of waste lands among a number of proprietors.

Fellowship is either *Single* or *Double*. It is single, when the shares or portions are to be proportioned each to one given number only; as when the stocks of partners are all employed for the same time: and double, when each portion is to be proportional to two or more numbers; as when the stocks of partners are employed for different times.

SINGLE FELLOWSHIP.

GENERAL RULE.

ADD together the numbers that denote the proportion of the shares: then say,

As the sum of the said proportional numbers,
is to the whole sum to be parted or divided,
so is each of the several proportional numbers,
to the corresponding share or part.

Or, As the whole stock, is to the whole gain or loss,
so is each man's particular stock,
to his particular share of gain or loss.

TO PROVE THE WORK. Add all the shares or parts together, and the sum will be equal to the whole number to be shared, when the work is right.

EXAMPLES.

1. To divide the number 240 into three such parts, as shall be in proportion to each other as the three numbers, 1, 2, and 3.

Here $1 + 2 + 3 = 6 =$ the sum of the numbers.
Then, as $6 : 240 :: 1 : 40 =$ the 1st part,
and as $6 : 240 :: 2 : 80 =$ the 2d part,
also as $6 : 240 :: 3 : 120 =$ the 3d part.

Sum of all 240, the proof.

2. Three persons, A, B, C. freighted a ship with 340 tuns of wine; of which A loaded 110 tuns, B 97, and C the rest: in a storm the seamen were obliged to throw overboard 85 tuns; how much must each person sustain of the loss?

Here $110 + 97 = 207$ tuns, loaded by A and B;
therefore $340 - 207 = 133$ tuns, loaded by C.
Hence, as $340 : 85 :: 110$
or as $4 : 1 :: 110 : 27\frac{1}{2}$ tuns $=$ A's loss;
and as $4 : 1 :: 97 : 24\frac{1}{4}$ tuns $=$ B's loss;
also as $4 : 1 :: 133 : 33\frac{1}{4}$ tuns $=$ C's loss;

Sum 85 tuns, the proof.

3. Two merchants, C and D, made a stock of 120*l*; of which C contributed 75*l*, and D the rest: by trading they gained 30*l*; what must each have of it?

Ans. C 18*l* 15*s*, and D 11*l* 5*s*.

4. Three merchants, E, F, G, make a stock of 700*l*; of which E contributed 123*l*, F 358*l*, and G the rest: by trading they gain 125*l* 10*s*; what must each have of it?

Ans. E must have 22*l* 1*s* 0*d* $2\frac{2}{35}$*q*.
F .... 64 3 8 $0\frac{32}{35}$.
G .... 39 5 3 $1\frac{1}{35}$.

5. A General imposing a contribution * of 700*l* on four villages, to be paid in proportion to the number of inhabitants contained in each; the first containing 250, the 2d 350, the 3d 400, and the 4th 500 persons; what part must each village pay?

Ans. the first to pay 116*l* 13*s* 4*d*.
the 2d ...... 163 6 8.
the 3d ...... 186 13 4.
the 4th ...... 233 6 8.

* Contribution is a tax paid by provinces, towns, or villages, to excuse them from being plundered. It is paid in provisions or in money, and sometimes in both.

6. A piece of ground, consisting of 37 ac 2 ro 14 ps, is to be divided among three persons, L, M, and N, in proportion to their estates: now if L's estate be worth 500*l* a year, M's 320*l*, and N's 75*l*; what quantity of land must each one have? Ans. L must have 20 ac 3 ro $39\frac{159}{179}$ pls.

M .... 13 1 $30\frac{46}{179}$.

N .... 3 0 $23\frac{173}{179}$.

7. A person is indebted to O 57*l* 15*s*, to P 108*l* 3*s* 8*d*, to Q 22*l* 10*d*, and to R 73*l*; but at his decease, his effects are found to be worth no more than 170*l* 14*s*; how must it be divided among his creditors?

Ans. O must have 37*l* 15*s* 5*d* $2\frac{5302}{10439}q$.

P .... 70 15 2 $2\frac{7498}{10439}$.

Q .... 14 8 4 $0\frac{4720}{10439}$.

R .... 47 14 11 $2\frac{3358}{10439}$.

8. A ship, worth 900*l*, being entirely lost, of which $\frac{1}{8}$ belonged to S, $\frac{1}{4}$ to T, and the rest to V; what loss will each sustain, supposing 540*l* of her were insured? Ans. S will lose 45*l*, T 90*l*, and V 225*l*.

9. Four persons, W, X, Y, and Z, spend among them 25*s*, and agree that W shall pay $\frac{1}{2}$ of it, X $\frac{1}{3}$, Y $\frac{1}{4}$, and Z $\frac{1}{5}$; that is, their shares are to be in proportion as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$: what are their shares? Ans. W must pay 9*s* 8*d* $3\frac{41}{77}q$.

X .... 6 5 $3\frac{53}{77}$.

Y .... 4 10 $1\frac{59}{77}$.

Z .... 3 10 $3\frac{1}{77}$.

10. A detachment, consisting of 5 companies, being sent into garrison, in which the duty required 76 men a day; what number of men must be furnished by each company, in proportion to their strength; the 1st consisting of 54 men, the 2d of 51 men, the 3d of 48 men, the 4th of 39, and the 5th of 36 men?

Ans., the 1st must furnish 18, the 2d 17, the 3d 16, the 4th 13, and the 5th 12 men *.

DOUBLE FELLOWSHIP.

Double Fellowship, as has been said, is concerned in cases in which the stocks of partners are employed or continued for different times.

Rule †. Multiply each person's stock by the time of its continuance; then divide the quantity, as in Single Fellowship, into shares, in proportion to these products, by saying,

As the total sum of all the said products,
Is to the whole gain or loss, or quantity to be parted,
So is each particular product,
To the corresponding share of the gain or loss.

---

* Questions of this nature frequently occurring in military service, General Haviland, an officer of great merit, contrived an ingenious instrument, for more expeditiously resolving them; which is distinguished by the name of the inventor, being called a Haviland.

† The proof of this rule is as follows: When the times are equal, the shares of the gain or loss are evidently as the stocks, as in Single Fellowship; and when the stocks are equal, the shares are as the times; therefore, when neither are equal, the shares must be as their products.

EXAMPLES.

1. A had in company 50*l* for 4 months, and B had 60*l* for 5 months; at the end of which time they find 24*l* gained: how must it be divided between them?

$$\begin{array}{rcrcl} \text{Here } 50 & & 60 & & \\ 4 & & 5 & & \\ \hline 200 & + & 300 & = & 500. \end{array}$$

Then as 500 : 24 :: 200 : $9\frac{3}{5}$ = 9*l* 12*s* = A's share.
and as 500 : 24 :: 300 : $14\frac{2}{5}$ = 14 8 = B's share.

2. C and D hold a piece of ground in common, for which they are to pay 54*l*. C put in 23 horses for 27 days, and D 21 horses for 39 days; how much ought each man to pay for the rent?

Ans. C must pay 23*l* 5*s* 9*d*.
D .... 30 14 3.

3. Three persons, E, F, G, hold a pasture in common, for which they are to pay 30*l* per annum; into which E put 7 oxen for 3 months, F put 9 oxen for 5 months, and G put in 4 oxen for 12 months; how much must each person pay of the rent?

Ans. E must pay 5*l* 10*s* 6*d* $1\frac{6}{19}$*q*.
F .... 11 16 10 $0\frac{8}{19}$.
G .... 12 12 7 $2\frac{6}{19}$.

4. A ship's company take a prize of 1000*l*, which they agree to divide among them according to their pay and the time they have been on board: now the officers and midshipmen have been on board 6 months, and the sailors 3 months; the officers have 40*s* a month, the midshipmen 30*s*, and the sailors 22*s* a month; moreover, there are 4 officers, 12 midshipmen, and 110 sailors: what will each man's share be?

Ans. each officer must have 23*l* 2*s* 5*d* $0\frac{93}{173}$*q*.
each midshipman .... 17 6 9 $3\frac{69}{173}$.
each seaman .... 6 7 2 $0\frac{8}{173}$.

5. H, with a capital of 1000*l*, began trade the first of January, and, meeting with success in business, took in I as a partner, with a capital of 1500*l*, on the first of March following. Three months after that they admit K as a third partner, who brought into stock 2800*l*. After trading together till the end of the year, they find there has been gained 1776*l* 10*s*; how must this be divided among the partners?

Ans. H must have 457*l* 9*s* $4\frac{1}{4}$*d* $\frac{23}{233}$*q*.
I . . . 571 16 $8\frac{1}{4}$ $\frac{97}{233}$.
K . . . 747 3 $11\frac{1}{4}$ $\frac{113}{233}$.

6. X, Y, and Z made a joint-stock for 12 months; X at first put in 20*l*, and 4 months after 20*l* more; Y put in at first 30*l*, at the end of 3 months he put in 20*l* more, and 2 months after he put in 40*l* more; Z put in at first 60*l*, and 5 months after he put in 10*l* more, 1 month after which he took out 30*l*; during the 12 months they gained 50*l*; how much of it must each have?

Ans. X must have 10*l* 18*s* 6*d* $3\frac{48}{61}$*q*.
Y . . . 22 8 1 $0\frac{13}{61}$.
Z . . . 16 13 4 0.

SIMPLE INTEREST.

Interest is the premium or sum allowed for the loan or forbearance of money. The money lent or forborn is called the Principal; and the sum of the principal

and its interest is called the Amount. Interest is allowed at so much per cent. per annum, or interest of 100*l* for a year, is called the rate of interest. Thus:

when interest is at 3 per cent. the rate is 3;
. . . . 4 per cent. . . 4;
. . . . 5 per cent. . . 5;
. . . . 6 per cent. . . 6.

But, by law, interest ought not to be taken higher than at the rate of 5 per cent.

Interest is of two sorts; Simple and Compound.

Simple Interest is that which is allowed for the principal lent or forborn only, for the whole time of forbearance. As the interest of any sum, for any time, is directly proportional to the principal sum, and also to the time of continuance; hence arises the following general rule of calculation.

As 100*l* is to the rate of interest, so is any given principal to its interest for one year. And again,

As 1 year is to any given time, so is the interest for a year, just found, to the interest of the given sum for that time.

Otherwise. Take the interest of 1 pound for a year, which multiply by the given principal, and this product again by the time of loan or forbearance, in years and parts, for the interest of the proposed sum for that time.

Note. When there are certain parts of years in the time, as quarters, or months, or days; they may be worked for, either by taking the aliquot or like parts of the interest of a year, or by the Rule of Three in the usual way. Also, the division by 100 is done by pointing off two figures for decimals.

EXAMPLES.

1. To find the interest of 230*l* 10*s*, for 1 year, at the rate of 4 per cent. per annum.

Here, as 100 : 4 :: 230*l* 10*s* : 9*l* 4*s* $4\frac{3}{4}$*d*.

```
              4
        ---------
1|00 ) 9|22   0
         20
        ----
        4|40
         12
        ----
        4|80          Ans. 9l 4s 4¾d.
          4
        ----
        3|20
```

2. To find the interest of 547*l* 15*s*, for 3 years, at 5 per cent. per annum.

As 100 : 5 :: 547·75
Or 20 : 1 :: 547·75 : 27·3875 interest for 1 year.

```
                      3
               ----------
             l 82·1625  ditto for 3 years.
                   20
               ----------
             s 3·2500
                 12
               ----------
             d 3·00      Ans. 82l 3s 3d.
```

3. To find the interest of 200 guineas, for 4 years, 7 months and 25 days, at $4\frac{1}{2}$ per cent. per annum.

	$210l$	
	$4\frac{1}{2}$	
	840	
	105	
	9·45	interest for 1 year.
	4	
	37·80	ditto 4 years.
6 mo = $\frac{1}{2}$	4·725	ditto 6 months.
1 mo = $\frac{1}{6}$	·7875	ditto 1 month.
	·6472	ditto 25 days.
	l 43·9597	
	20	
	s 19·1940	
	12	
	d 2·3280	
	4	
	q 1·3120	

	ds		£		ds		£
As	365	:	9·45	::	25	:	£
or	73	:	9·45	::	5	:	·6472

$$\begin{array}{r} 9{\cdot}45 \\ 5 \\ \hline 73)\ 47{\cdot}25\ (\ {\cdot}6472 \\ 345 \\ 530 \\ 19 \end{array}$$

Ans. $43l$ $19s$ $2\frac{1}{4}d$.

4. To find the interest of $450l$, for a year, at 5 per cent. per annum.
Ans. $22l$ $10s$.

5. To find the interest of $715l$ $12s$ $6d$, for a year, at $4\frac{1}{2}$ per cent. per annum.
Ans. $32l$ $4s$ $0\frac{3}{4}d$.

6. To find the interest of $720l$, for 3 years, at 5 per cent. per annum.
Ans. $108l$.

7. To find the interest of $355l$ $15s$, for 4 years, at 4 per cent. per annum.
Ans. $56l$ $18s$ $4\frac{3}{4}d$.

8. To find the interest of $32l$ $5s$ $8d$, for 7 years, at $4\frac{1}{4}$ per cent. per annum.
Ans. $9l$ $12s$ $1d$.

9. To find the interest of $107l$, for $1\frac{1}{2}$ year, at 5 per cent. per annum.
Ans. $12l$ $15s$.

10. To find the insurance on $205l$ $15s$, for $\frac{1}{4}$ of a year, at 4 per cent. per annum.
Ans. $2l$ $1s$ $1\frac{3}{4}d$.

11. To find the interest on $319l$ $6d$, for $5\frac{3}{4}$ years, at $3\frac{3}{4}$ per cent. per annum.
Ans. $68l$ $15s$ $9\frac{1}{2}d$.

12. To find the insurance on $107l$, for 117 days, at $4\frac{3}{4}$ per cent per annum.
Ans. $1l$ $12s$ $7d$.

13. To find the interest of $17l$ $5s$, for 117 days, at $4\frac{3}{4}$ per cent. per annum.
Ans. $5s$ $3d$.

14. To find the insurance on $712l$ $6s$ for 8 months, at $7\frac{1}{2}$ per cent. per annum.
Ans. $35l$ $12s$ $3\frac{1}{2}d$.

Note. The rules for Simple Interest serve also to calculate Insurances, or the Purchase of Stocks, or any thing else that is rated at so much per cent.

See also more on the subject of Interest with the algebraical expression and investigation of the rules, at the end of the Algebra.

COMPOUND INTEREST.

Compound Interest, called also interest upon interest, is that which arises from the principal and interest, taken together, as it becomes due, at the end of each stated time of payment. Though it be not lawful to lend money at compound interest, yet in purchasing annuities, pensions, or leases in reversion, it is usual to allow compound interest to the purchaser for his ready money.

Rule 1. Find the amount of the given principal, for the time of the first payment, by simple interest. Then consider this amount as a new principal for the second payment, whose amount calculate as before. Proceed thus through all the payments to the last, always accounting the last amount as a new principal for the next payment. The reason is evident from the definition of compound interest.

Otherwise,

Rule 2. Find the amount of 1 pound for the time of the first payment, and raise or involve it to the power of whose index is denoted by the number of payments. Then that power multiplied by the given principal, will produce the whole amount: from which the said principal being subtracted, leaves the compound interest of the same. This is evident from the first rule.

EXAMPLES.

1. To find the amount of 720*l*, for 4 years, at 5 per cent. per annum.

Here 5 is the 20th part of 100, and the interest of 1*l* for a year is $\frac{1}{20}$ or ·05, and its amount 1·05. Therefore,

1. *By the 1st rule.*

	£	*s*	*d*	
20)	720	0	0	1st year's principal
	36	0	0	1st year's interest
20)	756	0	0	2d year's principal
	37	16	0	2d year's interest
20)	793	16	0	3d year's principal
	39	13	9½	3d year's interest
20)	833	9	9½	4th year's principal
	41	13	5¾	4th year's interest
	£875	3	3¼	the whole amount, or answer required.

2. *By the 2d rule.*

1·05	amount of 1*l*.
1·05	
1·1025	2d power of it.
1·1025	
1·21550625	4th power of it.
720	
l 875·1645	
20	
s 3·2900	
12	
d 3·4800	

2. To find the amount of 50*l* in 5 years, at 5 per cent. per annum, compound interest. Ans. 63*l* 16*s* 3¼*d*.

3. To find the amount of 50*l* in 5 years, or 10 half-years, at 5 per cent. per annum, compound interest, the interest payable half-yearly. Ans. 64*l* 0*s* 1*d*.

4. To find the amount of 50*l* in 5 years, or 20 quarters, at 5 per cent. per annum, compound interest, the interest payable quarterly. Ans. 64*l* 2*s* 0¼*d*.

5. To find the compound interest of 370*l* forborn for 6 years, at 4 per cent. per annum. Ans. 98*l* 3*s* 4¼*d*.

6. To find the compound interest of 410*l* forborn for $2\frac{1}{2}$ years, at $4\frac{1}{2}$ per cent. per annum, the interest payable half-yearly. Ans. 48*l* 4*s* $11\frac{1}{4}$*d*.

7. To find the amount, at compound interest, of 217*l* forborn for $2\frac{1}{4}$ years, at 5 per cent. per annum, the interest payable quarterly. Ans. 242*l* 13*s* $4\frac{1}{4}$*d*.

ALLIGATION.

Alligation teaches how to compound or mix together several simples of different qualities, so that the composition may be of some intermediate quality or rate. It is commonly distinguished into two cases, Alligation Medial, and Alligation Alternate.

ALLIGATION MEDIAL.

Alligation Medial is the method of finding the rate or quality of the composition, from having the quantities and rates or qualities of the several simples given. It is thus performed :—

* Multiply the quantity of each ingredient by its rate or quality; then add all the products together, and add also all the quantities together into another sum; then divide the former sum by the latter, that is, the sum of the products by the sum of the quantities, and the quotient will be the rate or quality of the composition required.

EXAMPLES.

1. If three sorts of gunpowder be mixed together, viz. 50lb at 12*d* a pound, 44lb at 9*d*, and 26lb at 8*d* a pound; how much a pound is the composition worth?

Here 50, 44, 26 are the quantities,
and 12, 9, 8 the rates or qualities;

$$\begin{array}{lrcrcr} \text{then} & 50 & \times & 12 & = & 600 \\ & 44 & \times & 9 & = & 396 \\ & 26 & \times & 8 & = & 208 \\ \hline & 120) & & & & 1204 \end{array} \quad (10\tfrac{4}{120} = 10\tfrac{1}{30}.$$

Ans., the rate or price is $10\frac{1}{30}$*d* the pound.

* *Demonstration.* The rule is thus proved by Algebra.
Let a, b, c be the quantities of the ingredients,
and m, n, p their rates, or qualities, or prices;
then am, bn, cp are their several values,
and $am + bn + cp$ the sum of their values,
also $a + b + c$ is the sum of the quantities,
and if r denote the rate of the whole composition,
then $(a + b + c) \times r$ will be the value of the whole,
conseq. $(a + b + c) \times r = am + bn + cp$,
and $r = (am + bn + cp) \div (a + b + c)$, which is the rule.

Note. If an ounce or any other quantity of pure gold be reduced into 24 equal parts, these parts are called carats; but gold is often mixed with some base metal, which is called the alloy, and the mixture is said to be of so many carats fine, according to the proportion of pure gold contained in it: thus, if 22 carats of pure gold, and 2 of alloy be mixed together, it is said to be 22 carats fine.

If any one of the simples be of little or no value with respect to the rest, its rate is supposed to be nothing; as water mixed with wine, and alloy with gold and silver.

2. A composition being made of 5lb of tea at 7*s* per lb, 9lb at 8*s* 6*d* per lb, and $14\frac{1}{2}$lb at 5*s* 10*d* per lb; what is a lb of it worth? Ans. 6*s* $10\frac{1}{2}$*d*.

3. Mixed 4 gallons of wine at 4*s* 10*d* per gall, with 7 gallons at 5*s* 3*d* per gall, and $9\frac{3}{4}$ gallons at 5*s* 8*d* per gall; what is a gallon of this composition worth? Ans. 5*s* $4\frac{1}{4}$*d*.

4. Having melted together 7 oz of gold of 22 carats fine, $12\frac{1}{2}$ oz of 21 carats fine, and 17 oz of 19 carats fine: I would know the fineness of the composition? Ans. $20\frac{19}{73}$ carats fine.

ALLIGATION ALTERNATE.

Alligation Alternate is the method of finding what quantity of any number of simples, whose rates are given, will compose a mixture of a given rate. It is, therefore, the reverse of Alligation Medial, and may be proved by it.

RULE I*.

1. Set the rates of the simples in a column under each other. 2. Connect, or link with a continued line, the rate of each simple, which is less than that of the compound, with one, or any number, of those that are greater than the compound; and each greater rate with one or any number of the less. 3. Write the difference between the mixture rate, and that of each of the simples, opposite

---

* *Demonst.* By connecting the less rate with the greater, and placing the difference between them and the rate alternately, the quantities resulting are such, that there is precisely as much gained by one quantity as is lost by the other, and therefore the gain and loss upon the whole is equal, and is exactly the proposed rate: and the same will be true of any other two simples managed according to the rule.

In like manner, whatever the number of simples may be, and with how many soever every one is linked, since it is always a less with a greater than the mean price, there will be an equal balance of loss and gain between every two, and consequently an equal balance on the whole.

It is obvious, from the rule, that questions of this sort admit of a great variety of answers: for, having found one answer we may find as many more as we please, by only multiplying or dividing each of the quantities found, by 2, or 8, or any integer; the reason of which is evident: for, if two quantities, of two simples, make a balance of loss and gain, with respect to the mean price, so must also the double or treble, the $\frac{1}{2}$ or $\frac{1}{3}$ part, or any other ratio of these quantities, and so on *ad infinitum.*

These kinds of questions are called by algebraists *indeterminate* or *unlimited* problems; and by an analytical process, theorems may be deduced that will give all the *possible* answers.

Thus, taking for example the four simples A, B, C, D, which are to be mixed so as to produce the mean price m. Denote the prices of A, B, C, D by $m+a$, $m+b$, $m-c$, and $m-d$ respectively. Likewise let the quantities taken be x, y, z, v. Then

Mean	Prices.	Quantities.
	$m+a$	x
m	$m+b$	y
	$m-c$	z
	$m-d$	v

Then if each quantity be multiplied by its price, the sum of the products will evidently be the same as of all the quantities multiplied by the mean price, viz.:

$$(m+a)\,x + (m+b)\,y + (m-c)\,z + (m-d)\,v = m\,(x+y+z+v).$$

That is, $ax + by = cz + dv$.

But as there are four unknown quantities, and but one equation, we are at liberty to assume any other three conditions we please, and still the true result will be obtained. The three *simplest* that can be taken are those upon which the rule above given is founded, viz.

$$x = d,\ y = c,\ z = b,\ v = a;\ \text{or}\ x = c,\ y = d,\ z = a,\ v = b.$$ [The

the rate with which they are linked. 4. Then if only one difference stand against any rate, it will be the quantity belonging to that rate; but if there be several, their sum will be the quantity.

The examples may be proved by the rule for Alligation Medial.

EXAMPLES.

1. A merchant would mix wines at 16*s*, at 18*s*, and at 22*s* per gallon, so that the mixture may be worth 20*s* the gallon; what quantity of each must be taken?

$$\text{Here } 20\left\{\begin{array}{l} 16 \\ 18 \\ \\ 22 \end{array}\right. \qquad \begin{array}{l} 2 \text{ at } 16s \\ 2 \text{ at } 18s \\ \\ 4 + 2 = 6 \text{ at } 22s \end{array}$$

2. How much sugar at 4*d*, at 6*d*, and at 11*d* per lb, must be mixed together, so that the composition formed by them may be worth 7*d* per lb?

Ans. 1 lb. or 1 stone, or 1 cwt, or any other equal quantity of each sort.

3. How much corn at 2*s* 6*d*, 3*s* 8*d*, 4*s*, and 4*s* 8*d* per bushel must be mixed together, that the compound may be worth 3*s* 10*d* per bushel?

Ans. 2 at 2*s* 6*d*, 3 at 3*s* 8*d*, 3 at 4*s*, and 3 at 4*s* 8*d*.

RULE II.

When the whole composition is limited to a certain quantity: find an answer as before by linking; then say, as the sum of the quantities, or differences thus determined, is to the given quantity; so is each ingredient, found by linking, to the required quantity of each.

EXAMPLE.

1. How much gold of 15, 17, 18, and 22 carats fine, must be mixed together, to form a composition of 40 oz of 20 carats fine.

$$\text{Here } 20\left\{\begin{array}{l} 15 \\ 17 \\ 18 \\ \\ 22 \end{array}\right. \qquad \begin{array}{r} \ldots\ldots\; 2 \\ \ldots\ldots\; 2 \\ \ldots\ldots\; 2 \\ \\ 5 + 3 + 2 = 10 \\ \hline 16 \end{array}$$

Then as 16 : 40 :: 2 : 5

and 16 : 40 :: 10 : 25

Ans. 5 oz of 15, of 17, and of 18 carats fine, and 25 oz of 22 carats fine *.

The work will therefore stand in either of the following forms:

$$m\left|\begin{array}{l} m+a \\ m+b \\ m-c \\ m-d \end{array}\right. \quad \begin{array}{l} \ldots\ldots\; d \\ \ldots\ldots\; c \\ \ldots\ldots\; b \\ \ldots\ldots\; a \end{array} \qquad \text{or,} \quad m\left|\begin{array}{l} m+a \\ m+b \\ m-c \\ m-d \end{array}\right. \quad \begin{array}{l} \ldots\ldots\; c \\ \ldots\ldots\; d \\ \ldots\ldots\; a \\ \ldots\ldots\; b \end{array}$$

which is the rule, and any other three relations would give a different but a less practicable rule than this. The next in point of simplicity is

$$x = pd,\ y = qc,\ z = qb,\ v = pa;\ \text{or, } x = pc,\ y = qd,\ z = pa,\ v = qb.$$

See the Key to Keith's Arithmetic, by Maynard, from which valuable little treatise the latter part of this note is taken.

* A great number of questions might be here given relating to the specific gravities of bodies, but one of the most curious may suffice.

Hiero, king of Syracuse, gave orders for a crown to be made entirely of pure gold; but,

RULE III*.

When one of the ingredients is limited to a certain quantity: take the difference between each price, and the mean rate as before; then say, as the difference of that simple, whose quantity is given, is to the rest of the differences severally; so is the quantity given, to the several quantities required.

EXAMPLES.

1. How much wine at 5*s*, at 5*s* 6*d*, and 6*s* the gallon, must be mixed with 3 gallons at 4*s* per gallon, so that the mixture may be worth 5*s* 4*d* per gallon?

$$\text{Here } 64\begin{cases}48 & 8+2=10\\ 60 & 2+8=10\\ 66 & 4+16=20\\ 72 & 16+4=20\end{cases}$$

Then 10 : 10 :: 3 : 3
10 : 20 :: 3 : 6
10 : 20 :: 3 : 6 Ans. 3 gallons at 5*s*, 6 at 5*s* 6*d*, and 6 at 6*s*.

2. A grocer would mix teas at 12*s*, at 10*s*, and at 6*s* per lb, with 20lb at 4*s* per lb: how much of each sort must he take to make the composition worth 8*s* per lb? Ans. 20lb at 4*s*, 10lb at 6*s*, 10lb at 10*s*, and 20lb at 12*s*.

POSITION.

Position is a rule for performing certain questions, which cannot be resolved by the common direct rules. It is sometimes called False Position, or False Supposition, because it makes a supposition of false numbers, to work with in the same manner as if they were the true ones, and by their means discovers the true numbers sought. It is sometimes called Trial-and-Error, because it proceeds by *trials* of false numbers, and thence finds out the true ones by a comparison of the *errors*. Position is either Single or Double.

---

suspecting the workmen had debased it by mixing it with silver or copper, he recommended the discovery of the fraud to the famous Archimedes, and desired to know the exact quantity of alloy in the crown.

Archimedes, in order to detect the imposition, procured two other masses, the one of pure gold, the other of silver or copper, and each of the same weight with the former; and by putting each separately into a vessel full of water, the quantity of water expelled by them determined their specific gravities: from which, and their given weights, the exact quantities of gold and alloy in the crown may be determined.

Suppose the weight of each crown to be 10lb, and that the water expelled by the copper or silver was 92lb, by the gold 52lb, and by the compound crown 64lb: what will be the quantities of gold and alloy in the crown?

$$64 \left| \begin{matrix} 92 \\ 52 \end{matrix} \right. \quad \begin{matrix} 12 \text{ of copper} \\ 28 \text{ of gold.} \end{matrix}$$

And the sum of these is 12 + 28 = 40, which should have been 10; therefore by the Rule,

$$\left.\begin{matrix} 40 : 10 :: 12 : 3\text{lb of copper} \\ 40 : 10 :: 28 : 7\text{lb of gold} \end{matrix}\right\} \text{the answer.}$$

* In the very same manner questions may be resolved when several of the ingredients are limited to certain quantities, by finding first for one limit, and then for another. The last two rules can need no demonstration, as they evidently result from the first, the reason of which has been already explained.

SINGLE POSITION.

Single Position is that by which a question is resolved by means of one supposition only. Questions which have their result proportional to their supposition, belong to single position: such as those which require the multiplication or division of the number sought by any proposed number; or when it is to be increased or diminished by itself, or any parts of itself, a certain proposed number of times. The rule is as follows:

Take or assume any number for that which is required, and perform the same operations with it, as are described or performed in the question: then say, as the result of the said operation is to the position or number assumed; so is the result in the question to a 4th term, which will be the number sought*.

EXAMPLES.

1. A person after spending $\frac{1}{3}$ and $\frac{1}{4}$ of his money, has yet remaining $60l$; what had he at first?

Suppose he had at first $120l$.	Proof.
Now $\frac{1}{3}$ of 120 is 40	$\frac{1}{3}$ of 144 is 48
$\frac{1}{4}$ of it is 30	$\frac{1}{4}$ of 144 is 36
their sum is 70	their sum 84
which taken from 120	taken from 144
leaves 50	leaves 60 as per question.

Then, $50 : 120 :: 60 : 144$ the answer.

2. What number is that, which being increased by $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of itself, the sum shall be 75? Ans. 36.

3. A general, after sending out foraging $\frac{1}{2}$ and $\frac{1}{3}$ of his men, had yet remaining 1000: what number had he in command? Ans. 6000.

4. A gentleman distributed 52 pence among a number of poor people, consisting of men, women, and children; to each man he gave $6d$, to each woman $4d$, and to each child $2d$: moreover there were twice as many women as men, and thrice as many children as women: how many were there of each?

Ans. 2 men, 4 women, and 12 children.

5. One being asked his age, said, if $\frac{3}{5}$ of the years I have lived be multiplied by 7, and $\frac{2}{3}$ of them be added to the product, the sum will be 219: what was his age? Ans. 45 years.

DOUBLE POSITION.

Double Position is the method of resolving certain questions by means of two suppositions of false numbers.

---

* The reason of this rule is evident, because it is supposed that the results are proportional to the suppositions.

$$\text{Thus, } na : a :: nz : z, \text{ or } \frac{a}{n} : a :: \frac{z}{n} : z,$$

$$\text{or } \frac{a}{n} \pm \frac{a}{m} \pm \ldots : a :: \frac{z}{n} \pm \frac{z}{m} \pm \ldots : z.$$

To the double rule of position belong such questions as have their results not proportional to their positions: such are those, in which the number sought, or their parts, or their multiples, are increased or diminished by some given absolute number, which is no known part of the number sought.

RULE *.

TAKE or assume any two convenient numbers, and proceed with each of them separately, according to the conditions of the question, as in single position; and find how much each result is different from the result mentioned in the question, calling these differences the *errors*, noting also whether the results are too great or too little.

Then multiply each of the said errors by the contrary supposition, namely, the first position by the second error, and the second position by the first error.

Then, if the errors are like, divide the difference of the products by the difference of the errors, and the quotient will be the answer.

But if the errors are unlike, divide the sum of the products by the sum of the errors, for the answer.

Note. The errors are said to be like, when they are either both too great or both too little; and unlike, when one is too great and the other too little.

EXAMPLE.

1. What number is that, which being multiplied by 6, the product increased by 18, and the sum divided by 9, the quotient should be 20?

---

* *Demonstr.* The rule is founded on this supposition, namely, that the first error is to the second, as the difference between the true and first supposed number, is to the difference between the true and second supposed number; when that is not the case, the exact answer to the question cannot be found by this rule. The algebraist will know at once to what class of questions this property belongs, when it is stated that it is only to such as rise no higher than a simple equation. When it is by means of one equation, and one unknown, the solution can be obtained by *single position:* and when it involves two propositions, one of which cannot be mentally eliminated, we must have recourse to double position. When the conditions lead to a quadratic or higher equation, the condition above named, upon the hypothesis of which the rule is formed, does not take place, and the solution obtained by it, cannot therefore be more than an approximation. The degree of approximation, and some other particulars, may be seen discussed in a note under this head in the Algebra. That the rule is true, under this limitation, may be thus proved.

Let a and b be the two suppositions, and A and B their results, produced by similar operations; also r and s their errors, or the differences between the results A and B from the true result N; and let x denote the number sought, answering to the true result N of the question.

Then is $N - A = r$, and $N - B = s$, or $B - A = r - s$. And, according to the supposition on which the rule is founded, $r : s :: x - a : x - b$; hence, by multiplying extremes and means, $rx - rb = sx - sa$; then by transposition, $rx - sx = rb - sa$; and, by division, $x = \frac{rb - sa}{r - s}$ = the number sought, which is the rule when the results are both too little.

If the results be both too great, so that A and B are both greater than N; then $N - A = -r$, and $N - B = -s$, or r and s are both negative; hence $-r : -s :: x - a : x - b$, but $-r : -s :: +r : +s$, therefore $r : s :: x - a : x - b$; and the rest will be exactly as in the former case.

But if one result A only be too little, and the other B too great, or one error r positive, and the other s negative, then the theorem becomes $x = \frac{rb + sa}{r + s}$, which is the rule in this case, or when the errors are unlike.

Suppose the two numbers 18 and 30. Then,

	First Position.		Second Position.		Proof.
	18		30		27
	6	mult.	6		6
	108		180		162
	18	add	18		18
	9) 126	div.	9) 198		9) 180
	14	results	22		20
	20	true res.	20		
	+ 6	errors unlike	− 2		
2d pos.	30	mult.	18	1st pos.	
Errors { 2	180		36		
Errors { 6	36				
sum 8)	216	sum of products			
	27	= number sought.			

Or, by single position :—the increase given to the product of the number being 18, and this as well as the said product divided by 9, will give the question under the following form.

Required a number, to six-ninths of which if 2 be added, the sum shall be 20; or again, more simply, six-ninths of which is 18.

Then suppose 18 the number. Then,

$\frac{6}{9} \times 18 = 12$, which is too little by 6.

Then $12 : 18 :: 18 : \frac{324}{12} = 27$ Answer.

The advantage in practice of double position is, that instead of requiring any mental preparations similar to those above mentioned, it renders the whole process mechanical; and which are indeed tantamount to as many algebraical ones, bearing in fact a great resemblance to the unsymbolic algebra of the Arabians and Persians and Indians.

RULE II.

Find, by trial, two numbers, as near the true number as convenient, and work with them as in the question; marking the errors which arise from each of them.

Multiply the difference of the two numbers assumed, or found by trial, by one of the errors, and divide the product by the difference of the errors, when they are like, but by their sum when they are unlike. Or thus, by proportion: as the difference of the errors, or of the results (which is the same thing), is to the difference of the assumed numbers, so is either of the errors, to the correction of the assumed number belonging to that error.

Add the quotient, or correction last found, to the number belonging to the said error, when that number is too little, but subtract it when too great, and the result will give the true quantity sought *.

* For since, by the supposition, $r : s :: x - a : x - b$, therefore by division, $r - s : s :: b - a : x - b$, or as $B - A : b - a :: s : x - b$, for $B - A$ is $= r - s$; which is the 2nd rule. Of course the remarks upon the approximation of the first rule apply likewise to the present one.

EXAMPLES.

1. Thus, the foregoing example, worked by this 2nd rule, will be as follows :

30 positions 18 ;		their diff. 12
−2 errors +6 ;		least error 2
	sum of errors 8) 24 (3 subtr.	
	from the position 30	
	leaves the answer 27	

Or, as 22 — 14 : 30 — 18, or as 8 : 12 :: 2 : 3 the correction, as above.

2. A son asking his father how old he was, received this answer : your age is now one-third of mine ; but 5 years ago, your age was only one-fourth of mine. What then are their two ages ? Ans. 15 and 45.

3. A workman was hired for 20 days, at 3*s* per day, for every day he worked ; but with this condition, that for every day he did not work, he should forfeit 1*s*. Now it so happened, that upon the whole he had 2*l* 4*s* to receive : how many of the days did he work ? Ans. 16.

4. A and B began to play together with equal sums of money : A first won 20 guineas, but afterwards lost back $\frac{2}{3}$ of what he then had ; after which B had four times as much as A : what sum did each begin with ?

Ans. 100 guineas.

5. Two persons, A and B, have both the same income. A saves $\frac{1}{5}$ of his ; but B, by spending 50*l* per annum more than A, at the end of 4 years finds himself 100*l* in debt : what does each receive and spend per annum ?

Ans., they receive 125*l* per annum ; also A spends 100*l*, and B spends 150*l* per annum.

PRACTICAL QUESTIONS IN ARITHMETIC.

Quest. 1. The swiftest velocity of a cannon-ball is about 2000 feet in a second of time. Then in what time, at that rate, would such a ball move from the earth to the sun, admitting the distance to be 100 millions of miles, and the year to contain 365 days 6 hours ? Ans. $8\frac{4808}{13149}$ years.

Quest. 2. What is the ratio of the velocity of light to that of a cannon-ball, which issues from the gun with a velocity of 1500 feet per second ; light passing from the sun to the earth in $8\frac{1}{3}$ minutes ? Ans. the ratio of 704000 to 1.

Quest. 3. The slow or parade-step being 70 paces per minute, at 28 inches each pace, it is required to determine at what rate per hour that movement is ?

Ans. $1\frac{113}{132}$ miles.

Quest. 4. The quick-time or step, in marching, being 2 paces per second, or 120 per minute, at 28 inches each ; at what rate per hour does a troop march on a route, and how long will they be in arriving at a garrison 20 miles distant, allowing a halt of one hour by the way to refresh ?

Ans. $\begin{cases} \text{the rate is } 3\frac{2}{11} \text{ miles an hour.} \\ \text{and the time } 7\frac{2}{7} \text{ hr, or 7h } 17\frac{1}{7} \text{ min.} \end{cases}$

Quest. 5. A wall was to be built 700 yards long in 29 days. Now, after 12 men had been employed on it for 11 days, it was found that they had com-

pleted only 220 yards of the wall. It is required to determine how many men must be added to the former, that the whole number of them may just finish the wall in the time proposed, at the same rate of working.

Ans. 4 men to be added.

QUEST. 6. Determine how far 500 millions of guineas will reach, when laid down in a straight line touching one another; supposing each guinea to be an inch in diameter, as it is very nearly. Ans. 7891 miles, 728 yds, 2 ft, 8 in.

QUEST. 7. Two persons, A and B, being on opposite sides of a wood, which is 536 yards about, begin to go round it, both the same way, at the same instant of time; A goes at the rate of 11 yards per minute, and B 34 yards in 3 minutes; the question is, how many times will the wood be gone round before the quicker overtake the slower? Ans. 17 times.

QUEST. 8. A can do a piece of work alone in 12 days, and B alone in 14; in what time will they both together perform a like quantity of work?

Ans. $6\frac{6}{13}$ days.

QUEST. 9. A person who was possessed of a $\frac{3}{5}$ share of a copper mine, sold $\frac{3}{4}$ of his interest in it for 1800*l*; what was the reputed value of the whole at the same rate? Ans. 4000*l*.

QUEST. 10. A person after spending 20*l* more than $\frac{1}{4}$ of his yearly income, had then remaining 30*l* more than the half of it; what was his income?

Ans. 200*l*.

QUEST. 11. The hour and minute hands of a clock are exactly together at 12 o'clock; when are they next together? Ans. $1\frac{1}{11}$ hr or 1 hr $5\frac{5}{11}$ min.

QUEST. 12. If a gentleman whose annual income is 1500*l*, spend 20 guineas a week; whether will he save or run in debt, and how much in the year?

Ans. he saves 408*l*.

QUEST. 13. A person bought 180 oranges at 2 a penny, and 180 more at 3 a penny; after which he sold them out again at 5 for 2 pence: did he gain or lose by the bargain? Ans. he lost 6 pence.

QUEST. 14. If a quantity of provisions serves 1500 men 12 weeks, at the rate of 20 ounces a day for each man; how many men will the same provisions maintain for 20 weeks, at the rate of 8 ounces a day for each man?

Ans. 2250 men.

QUEST. 15. In the latitude of London, the distance round the earth, measured on the parallel of latitude, is about 15550 miles; now as the earth turns round in 23 hours 56 minutes, at what rate per hour is the city of London carried by this motion from west to east? Ans. $649\frac{259}{359}$ miles an hour.

QUEST. 16. A father left his son a fortune, $\frac{1}{4}$ of which he ran through in 8 months: $\frac{3}{7}$ of the remainder lasted him 12 months longer; after which he had 820*l* left: what sum did the father bequeath his son? Ans. 1913*l* 6*s* 8*d*.

QUEST. 17. If 1000 men, besieged in a town, with provisions for 5 weeks, allowing each man 16 ounces a day, be reinforced with 500 men more; and supposing that they cannot be relieved till the end of 8 weeks, how many ounces a day must each man have, that the provision may last that time?

Ans. $6\frac{2}{3}$ ounces.

QUEST. 18. A younger brother received 8400*l*, which was just $\frac{7}{9}$ of his elder brother's fortune: what was the father worth at his death? Ans. 19200*l*.

QUEST. 19. A person, looking on his watch, was asked what was the time of the day, who answered, "It is between 5 and 6;" but a more particular answer being required, he said that "the hour and minute hands are exactly together:" what was the time? Ans. 27 $\frac{3}{11}$ min. past 5.

Quest. 20. If 20 men can perform a piece of work in 12 days, how many men will accomplish another thrice as large in one-fifth of the time? Ans. 300.

Quest. 21. A father devised $\frac{7}{18}$ of his estate to one of his sons, and $\frac{7}{18}$ of the residue to another, and the surplus to his relict for life. The children's legacies were found to be 514*l* 6*s* 8*d* different: what money did he leave the widow the use of? Ans. 1270*l* 1*s* $9\frac{11}{49}$*d*.

Quest. 22. A person, making his will, gave to one child $\frac{13}{20}$ of his estate, and the rest to another. When these legacies came to be paid, the one turned out 1200*l* more than the other: what did the testator die worth? Ans. 4000*l*.

Quest. 23. Two persons, A and B, travel between London and Lincoln, distant 100 miles, A from London, and B from Lincoln at the same instant. After 7 hours they met on the road, when it appeared that A had rode $1\frac{1}{2}$ miles an hour more than B. At what rate per hour then did each of the travellers ride? Ans. A $7\frac{25}{28}$, and B $6\frac{11}{28}$ miles.

Quest. 24. Two persons, A and B, travel between London and Exeter. A leaves Exeter at 8 o'clock in the morning, and walks at the rate of 3 miles an hour, without intermission; and B sets out from London at 4 o'clock the same evening, and walks for Exeter at the rate of 4 miles an hour constantly. Now, supposing the distance between the two cities to be 130 miles, where will they meet? Ans. $69\frac{3}{7}$ miles from Exeter.

Quest. 25. One hundred eggs being placed on the ground, in a straight line, at the distance of a yard from each other: how far will a person travel who shall bring them one by one to a basket, which is placed at one yard from the first egg? Ans. 10100 yards, or 5 miles and 1300 yards.

Quest. 26. The clocks of Italy go on to 24 hours: how many strokes do they strike in one complete revolution of the index? Ans. 300.

Quest. 27. One Sessa, an Indian, having invented the game of chess, showed it to his prince, who was so delighted with it, that he promised him any reward he should ask; on which Sessa requested that he might be allowed one grain of wheat for the first square on the chess-board, 2 for the second, 4 for the third, and so on, doubling continually to 64, the whole number of squares. Now, supposing a pint to contain 7680 of these grains, and one quarter or 8 bushels to be worth 27*s* 6*d*, it is required to compute the value of all the corn.

Ans. 6450468216285*l* 17*s* 3*d* $3\frac{32757}{32768}$*q*.

Quest. 28. A person increased his estate annually by 100*l* more than the $\frac{1}{4}$ part of its value at the beginning of that year; and at the end of 4 years found that his estate amounted to 10342*l* 3*s* 9*d*: what had he at first?

Ans. 4000*l*.

Quest. 29. Paid 1012*l* 10*s* for a principal of 750*l*, taken in 7 years before: at what rate per cent. per annum did I pay interest? Ans. 5 per cent.

Quest. 30. Divide 1000*l* among A, B, C; so as to give A 120 more, and B 95 less than C. Ans. A 445, B 230, C 325.

Quest. 31. A person being asked the hour of the day, said, the time past noon is equal to $\frac{4}{3}$ths of the time till midnight: what was the time?

Ans. 20 min. past 5.

Quest. 32. Suppose that I have $\frac{3}{16}$ of a ship whose whole worth is 1200*l*; what part of her have I left after selling $\frac{2}{5}$ of $\frac{4}{9}$ of my share, and what is it worth?

Ans. $\frac{37}{240}$, worth 185*l*.

Quest. 33. Part 1200 acres of land among A, B, C; so that B may have 100 more than A, and C 64 more than B. Ans. A 312, B 412, C 476.

Quest. 34. What number is that, from which if there be taken $\frac{2}{7}$ of $\frac{3}{8}$, and to the remainder be added $\frac{6}{15}$ of $\frac{5}{16}$, the sum will be 10? Ans. $9\frac{20}{21}$.

Quest. 35. There is a number which, if multiplied by $\frac{2}{3}$ of $\frac{4}{5}$ of $1\frac{1}{2}$, will produce 1: what is the square of that number? Ans. $1\frac{9}{16}$.

Quest. 36. What length must be cut off a board, $8\frac{1}{2}$ inches broad, to contain a square foot, or as much as 12 inches in length and 12 in breadth?

Ans. $16\frac{16}{17}$ inches.

Quest. 37. What sum of money will amount to 138*l* 2*s* 6*d*, in 15 months, at 5 per cent. per annum simple interest? Ans. 130*l*.

Quest. 38. A father divided his fortune among his three sons, A, B, C, giving A 4 as often as B 3, and C 5 as often as B 6; what was the whole legacy, supposing A's share was 4000*l*? Ans. 9500*l*.

Quest. 39. A young hare starts 40 yards before a grey-hound, and is not perceived by him till she has been up 40 seconds; she scuds away at the rate of 10 miles an hour, and the dog, on view, makes after her at the rate of 18: how long will the course hold, and what ground will be run over, counting from the outsetting of the dog? Ans. $60\frac{5}{22}$ sec. and 530 yards run.

Quest. 40. Two young gentlemen, without private fortune, obtain commissions at the same time, and at the age of 18. One thoughtlessly spends 10*l* a year more than his pay; but, shocked at the idea of not paying his debts, gives his creditor a bond for the money, at the end of every year, and also insures his life for the amount; each bond costs him 30 shillings, besides the lawful interest of 5 per cent. and to insure his life costs him 6 per cent.

The other, having a proper pride, is determined never to run in debt; and, that he may assist a friend in need, perseveres in saving 10*l* every year, for which he obtains an interest of 5 per cent. which interest is every year added to his savings, and laid out, so as to answer the effect of compound interest.

Suppose these two officers to meet at the age of 50, when each receives from Government 400*l* per annum; that the one, seeing his past errors, is resolved in future to spend no more than he actually has, after paying the interest for what he owes, and the insurance on his life.

The other, having now something beforehand, means in future to spend his full income, without increasing his stock.

It is desirable to know how much each has to spend per annum, and what money the latter has by him to assist the distressed, or leave to those who deserve it?

Ans. The reformed officer has to spend 66*l* 19*s* 1*d* 2·6583*q* per annum.
The prudent officer has to spend 437*l* 12*s* 11*d* 3·4451*q* per annum, and
The latter has saved, to dispose of, 752*l* 19*s* 9·2256*d*.

ALGEBRA.

I. *Introductory explanation of the character and objects of this branch of Mathematics.*

1. If a series of given numbers be directed to be combined in any specified manner, that specification may either be expressed in words at length, or by means of the usual symbols of arithmetical operations, and such other contrivances as have been already explained in the treatise on arithmetic. When the symbolic or abbreviated mode of expression is employed, the collection of numbers and symbols constitute what is called *an arithmetical expression.* Thus, if from the sum of six and seven we were directed to take three, and multiply the remainder by one-half the defect of six from ten, then the *arithmetical expression* for this would be

$$(6 + 7 - 3) \times \frac{10 - 6}{2},$$

all the symbols and modes of writing employed in this expression having been already defined and rendered familiar (page 6).

2. If we *actually perform the several operations here indicated or directed,* we shall obtain what is called the *value of the expression,* which in the present case is 20, and in each case is dependent upon the arithmetical conditions of the given expression.

When we express that 20 is the equivalent or value of such an expression, we form an *arithmetical equation,* viz.

$$(6 + 7 - 3) \times \frac{10 - 6}{2} = 20.$$

3. In the solution of any arithmetical question, we are enabled for the most part, with a little consideration, to refer to a class for solving which *rules* have been already invented. These rules consist in the substitution of a series of arithmetical operations of a simple kind, to be performed in a specified order upon the several numbers given in the conditions of the question. Thus, in questions which are reducible to "the Rule of Three terms," or simply the "Rule of Three," the answer is found by arranging the given terms in a particular and specified order, and then dividing the product of the second and third by the first of the terms so arranged. Now as this *rule* is the same whatever the first, second, and third terms may happen to be, it could not be expressed without *some symbols* to stand for those terms, which, whilst expressing the fact of their being numbers so arranged, would yet not confine them to any particular values as numbers, nor to any particular class as objects.

The letters of the Alphabet have been used for this purpose throughout Europe, and those regions which have received their science from Europeans, without a single exception. Sometimes they have been so chosen as to be the initial letter of the kind of quantity whose numbers they stood in place of, as t, s, v, for time, space, and velocity: but they have been generally so selected, that the earlier letters of the alphabet, a, b, c, ... should stand in *the place of the numbers which in every actual question are given, or express the given conditions;*

whilst the final letters, $z, y, x, w, v, \ldots$ have been employed to occupy the place in the immediate expression of the question, of those numbers which *till the rule has been put into execution are unknown, and which therefore it is the object of the problem to discover.* Thus, in the Rule of Three, if a, b, c taken in order be selected to designate generally the first, second, and third of the given terms taken in order, and which are all given in each actual case, and if the answer, as yet unknown, be denoted by x, we shall have the condition, when simplified from all extraneous considerations, expressed thus:

$$a : b :: c : x,$$

and the formula or rule for solution would take this form:

$$x = \frac{b \times c}{a}.$$

4. So likewise, if in the case of the numbers 10, 6, 7, 3, which occur in the arithmetical formula in the beginning of this chapter, these numbers had been the particular values of four quantities which in the solution of some specific class of questions were always given amongst its conditions, and that by some means or other, the rule for solution had been discovered to be, that the third number (taking them in the order of their occurrence in the question, or of some arrangement to which the rule always subjected them) must be taken from the sum of the second and fourth, and the remainder multiplied by half the defect of the second from the first: then, writing as the symbols of the first, second, third, and fourth terms of the equation in art 2. the letters a, b, c, d, and for the yet unknown number, that is, the answer sought, writing the symbol x, the expression of this rule in an algebraic formula will be

$$(b + d - c) \times \frac{a - b}{2} = x.$$

5. To take another example, suppose it were proposed to find that number to the square of which we add the number b, then the sum shall be equal to a times the number itself; then the formula of solution would be

$$x = \frac{a}{2} + \sqrt{\frac{a \times a}{4} - b},$$

$$\text{and } x = \frac{a}{2} - \sqrt{\frac{a \times a}{4} - b}^{*}.$$

The problem itself would take the following form:

$$x \times x + b = a \times x:$$

and it is very obvious, at first sight, that the method of solution falls under no rule that has been given in the treatise on arithmetic; and therefore such rule must be sought for by some new method of investigation, either analogous to those by which the arithmetical rules were discovered, or possibly by some process altogether different from them in principle as well as in plan.

---

* The student may be surprised to see that two different methods of calculating the result have been given; but he will see the reason of this hereafter, and also that often three, four, or indeed any number whatever of answers to different questions are possible. To take an instance, the values 4 and 3, as those of a and b, would find $x = 2 + \sqrt{1}$, and $x = 2 - \sqrt{1}$, or 3 and 1. He can verify these by actual trial.

In trying other numbers he may hit upon some conditions that will not admit of any solution whatever; as, if $a = 2$ and $b = 2$, the solutions would be $x = 1 + \sqrt{-1}$, and $x = 1 - \sqrt{-1}$, results which he is not in possession of methods capable of interpreting. Such cases are said to be *impossible*, and their explanation will be found in a note on "Quadratic Equations" in the present volume.

6. The investigation of such rules for calculation is one of the two objects of Algebra.

The other object, which is subservient to the former, is the discovery of the different operations which may be performed with the same given numbers, and shall produce the same ultimate numerical results as any given operations different from these shall produce, without regard to what those numbers may chance to be. Thus, if the square of the sum of a and b were sought in another form, it may be exhibited thus $a \times a + 2 \times a \times b + b \times b$. And the statement of this fact is thus written:

$$(a + b) \times (a + b) = (a \times a) + (2 \times a \times b) + (b \times b.)$$

Shorter modes of writing it will be exhibited presently; but here the simple symbols used in the arithmetic have been alone employed, for the purpose of showing the nature of algebraic notation in its earliest forms, and to illustrate the objects for which it was devised.

7. The discovery of formulæ for the solution of questions constitutes the *algebraical problem;* and the discovery of formulæ of transformation, or of those which give equivalent results independently of the particular value of the quantities which enter into their composition, constitute the *algebraical theorem.*

8. The motives which gave rise to the use of alphabetic letters as symbols of number in preference to any other system of symbols, arbitrarily selected for the same purpose, are principally the following. *First,* As they have no *numerical* signification in themselves, they are subject to no ambiguity, having in reference to numbers no other signification than they are defined to have in the outset of each problem, or either defined, or understood from general practice, to have in each theorem. *Secondly,* Being familiar to the eye, the tongue, the hand, and the mind, that is, having a well-known form and name, they are easily read, written, spoken, remembered, and discriminated from one another, which could not be the case were they mere arbitrary marks, formed according to the caprice of each individual who used them, and always different, as in such case they must almost of necessity be, at each different time that the same person required to use them. *Thirdly,* The order in which the letters are arranged in the alphabet, facilitates the classification of them into groups much more easy to survey and comprehend in the expressions which arise from the performance of any assigned operations, and thereby renders the investigator much less likely to omit any of them by an imperfect enumeration, than if they were composed of marks that were used for that purpose only, and selected for each individual occasion from the various combinations that could be formed of such simple linear elements as the hand could readily trace, and the eye readily distinguish from all other combinations.

II. *Definitions, Notation, and Fundamental Principles.*

The principal symbols which are employed to designate the operations of algebra and arithmetic, and the relations which subsist between quantities, are the following. Their object is to abbreviate.

I. 1. $+$ signifies *addition,* and is read *plus.* Thus $2 + 3$ or $a + b + c$ respectively signify that 3 is to be added to 2, and that b is to be added to a, and that then c is to be added to the sum of a and b.

A quantity to which the symbol $+$ is prefixed, is called a *positive* or *affirmative quantity.*

2. $-$ signifies *subtraction,* and is read *minus.* Thus, $3 - 1$, or $b - a$, signify

respectively that 1 is to be subtracted from 3, and a from b. The number to be subtracted is always placed after the symbol.

A quantity to which the sign — is prefixed is called a *negative quantity* *.

3. $\backsim$ signifies the *difference* of the quantities between which it is placed; and is used either when it is not known or is not necessary to specify which is the greater of them. In this case $a \backsim b$, or $b \backsim a$, signify the same thing.

4. $\times$ is the symbol of *multiplication*, and is placed between the factors which are to be multiplied together. Sometimes a point . (placed at the lower part of the line, to distinguish it from the decimal point, which is placed at the upper part of the line,) is employed for the same purpose, and especially between the numerical factors, as $3 \,.\, 5 \,.\, 7$, or $1 \,.\, 2 \,.\, 3 \,.\, 4$: and in the case of simple literal factors, the practice is now almost universal to drop all marks between the simple factors, and write them in consecutive juxta-position. Thus $a \times b \times c \times x$, or $a \,.\, b \,.\, c \,.\, x$, or $abcx$ designate the same thing, viz. the continued product of the numbers which a, b, c, and x are put to represent †.

When one of the factors is a number, it is called a *coefficient:* thus in $2 \times a \times b$ or $2ab$, the 2 is called the coefficient of ab, and in $53xyz$, 53 is called the coefficient of xyz. When no coefficient is written, 1 is understood to be meant, the quantity being taken *once*.

Also, in some cases where letters are put for numbers, the letters which represent given or known numbers are likewise called coefficients; as in $3\,axz$, $3a$ is called the coefficient of xz. In the case of a number being actually given, the coefficient is said to be a *numeral coefficient*; but when it is given in literal symbols, it is called a *literal* coefficient.

Moreover it may be remarked that cases of algebraical investigation sometimes present themselves in which even the symbols of the unknown quantities are conveniently considered as coefficients: but these will be pointed out when they arise.

Though, as is proved in the note, the order of the factors in multiplication, so

---

* Quantities affected with the signs + and + or — and —, are said to have *like signs*; and those affected with — and +, or + and —, are said to have *unlike signs*.

It is manifest from the nature of addition and subtraction, that the disposition of the quantities as to *order* is immaterial; for $a + b$, or $b + a$, is the same thing, and $a + b - c$, $a - c + b$, or $-c + b + a$, express the same quantity, only under a different arrangement, as to relative position.

† It may be readily shown that it is immaterial in what order the factors are taken for the purpose of multiplication: that is, which is made the multiplicand and which the multiplier.

For if a number of dots (or units) be placed *horizontally* equal in number to the units in the factor selected as the multiplicand, and this be repeated under this horizontal band till there are as many bands as there are units in the other factor: then the same number of dots considered as forming *vertical* columns will be constituted of as many times the number there is in the multiplier as there are units in the multiplicand, and representing therefore the result of a multiplicand with the *order* of the factors inverted. Thus if we take four times three, the dots will stand

• • •
• • •
• • •
• • •

and if we turn the column which is vertical into a horizontal position, it becomes

• • • •
• • • •
• • • •

And in the same way it is shown of m and n as factors.

far as the *value* of the product is concerned, is immaterial; yet in the disposition of them as algebraic symbols, it is found convenient generally to arrange them in alphabetic order. Thus the quantity $abcxy$ is the same in point of value as $bcxya$, or any other arrangement that can be made of them, still the form $abcxy$ is preferable, for many reasons, to any other that can be given to the same quantity.

5. $\div$ is the symbol of *division*. Sometimes the dividend is put before and the divisor after the mark, and sometimes they are placed respectively above and below the line in the place of the two dots, after the manner of an arithmetical fraction. Thus $a \div b$ or $\frac{a}{b}$ alike signify that the number a is to be divided by the number b.

6. $=$ is the symbol for the words "*is equal to*," and is generally read "*equals*." It is used to signify that the value of the aggregate of the terms which precede it is equal to the value of the aggregate of those which follow it. The whole expression is called *an equation*, and the quantities which stand to the left of the symbol are said to constitute *the first side of the equation*, and in like manner those which stand on the other constitute the *second side of the equation*. Thus $ax + b = cxx + dx - e$ is an equation, the first side of which is $ax + b$, and the second side $cxx + dx - e$.

7. The symbols $>$, $<$, and sometimes $\neq$, are used to express the *inequality* of the quantities between which they are placed. The opening of the symbol is always turned towards the greater quantity. Thus $a > b$ signifies, that a is *greater* than b; and $f < g$ signifies, that f is *less* than g.

When it either is not known, or is not necessary to state which is the greater of the two quantities, which are nevertheless unequal, the symbol $\neq$ is used.

8. : signifies that it is the *ratio* of the two quantities between which it stands which is the subject of consideration. Thus $x : y$ designates the ratio of x to y. It is read x to y, or *the ratio of x to y*.

When two ratios are equal; that is, when some two numbers have the same ratio that two others have, it is expressed in one of these two ways:—

$$x : y :: u : v \text{ or } x : y = u : v$$

The former is most usual, and adhered to in this Course. The phrases which they represent are

x is to y as u is to v;
or, x has the same ratio to y that u has to v:
or, the ratio of x to y, is the same as that of u to v.
or, again, the ratio of x to y is equal to that of u to v.

9. It is often found necessary to class the quantities of which an expression is composed, into sets combined in some particular way. This classification is effected by enclosing them under a horizontal bar (called *a vinculum*), or between *parentheses*, or *braces*, or *brackets*. This is always done when the actual operations indicated are to be changed for others which shall produce equivalent results, and which are more easy to perform than those which were originally indicated. Nearly the whole of algebra consists in discovering these *equivalent operations*.

Thus, referring to the example given in the Introductory Chapter, we might have put it in the more complicated but equivalent form,

$$\frac{6 \times 10 + 7 \times 10 - 3 \times 10 - 6 \times 6 - 6 \times 7 + 6 \times 3}{2}:$$

but by taking the form there given, the actual labour of computation is not above a fourth of that which would attend upon all these. Or, again, in general

symbols, the expression $(a + 2b - 3c)\ (4a - 2b + 3c)$ indicates that the sum of a and twice b being taken, and three times c being subtracted from the sum, and that to the difference of four times a and twice b, three times c is added, then the product of these two results is to be taken. This effect might be produced by other means much more complicated, but which are avoided by the contrivance indicated above.

When the terms which are collected together are also employed to form the numerator and denominator of an expression in a fractional form, no ambiguity can arise, except the said numerator and denominator are composed of vinculated factors, in dropping the vinculum. Thus $\frac{(a + b)}{(a - b)}$ may be written simply $\frac{a + b}{a - b}$.

10. When several of the factors which compose a quantity are equal to one another, a considerable abbreviation of the trouble of writing, and of the space occupied, has been devised, by simply writing the common factor in its place, and a small figure above it and to the right hand, expressing how many times that factor occurs in the product. Thus, instead of $a \times a \times a \times a$ or $aaaa$, it is usual to write a^4, the a signifying the common factor, and the 4 the number of times of its occurrence. So likewise, instead of the expression $3aaabbxxxx$, is written $3a^3b^2x^4$; and $(a + x)\ (a + x)\ (a + x)$ is written $(a + x)^3$: and so on for any number or form of the component equal factors of an expression.

The number affixed is called the *index* or *exponent*, and the quantity is said to be *raised to the power* denoted by that exponent. Thus a^4 signifies the fourth power of a, and 4 is called the index or exponent of the power of a: and a^n is called the nth power of a, and n is the exponent of that power.

In conformity with this notation, when no index is annexed to a quantity, 1 is understood: thus, by a, the first power of a is meant, that is, a^1.

It will obviously follow, that when *two different powers of the same quantity* are to be multiplied together, the symbolical result may be written as a power whose exponent is the sum of the exponents of the several factors. Thus $a^4 \times a^2 \times a^1 \times a^3 = a^{4+2+1+3} = a^{+10}$; for it is the same thing as $aaaa \times aa \times a \times aaa$, that is, $aaaaaaaaaa$ or a^{10}. On the same principle a^2b^3c multiplied by a^3bc^5 may be more briefly written $a^{2+3}b^{3+1}c^{\cdot+5}$, or still more briefly $a^5b^4c^6$.

Similarly, in general $a^m \times a^n \times a^r \ldots = a^{m+n+r \ldots}$, where the dots after the quantities express the continuation of the same class of quantities to which they are respectively annexed to any assignable extent.

It moreover follows, that the quotient of one power of a quantity by another is symbolically expressible by a power of that quantity whose exponent is equal to that denoted by the remainder left after subtracting the exponent of the divisor from that of the dividend. For since multiplication increases the number of factors, division will decrease them. Thus, since $a^5 \times a^3 = a^8$, so also $\frac{a^8}{a^3} = a^5$; and in like manner generally $\frac{a^m}{a^n} = a^{m-n}$.

The two modes of notation just employed to designate the division of one power of a quantity by another power of the same quantity may be used indifferently. By remarking, however, the corresponding forms in some particular cases, we shall be led to some simple but important conclusions. And first,

When $m = n$, we have $\frac{a^m}{a^m} = 1$, and $a^{m-m} = a^0$, which are of course iden-

tical in signification. Hence we learn that a^o always signifies 1, whatever a may be. Indeed a^m signifying 1 a^m, if $m = o$ we have 1 $a^o = 1$, or $1 \times a^o = 1$, or 1 time a taken of no power * whatever. The result is therefore perfectly consistent with first principles and the adopted notations.

When $m < n$, then if we put $m = n - r$, we have $\frac{a^{n-r}}{a^n} = a^{n-n-r} = a^{o-r}$ $= a^{-r}$. But this also signifies, when $n = m + r$ †, the following expression, $\frac{a^m}{a^{m+r}} = \frac{a^m}{a^m a^r} = \frac{1}{a^r}$. Hence, $a^{-r} = \frac{1}{a^r}$. We learn from this, that when in any assigned series of operations upon the powers of a quantity, we arrive at an index of the form $-r$, then the expression signifies the reciprocal of the rth power of the factor a. Thus a^{-2} signifies $\frac{1}{a^2}$, and 2×10^{-4} signifies $\frac{2}{10^4}$, or $\frac{2}{10000}$, or ·0002, and so on.

11. As we have occasion to calculate the *powers* of given *numbers*, considered as roots, so we have often to find the roots of given numbers considered as powers. The operations are considered as the inverse of each, and are denoted by inverse notations.

Thus to cube the quantity aa or a^2 we have $aa \times aa \times aa$ or $a^2 \,.\, a^2 \,.\, a^2$ or a^6; so likewise to find the cube root of a^6 we have to separate it into three equal factors. This operation, in all such cases, is indicated by dividing the exponent of the power by the exponent of the root: thus the cube root of a^6 is $a^{\frac{6}{3}}$, where 6 is the exponent of the given quantity, and 3 the exponent of the root. In the same manner the square root of $a^6 b^4 c^2 d^{-4}$, is $a^{\frac{6}{2}} b^{\frac{4}{2}} c^{\frac{2}{2}} d^{\frac{-4}{2}}$, or $a^3 b^2 c d^{-2}$ ‡. If there be a numerical coefficient, the root of that is either actually extracted or indicated like the rest; as in the square root of $25a^4$ we may either put $25^{\frac{1}{2}}a^2$ or $5a^2$. Most commonly the numerical root is actually extracted when the given number admits of an accurate root, but indicated when the value cannot be assigned in a finite form.

It will be obvious from the signification given above to this notation, that the numerator of a fractional exponent expresses the power to which a root is raised, and the denominator the root which is to be taken of that result.

It also follows from the nature and relation of roots and powers, that it is immaterial whether we first extract the indicated root, and then raise it to the indicated power, or conversely, we first raise the indicated power, and then

---

* The term *dimension* is often employed instead of the word *power*. It is derived from the analogy which the dimensions of *line*, *square*, and *cube* in geometry, when they are expressed *numerically*, have to the *first*, *second*, and *third* powers. Beyond the third power, geometry furnishes nothing analogous to the powers of quantities; and hence the terms fourth, fifth, &c. dimensions though generally used, are hardly accurate.

† This is the same as the former, $m = n - r$, having r added to both sides; and hence $m + r = n - r + r = n$.

‡ When the index of the power is not divisible by the index of the root, the fractional form of the index which results is retained; as in the cube root of $a^2 b^7$, we write it $a^{\frac{2}{3}} b^{\frac{7}{3}}$; or sometimes the given quantity is conceived to be put under the form $a^2 b^6 b^1$, or simply, $a^2 b^6 b$, and the root written $a^{\frac{2}{3}} b^2 b^{\frac{1}{3}}$. The ultimate purpose of the particular inquiry in question is the only guide to the Algebraist in his choice amongst these notations.

extract the indicated root. In actual *numerical* practice, however, it is always best to extract the root first, when the number admits of an exact root: but when it does not admit of an exact one, it is better to raise the indicated power first, and then extract the indicated root. The reason is, that the number of figures necessary is always less than would be required in taking the contrary course.

Respecting the verbal enunciation of these expressions, it should be remarked that

> $a^{\frac{1}{3}}$ is generally read the *third root* of a, or the *one-third power* of a.
>
> $\sqrt[3]{a}$ is always read the *third root* of a.
>
> $a^{\frac{2}{3}}$ is read the *two-thirds power* of a, or the *cube root* of a^2, or the *square* of the *cube root* of a,

and so on with other expressions.

It ought also to be understood that the fractional indices may be, and often are, converted into decimals; as for $a^{\frac{1}{2}}$ may be written $a^{\cdot 5}$, for $a^{-\frac{3}{2}}$ may be written either $a^{-1 \cdot 5}$, or $\frac{1}{a^{1 \cdot 5}}$.

12. Another method of indicating the extraction of roots is by the symbol $\sqrt{\ }$ prefixed (being only a modification of the form of the old manuscript *r*, the initial letter of the word *radix*); and the order of the root expressed by a small number written over it, and lying a little to the left: as $\sqrt[2]{a}$, $\sqrt[3]{b}$, $\sqrt[4]{(a+x)}$, $\sqrt[n]{a^2 - b^2}$; which signify respectively the same thing as $a^{\frac{1}{2}}$, $b^{\frac{1}{3}}$, $(a+x)^{\frac{1}{4}}$, $(a^2 - b^2)^{\frac{1}{n}}$, or in words, the square (or second) root of a, the cube (or third) root of b, the fourth root of $a + x$, and the nth root of $a^2 - b^2$.

The distinction of *rational* and *irrational*, in respect to quantities whose roots can or cannot be respectively extracted, is a very convenient one. An irrational quantity is often called a *surd*. Rational quantities are sometimes put under an *irrational form*, to *facilitate their combination with irrational quantities into one expression*.

13. Quantities receive different designations which are found useful in algebraical enunciations, according to peculiar circumstances. The following are the principal ones:

A *simple quantity* * is that which consists of a single term, or of several factors only, each of which is a single term. As $3a$, or $5ab$, or $6a^2b^3c^7$, or $3a^{-5}$. It is often called a *monome* or a *monomial quantity*.

A *compound quantity* is composed of the aggregation of two or more simple quantities connected together by addition or subtraction: as $a + b$, $2a - 3c$, or $a + 2b - 3c$.

Of compound quantities, that which is composed of two terms, as $x + y$, $xy + ab$, or $x^2 - y^2$ †, or $3x + 4aby$ is called a *binomial quantity;* when three terms, $x^2 + ax + b$, it is called a *trinomial quantity*; when four or more terms, a *polynomial quantity*, or simply a *polynomial*.

When the binomial, trinomial, or polynomial expressions are so related that their terms counterbalance or mutually destroy one another in the aggregate, the

* The term *expression* is often applied to any combination of algebraical quantities.

† The term *residual* was formerly applied to that form of the binomial expression when they were connected by the sign —, as $a - b$, $x^2 - y^2$. The distinction is now, however, fallen greatly into disuse.

total expression is put equal to zero, and the expression in this form is called a binomial equation, a trinomial equation, or a polynomial equation. Sometimes the word equation is understood, and the terms *binome* or *binomial*, *trinome* or *trinomial*, *polynome* or *polynomial*, are used instead of the compound phrase.

14. When any general numerical relation is exhibited in the form of an equation, the expression is called a *formula.*

15. When the numerator and denominator of any quantity are interchanged, the resulting expression is called the *reciprocal* of the former. If the quantity be in the integer form, it may be put in a fractional form, and its denominator is understood to be unity or 1. In that case the result or reciprocal is 1 divided by the given quantity. Thus $\frac{a+b}{a-b}$ is the reciprocal of $\frac{a-b}{a+b}$, and $\frac{1}{cd}$ is the reciprocal of cd.

16. There is also a distinction to be made between the notation of factors when writing algebraically and numerically. It was explained in the third definition, that when algebraical symbols of factors were written, they were generally put in juxtaposition, without any mark between them. In arithmetic, the figures placed in juxtaposition have each a relative value derived from the particular position they occupy, which is generally called the *local value*, and are thereby rendered units, tens, hundreds, &c. or tenths, hundredths, &c. To represent the compound number, then, when the component figures are supplied in literal notation, each one must have for its coefficient such a power of 10 as will raise or lower it to its proper locality in the decimal scale. Thus to represent 5305·2906 without employing the artifice of local value, we must write it thus: $5 \times 10^3 + 3 \times 10^2 + 0 \times 10^1 + 5 \times 10^0 + 2 \times 10^{-1} + 9 \times 10^{-2} + 0 \times 10^{-3} + 6 \times 10^{-4}$.

Or, to designate a number whose digits to the left of the decimal point were x, y, z, and the right of it were t and u, we should have

$$10^2x + 10^1y + 10^0z + 10^{-1}t + 10^{-2}u.$$

17. An algebraical expression is said to be *ranged according to the powers* (or dimensions) of some quantity, when the term containing the highest power of that letter is placed first, the term containing the next power next to it in order, and so on to the lowest. Sometimes also it is so ranged as to begin with the lowest and ascend to the higher, in order. In expressions containing a finite number of terms the former plan is most commonly adopted; but in expressions where the terms run out to infinity (as it will appear in the course of the work is often the case), the latter is necessarily employed. Thus, in the expression $x^3 + 3x^2 - 9x + 10 - 8x^{-1} + 3x^{-2}$, the arrangement is according to the powers of x, and so also is $3x^{-2} - 8x^{-1} + 10 - 9x + 3x^2 + x^3$, so arranged. The former in *a descending series of powers*, the latter in *an ascending series of powers*. In $x^2 + 2xy + y^2$ they are ranged according to the *descending powers* of x, and in $y^2 + 2xy + x^2$ in *descending powers* of y. Of the infinite series of *ascending powers* the following is an example—

$$1 - 2x + 2x^2 - 2x^3 + \dots \text{ ad inf.}$$

and of *descending powers*, the following—

$$1 - \frac{c}{x} + \frac{c^2}{x^2} + \frac{c^3}{x^3} + \dots, \text{ or } 1 - cx^{-1} + c^2x^{-2} - c^2x^{-3} + \dots$$

III. *Exercises in Notation and its interpretation.*

I. Write in words the signification of the expressions $x^2 - 10x = 119$; $y^2 - x^2 = 40$; $x\sqrt{y} - 4ab = c^2$; $(a + x - y)^{\frac{1}{7}} \times \left\{a - x + \frac{3}{y}\right\}^{\frac{1}{3}}$ $\sqrt{x^2 + y^2} \pm \sqrt{x^2 - y^2}$; $a^x \pm a^{-x}$; and $x^y - y^x = 0$.

II. Write in the common numerical forms the quantities $10^5 \times 3 + 10^0 \times 7 + 10^{-4} \times 2$; and form an expression whose digits are decimals, beginning at the third place from the unit's place, are z, o, y, o, x, o, o, t.

III. How are the following expressions to be interpreted verbally?

$(a^{\frac{1}{3}})^3$; $(a^3)^{\frac{1}{3}}$; $\sqrt[3]{(a - x)} = (a - 5x)^{\frac{2}{3}}$; $3a - 6b = -4a + 15b$;

$-a - b = -(a + b)$; $6(ax - by) = 6ax - 6by$; $\sqrt[n]{a + b}$; $4z^{\frac{m}{n}}$;

$(4z)^{\frac{m}{n}}$; $4^{\frac{m}{n}} z^{\frac{m}{n}}$; $(abc)^3$, a^3bb^2ccc, $(ab)^2abc^2$. And point out any which are identical in value though different in their forms.

IV. Put into algebraical symbols the following statements:

1. There is a certain number at present unknown, to the square root of which if we add its square we shall obtain 18: and another whose half exceeds its third part by three-tenths of an unit.

2. Of three unknown numbers, the sum of the first two is equal to 5, the difference of the second and third is 1, and the sum of all three is 9.

3. Add three given numbers together, and indicate the cube of the square root of their sum, multiplied by the product of the three numbers themselves.

4. Express that three times the square of a certain unknown quantity is equal to half the product last mentioned.

5. The sum of two unknown numbers is equal to the cube of the square root of their product; and their difference is equal to 10.

6. The number 6 is subtracted from 5, and the square root of the remainder is taken from the cube root of their sum; how is this to be expressed algebraically, and how in the condensed form of arithmetical notation?

7. Express the cube of the cube root of a known quantity, and the cube root of the cube of the same quantity.

8. Express that the sum of a geometrical series whose first term, ratio, and number of terms at present unknown is to be computed:—by both methods of notation.

9. The three leading terms of a proportion are always supposed given: express that the fourth (yet unknown) is equal to the product of the second and third divided by the first. And express that the square roots of four given quantities are *reciprocally* proportional: and also that the reciprocals of the squares of the fifth roots of four other quantities are *directly* proportional.

10. Of three given quantities express the sum of each two diminished by the remaining one: and the product of the three resulting quantities.

11. Denote that the difference of the cubes of two unknown quantities divided by the difference of the quantities themselves, is equal to the sum of the squares added to the product of those two quantities.

12. Express the theorem that if the sum of any number of quantities (first supposed given and then supposed unknown) be multiplied by another given quantity, the product is equal to the sum of the products made by multiplying each of the first-named (whether given or unknown) quantities by the one last-named.

13. The following formulæ are to be explained in words :

$$(a+b-c)^n;\ (2-4+6)^{\frac{1}{2}}=2;\ \sqrt{x^2+2xy+y^2}=\pm(x+y)$$
$$\sqrt[3]{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}};\ \left\{(x^2+y^2)^{\frac{1}{2}}-(x^2-y^2)^{\frac{1}{2}}\right\}^{\frac{1}{3}}$$

In what do these last expressions differ?

14. How is the following theorem to be expressed :—the product of two known powers of an unknown quantity is equal to that power of the same quantity which is equal to the sum of the two known powers? Express the same when all the quantities are known, when all are unknown, and when two unknown powers of a known quantity are substituted in the theorem.

15. Suppose that in the conditions of some particular example that was proposed, the known indices were found to be 3 and 2, and from some additional conditions it was otherwise found that the base a was ·001, what would be the value of the expression? And what if the given powers were ·3 and ·2?

16. Indicate the extraction of the 10th root of the 3d root of $\frac{b^2}{1000}$, and of $\frac{10^5\times a}{10^2\,b}$, both by radicals and indices.

17. Express the product of the sum of the square roots of three given quantities into the negative square of an unknown quantity being equal to the product of all the quantities mentioned.

18. Interpret the expression $a : x :: b^{\frac{5}{3}} : c^{\frac{5}{3}}$; and admitting the rules given for the "rule of three" to be true, how is the solution to be expressed in letters? and, also, if $a=10$, $b=·008$, and $c=27000$, assign the value of x.

19. A certain number is unknown : but it is known that if three times its defect from 10 be divided by 2, the quotient will be equal to one-third of double the square of the number multiplied by its square root. Express this statement in appropriate symbols.

20. What kind of symbol is $\sqrt[3]{\ }$? What kind is $(\quad)^{\frac{1}{3}}$? Point out and distinguish the symbols of operation from those of quantity in the expression $\{\sqrt[3]{(a+b)^2-(a-b)^2}\}^{\frac{5}{6}}\times 4\,cdx\div(a+b)^{\frac{1}{3}}(a-b)^{\frac{1}{3}}=x^2$.

21. If the *figures* which compose a number were $s, r, q \ldots d, c, b, a$, reckoning a that to the right hand, and suppose the decimal point fall between e and d: How is that number to be represented algebraically?

V. Range — $x^3-10x+15x^2-8x^4-3x^0+4x^{-1}-6x^{-2}$, according to powers of x, both ascending and descending; and $4x^3y-3xy^3+9x^2y^2+4x^4-4y^4$, according to ascending and descending powers first of x and then of y.

Arrange $x^3yz+xy^2z^2+x^4z+x^4z-3x^2y^2z+4x^3yz+9x^0y^0z^5-10x^5y^0+3z^4x+z^4y$, according to powers of z, and the several multipliers in terms of x and y collected in vincula, and each of these arranged respectively in ascending terms of y, and in ascending terms of x.

The intelligent teacher can select a few, (such as may suit his purpose, and the defects he observes in his pupils) from the questions in the application of simple and quadratic equations : by which means the nature of algebraical notation will be completely illustrated.

EXAMPLES FOR PRACTICE.

In finding the numeral values of various expressions, or combinations, of quantities.

Supposing $a=6$, $b=5$, $c=4$, $d=1$, and $e=0$. Then will

1. $a^2+3ab-c^2=36+90-16=110$.

2. And $2a^3 - 3a^2b + c^3 = 432 - 540 + 64 = -44$.

3. And $a^2 \times (a + b) - 2abc = 36 \times 11 - 240 = 156$.

4. And $\dfrac{a^3}{a + 3c} + c^2 = \dfrac{216}{18} + 16 = 12 + 16 = 28$.

5. And $\sqrt{2ac + c^2}$ or $(2ac + c^2)^{\frac{1}{2}} = \sqrt{64} = 8$.

6. And $\sqrt{c} + \dfrac{2bc}{\sqrt{(2ac + c^2)}} = 2 + \dfrac{40}{8} = 7$.

7. And $\dfrac{a^2 - \sqrt{b^2 - ac}}{2a - \sqrt{b^2 + ac}} = \dfrac{36 - 1}{12 - 7} = \dfrac{35}{5} = 7$.

8. And $\sqrt{b^2 - ac} + \sqrt{2ac + c^2} = 1 + 8 = 9$.

9. And $\sqrt{b^2 - ac + \sqrt{2ac + c^2}} = \sqrt{25 - 24 + 8} = 3$.

10. And $a^2b + c - d = 183$; and $9ab - 10b^2 + c = 24$.

11. Likewise $\dfrac{a^2b}{c} \times d = 45$; and $\dfrac{a + b}{c} \times \dfrac{b}{d} = 13\frac{3}{4}$; and $\dfrac{a + b}{c} - \dfrac{a - b}{d} = 1\frac{3}{4}$; $\dfrac{a^2b}{c} + e = 45$; $\dfrac{a^2b}{c} \times e = 0$.

12. Show which of the two quantities $\dfrac{2x + 1}{3}$ and $\dfrac{7x}{2}$ is the greater, when x is used to signify the numbers $-\cdot001$, $-\cdot1$, $-\cdot10$, -100; $+\cdot001$, $+\cdot1$, $+10$, and $+100$, successively.

13. Also keeping the old values of a, b, c, d, e, show that $(b - c) \times (d - e) = 1$; $(a + b) - (c - d) = 8$; and $(a + b) - c - d = 6$.

14. Also that $a^2c \times d^3 = 144$; $acd - d = 23$; $a^2e + b^2e + d = 1$; and $\dfrac{b - e}{d - e} \times \dfrac{a + b}{c - d} = 18\frac{1}{3}$.

15. And that $\sqrt{a^2 + b^2} - \sqrt{a^2 - b^2} = 4\cdot4936249$; $3ac^2 + \sqrt[3]{a^3 - b^3} = 292\cdot497942$; and $4a^2 - 3a\sqrt{a^2 - \frac{2}{3}ab} = 72$.

16. Suppose $a = 6 \times 10^4$; $b = 5 \times 10^3$; $c = 4 \times 10^2$; $d = 8 \times 10^1$; and $e = 1$: what will be the values of the expressions in examples 11, 12, 13, and 14?

17. If we have in any algebraical problem reason to know that $a = 0$, $b = 6$, $c = \cdot1$, $d = \cdot001$, and $e = \frac{1}{2}c$, show what the values of the expressions 1, 2, 3, 4, 5, and 6, would then be.

18. If $a = \cdot6$; $b = \cdot5$; $c = \cdot4$; $d = \cdot1$; $e = 0$; show what the results of the first eleven expressions would be. Likewise when a, b, c, d, e, were half these last-named values, and also four-fifths of them; three times as much, and ten thousand times as much; and write down these results adapted to each of these cases.

19. Find the values of the expressions in 11, 12, 13, when $a = \frac{3}{4}$; $b = \frac{1}{3}$; $c = \frac{1}{2}$; $d = \frac{3}{8}$; and $e = \frac{1}{6}$.

20. Let $a = 10$, what are the values of $3 \times a^5$, $2 \times a^2$, $6 \times a^1$, $3 \times a^{-2}$, and $9 \times a^{-3}$? Also assign their sum.

ADDITION.

Addition, in Algebra, is the connecting the quantities together by their proper signs, and incorporating or uniting into one term or sum, such as are similar, and can be united. As $3a + 2b - 2a = a + 2b$, the sum.

The rule of addition, in algebra, may be divided into three cases:
(1.) When the quantities are like, and have like signs:
(2.) When the quantities are like, and have unlike signs:
(3.) When the quantities are unlike *.

CASE I.

When the quantities are like, and have like signs.

Add the co-efficients together, and set down the sum; after which set the common letter or letters of the like quantities, and prefix the common sign $+$ or $-$.

Thus $3a$ added to $5\,a$, makes $8a$.

And $-\ 2ab$ added to $-\ 7ab$, makes $-\ 9ab$.

And $5a + 7b$ added to $7a + 3b$, makes $12a + 10b$.

---

* The reasons on which these operations are founded will readily appear, by a little reflection on the nature of the quantities to be added or collected together; for, with regard to the first example, where the quantities are $3a$ and $5a$, whatever a represents in the one term, it will represent the same thing in the other; so that 3 times any thing and 5 times the same thing, collected together, will make 8 times that thing. Thus, if a denote a shilling; then $3a$ is 3 shillings, and $5a$ is 5 shillings, and their sum 8 shillings. In like manner, $-\ 2ab$ and $-\ 7ab$, or $-$ 2 times any thing, and $-$ 7 times the same thing, make $-$ 9 times that thing.

As to the second case, in which the quantities are like, but the signs unlike; the reason of its operation will easily appear, by reflecting, that addition means only the uniting of quantities together by means of the arithmetical operations denoted by their signs $+$ and $-$, or of addition and subtraction; which, being of contrary or opposite natures, the one co-efficient must be subtracted from the other, to obtain the incorporated or united mass.

As to the third case, where the quantities are unlike, it is plain that such quantities cannot be united into one, or otherwise added, than by means of their signs: thus, for example, if a be supposed to represent a crown, and b a shilling; then the sum of a and b can be neither $2a$ nor $2b$, that is, neither 2 crowns nor 2 shillings, but only 1 crown plus 1 shilling, that is, $a + b$.

In this rule, the word *addition* is not very properly used; being much too limited to express the operation here performed. The business of this operation is to incorporate into one mass or algebraic expression, different algebraic quantities, as far as an actual incorporation or union is possible; and to retain the algebraic marks for doing it, in cases where the former is not possible. When we have several quantities, some affirmative and some negative; and the relation of these quantities can in the whole or in part be discovered; such incorporation of two or more quantities into one, is plainly effected by the foregoing rules.

It may seem a paradox, that what is called addition in algebra should sometimes mean addition, sometimes subtraction, and sometimes both. But the paradox wholly arises from the scantiness of the name given to the algebraic process; from employing an old term in a new and more enlarged sense. Instead of addition, call it *incorporation*, or *union*, or *striking a balance*, or give it any name to which a more extensive idea may be annexed, than that which is usually implied by the word addition, and the paradox will vanish.

OTHER EXAMPLES FOR PRACTICE.

1.	2.	3.	4.
$3a$	$-3bx$	bxy	$3z$
$9a$	$-5bx$	$2bxy$	$2z$
$5a$	$-4bx$	$5bxy$	$2z$
$12a$	$-2bx$	bxy	$3z$
a	$-7bx$	$3bxy$	z
$2a$	$-bx$	$6bxy$	$4z$
$32a$	$-22bx$		

5.	6.	7.	8.
$2ax-4y$	$3x^2+5xy$	$5xy$	$4a-4b$
$4ax-y$	x^2+xy	$14xy$	$5a-5b$
$ax-3y$	$2x^2+4xy$	$22xy$	$6a-b$
$5ax-5y$	$5x^2+2xy$	$17xy$	$3a-2b$
$7ax-2y$	$4x^2+3xy$	$1\frac{1}{2}xy$	$2a-7b$
		$\frac{1}{2}xy$	$8a-b$
$19ax-15y$			

9.	10.	11.
$-12yy$	$30-13x^{\frac{1}{2}}-3xy$	$5xy-3x+4ab$
$-\cdot 7y^2$	$23-10x^{\frac{1}{2}}-4xy$	$8xy-4x+3ab$
$-2y^2$	$14-14\sqrt{x}-7xy$	$3xy-5x+5ab$
$-\cdot 4yy$	$10-16x^{\frac{1}{2}}-5xy$	$xy-2x+ab$
$-y^2$	$16-20\sqrt{x}-xy$	$4xy-x+7ab$
$-3yy$		

CASE II.

When the quantities are like, but have unlike signs.

Add all the affirmative coefficients into one sum, and all the negative ones into another, when there are several of a kind: then subtract the less sum, or the less coefficient, from the greater, and to the remainder prefix the sign of the greater, and subjoin the common quantity or letters.

Thus, $+5a$ and $-3a$, united, make $+2a$.

And $-5a$ and $+3a$, united, make $-2a$.

OTHER EXAMPLES FOR PRACTICE.

1.	2.	3.	4.
$-5a$	$+8x^3+3y$	$+3ax^2$	$-3a^2$
$+4a$	$-5x^3+4y$	$+4ax^2$	$-5a^2$
$+6a$	$-16x^3+5y$	$-8ax^2$	$-10a^2$
$-3a$	$+3x^3-7y$	$-6ax^2$	$+10a^2$
$+a$	$+2x^3-2y$	$+5ax^2$	$+14a^2$
$+3a$	$-8x^3+3y$		

5.

$$\begin{array}{r} +4ab+4 \\ -4ab+12 \\ +7ab-14 \\ +ab+3 \\ -5ab-10 \\ \hline \\ \hline \end{array}$$

6.

$$\begin{array}{l} -3ax^{\frac{1}{2}} \\ +a\sqrt{x} \\ +5ax^{\frac{1}{2}} \\ +6ax^{\frac{1}{2}} \\ \hline \\ \hline \end{array}$$

7.

$$\begin{array}{l} +10\sqrt{a}\times\sqrt{x} \\ -3\sqrt{ax} \\ +4\sqrt{ax} \\ +12a^{\frac{1}{2}}x^{\frac{1}{2}} \\ \hline \\ \hline \end{array}$$

8.

$$\begin{array}{l} +3y+4ax^{\frac{1}{2}} \\ -y-5ax^{\frac{1}{2}} \\ +4y+2ax^{\frac{1}{2}} \\ -2y+6a\sqrt{x} \\ \hline \\ \hline \end{array}$$

9.

$$\begin{array}{r} 5x\sqrt[3]{a+y}-2x\sqrt[4]{y}+\sqrt{2} \\ 3x(a+y)^{\frac{1}{3}}+6xy^{\frac{1}{4}}+2^{\frac{1}{2}} \\ -8x(a+y)^{\frac{1}{3}}-4xy^{\frac{1}{4}}+3\sqrt{2} \\ 7x\sqrt[3]{a+y}+3x\sqrt[4]{y}+2\sqrt{2} \\ 2x(a+y)^{\frac{1}{3}}+5x\sqrt[4]{y}+\frac{5}{2}\times 2^{\frac{1}{2}} \\ -9x\sqrt[3]{a+y}-8xy^{\frac{1}{4}}-8\sqrt{2} \\ \hline \\ \hline \end{array}$$

10.

$$\begin{array}{r} -3(ax+by+cz)^{\frac{1}{4}}-\sqrt{x^2+y^2}+a-b \\ 2\sqrt[4]{ax+by+cz}+(x^2+y^2)^{\frac{1}{2}}-3(a-b) \\ 7\,\overline{ax+by+cz}^{\frac{1}{4}}-\sqrt{x^2+y^2}+2(a-b) \\ 3\sqrt[4]{(ax+by+cz)}+(x^2+y^2)^{\frac{1}{2}}+a-b \\ -5\sqrt[4]{(ax+by+cz)}+\overline{x^2+y^2}^{\frac{1}{2}}-2(a-b) \\ (ax+by+cz)^{\frac{1}{4}}-\sqrt{x^2+y^2}-3(a-b) \\ \hline \\ \hline \end{array}$$

CASE III.

When the quantities are unlike.

Having collected together all the like quantities, as in the two foregoing cases, set down those that are unlike, one after another, with their proper signs.

EXAMPLES.

1.

$$\begin{array}{r} 3xy \\ 2ax \\ -5yx \\ 6ax \\ \hline -2xy+8ax \\ \hline \end{array}$$

2.

$$\begin{array}{r} 6yx-12x^2 \\ -4x^2+3xy \\ +4x^2-2yx \\ -3xy+4x^2 \\ \hline 4xy-8x^2 \\ \hline \end{array}$$

3.

$$\begin{array}{l} 4ax-130+3x^{\frac{1}{2}} \\ 5x^2+3ax+9x^2 \\ 7xy-4x^{\frac{1}{2}}+90 \\ \sqrt{x}+40-6x^2 \\ \hline 7ax+8x^2+7xy \\ \hline \end{array}$$

4.

$$\begin{array}{r} 9x^2y^2 \\ -7x^2y \\ +3axy \\ -4x^2y \\ \hline \\ \hline \end{array}$$

5.

$$\begin{array}{l} 14ax-2x^2 \\ 5ax+3xy \\ 8y^2-4ax \\ 3x^2+26 \\ \hline \\ \hline \end{array}$$

6.

$$\begin{array}{l} 9+10\sqrt{ax}-5y \\ 2x+7\sqrt{xy}+5y \\ 5y+3x\sqrt{a}-4y \\ 10-4a^{\frac{1}{2}}x+4y \\ \hline \\ \hline \end{array}$$

7.	8.	9.
$4yx^2$	$4\sqrt{x} - 3y$	$3a^2 + 9 + x^{\frac{1}{2}} - 4$
$-6xy^2$	$2\sqrt{xy} + 14x$	$2a - 8 + 2a^2 - 3x$
$+3y^2x$	$3x + 2y$	$3x^2 - 2a^2 + 18 - 7$
$-7x^2y$	$-9 + 3y\sqrt{x}$	$-12 + a - 3x^2 - 2y$

Add $a + b$ and $3a - 5b$ together; and also $5a - 8x$ and $3a - 4x$ together.
Add $6x - 5b + a + 8$ to $-5a - 4x + 4b - 3$.
Add $a + 2b - 3c - 10$ to $3b - 4a + 5c + 10$ and $5b - c$.
Add $a + b$ and $a - b$ together; and $\frac{1}{2}x + \frac{1}{2}y$ to $-\frac{1}{2}x - \frac{1}{2}y$.
Add $3a + b - 10$ to $c - d - a$ and $-4c + 2a - 3b - 7$.
Add $3a^2 + b^2 - c$ to $2ab - 3a^2 + bc - b$.
Add $a^3 + b^2c - b^2$ to $ab^2 - abc + b^2$.
Add $9a - 8b + 10x - 6d - 7c + 50$ to $2x - 3a - 5c + 4b + 6d - 10$.
Add $a + b + c$, $-a + b + c$, $a - b + c$, and $a + b - c$ together; and likewise $a + b + c - d$, $d + a + b - c$, $c + d + a - b$, and $b + c + d - a$ together.

Note. It often happens that some one letter is considered the principal object in the calculation, and that the others are viewed as coefficients of this one. In this case their sums will assume a compound form: as in the following examples.

$$\begin{array}{l} 2ax + 3by^2 \\ cdx + 8ady^2 \\ -6bx - cy^2 \\ \hline (2a+cd-6b)\,x + (3b+8ad-c)\,y^2 \end{array} \qquad \begin{array}{l} ax + 3dy^2 \\ 2by - 3dx \\ -by^2 + 4my \\ \hline (a-3d)\,x + (3d-b)\,y^2 + (2b+4m)\,y \end{array}$$

In these cases, $2a$, cd, $-6b$, &c. instead of being considered to form part of the components of the respective terms in which they appear, are collected under vincula, and the collection under each vinculum treated as a single quantity. Two other examples are added for the student's exercise.

$$\begin{array}{l} -a\sqrt{x^2-y^2} + b\sqrt{x^2+y^2} \\ -5c\sqrt{x^2+y^2} - 3d\sqrt{x^2+y^2} \\ -f(x^2+y^2)^{\frac{1}{2}} - 2c(x^2-y^2)^{\frac{1}{2}} \\ 2\sqrt{x^2+y^2} + 4a\sqrt{x^2-y^2} \\ \sqrt{x^2-y^2} - (x^2+y^2)^{\frac{1}{2}} \\ \hline \end{array} \qquad \begin{array}{l} (a+b)\sqrt{x} - (2+m)\sqrt{y} \\ 4y^{\frac{1}{2}} + (a+c)\,x^{\frac{1}{2}} \\ 3x\sqrt{y} - (2d-e)\,x^{\frac{1}{2}} \\ (m+n)\,y^{\frac{1}{2}} + (b+2c)\sqrt{x} \\ -2x\sqrt{x} + 12a\sqrt{y} \\ \hline \end{array}$$

Sometimes these literal coefficients, instead of being collected horizontally, are placed vertically under each other. Specimens may be seen in multiplication and division.

SUBTRACTION.

Set down in one line the first quantities from which the subtraction is to be made; and underneath them place all the other quantities composing the subtrahend; ranging the like quantities under each other, as in addition.

Then change all the signs (+ and —) of the lower line, or conceive them to be changed; after which, collect all the terms together as in the cases of addition *.

Note. When the sign — is prefixed to a compound expression, it indicates that if the parenthesis is removed, the signs of all the terms must be changed.

Thus, $-(ax - bx + 2cx^2 - 3dx^3) = -ax + bx - 2cx^2 + 3dx^3$. For otherwise the sign — would not affect all the terms within the parenthesis equally.

EXAMPLES.

1.	2.	3.
From $7a^2 - 3b$	$9x^2 - 4y + 8$	$8xy - 3 + 6x - y$
Take $2a^2 - 8b$	$6x^2 + 5y - 4$	$4xy - 7 - 6x - 4y$
Rem. $5a^2 + 5b$	$3x^2 - 9y + 12$	$4xy + 4 + 12x + 3y$

4.	5.	6.
From $5xy - 6$	$4y^2 - 3y - 4$	$-20 - 6x - 5xy$
Take $-2xy + 6$	$2y^2 + 2y + 4$	$3xy - 9x - 8 - 2ay$
Rem.		

7.	8.	9.
From $8x^2y + 6$	$5\sqrt{xy} + 2x\sqrt{xy}$	$7x^2 + 2\sqrt{x} - 18 + 3b$
Take $-2x^2y + 2$	$7\sqrt{xy} + 3 - 2xy$	$9x^2 - (12 - 5b) + x^{\frac{1}{2}}$
Rem.		

10.	11.	12.
$5xy - 30$	$7x^3 - 2(a + b)$	$4xy^3 + 20a\sqrt{(xy + 10)}$
$7xy - 50$	$2x^2 - 4(a + b)$	$3x^2y^2 + 12a\sqrt{(xy - 10)}$

Note. If *literal coefficients* occur, they must be collected (the subtractive ones with their signs changed) as directed in note upon case iii. of addition.

From $a + b$, take $a - b$.
From $4a + 4b$, take $b + a$.
From $4a - 4b$, take $3a + 5b$.
From $8a - 12x$, take $4a + 3x$.
From $2x - (4a + 2b - 5)$, take $8 - (5b - a - 6x)$.

* This rule is founded on the consideration, that addition and subtraction are opposite to each other in their nature and operation, as are the signs + and —, by which they are expressed and represented : hence, since to unite a negative quantity with a positive one of the same kind, has the effect of diminishing it, or subducting an equal positive one from it, therefore to subtract a positive (which is the opposite of uniting or adding) is to add the equal negative quantity. In like manner, to subtract a negative quantity, is the same in effect as to add or unite an equal positive one. So that, changing the sign of a quantity from + to —, or from — to +, changes its nature from a subductive quantity to an additive one; and any quantity is in effect subtracted by barely changing its sign.

From $3a + b + c - (d + 10)$, take $c + 2a - d$.
From $3a + b + c - (d - 10)$, take $b - (10 - 3a)$.
From $2ab + b^2 - 4c + bc - b$, take $3a^2 - c + b^2$.
From $a^3 + 3b^2c + ab^2 - abc$, take $b^2 + ab^2 - abc$.
From $12x + 6a - (4b - 40)$, take $4b - (3a - 4x - 6d) - 10$.
From $2x - (3a - 4b) + 6c - 50$, take $9a + x + (6b - 6c - 40)$.
From $6a - (4b + 12c - 12x)$, take $2x - (8a - 4b) - 5c$.
From $\frac{1}{2}(a + b + c)$, take a, b, c, separately and successively.

Subtract $-3\sqrt{a + x} + 4(x^2 - y^2)^{\frac{1}{2}} - 1$, from $\sqrt{x^2 + y^2} - 2(a+x)^{\frac{1}{2}} + 3$;
and $-17axy + 11abc - x\sqrt[4]{x + y}$, from $2x(x + y)^{\frac{1}{4}} - 3axy$.

Subtract $sx^2 - pxy + qx - c$, from $ax^2 + mxy + nx + b$; and $mxy - pqxz - n(z^2 + a)$, from $pxy + qrx - r(z^2 + a)$.

From $a(n - y)^{\frac{1}{2}} + bxy + c(a + x)^2$, take $(n - y)^{\frac{1}{2}} - bxy + (a + c)(a + x)^2$; and take $(a - b)(x + y) + (c - d)(x - y) - n$, from $(a + b)(x + y) - (c + d)(x + y) + m$.

From $(a-b)xy - hx^2$ subtract $(2p - 3q)(x+y)^{\frac{1}{2}}$; and from $-(p + q)\sqrt{x + y}$, subtract $-axy - 3x^2 - hx^2$: and add the two remainders together.

MULTIPLICATION.

This consists of several cases, according as the factors are simple or compound quantities.

CASE I.

When both the factors are simple quantities.

1. Multiply the co-efficients of the two terms together; then, to the product annex all the letters in those terms, which will give the whole product required.

2. When the same letter is repeated in the product, the result may be simplified by writing the sum of the indices instead of the separate factors, agreeably to def. p. 105. Thus, for $a^3 \times a^2$, write a^5.

3. The same rule holds good if there be fractional indices, or their equivalent radicals, in the product: as for $a^{\frac{1}{3}}a^{\frac{1}{2}}$, write $a^{\frac{1}{3} + \frac{1}{2}}$, or $a^{\frac{5}{6}}$; and for $a^{\frac{1}{2}} \times a^{\frac{1}{2}}$, write a^1 or a.

This is true whatever a may represent, as for instance, if $a = -1$, then $(-1)^{\frac{1}{2}} \times (-1)^{\frac{1}{2}}$ or $\sqrt{-1} \times \sqrt{-1} = -1$.

Note *. Like signs, in the factors, produce $+$, and unlike signs $-$, in the products.

---

* That this rule for the signs is true, may be thus shown.

1. When $+a$ is to be multiplied by $+c$; the meaning is, that $+a$ is to be taken as many times as there are units in c; and since the sum of any number of positive terms is positive, it follows that $+a \times +c$ makes $+ac$.

2. When two quantities are to be multiplied together, the result will be exactly the same, in whatever order they are placed; for a times c is the same as c times a, and therefore, when $-a$

EXAMPLES.

1.	2.	3.	4.
$10a$	$-3a$	$7a$	$-6x$
$2b$	$+2b$	$-4c$	$-4a$
$20ab$	$-6ab$	$-28ac$	$+24ax$

5.	6.	7.	8.
$4ac$	$9a^2x$	$-2x^2y$	$-4xy$
$-3ab$	$4x^{\frac{3}{2}}$	$3x^{\frac{1}{2}}y^2$	$-xy^{\frac{1}{2}}$

9.	10.	11.	12.
$-3ax\sqrt{-1}$	$-ax^{\frac{1}{2}}$	$(+3xy)^{\frac{1}{2}}$	$-5xyz\sqrt{-1}$
$4x\sqrt{-1}$	$-6c$	$(-4)^{\frac{1}{2}}$	$-4ax\sqrt{-1}$

Though only two factors have been proposed, there may be any number. The process is however still the same, repeating the operation with every successive term upon the result of all the preceding.

When, however, the factors are all equal, the literal parts may be more readily assigned; viz. multiply the index of each letter in the common factor by the index of the number of factors. The products of these are the indices of the literal parts. Thus,

$$a^m\, b^n\, c^p \times a^m\, b^n\, c^p \times a^m\, b^n\, c^p = a^{3m}\, b^{3n}\, c^{3p}, \text{ or } (a^m\, b^n\, c^p)^3.$$

When there are numeral coefficients, the powers of these must be found as at p 65. Thus, suppose we sought the products of

$$3a^2b \times 3a^2b, \text{ and of } -2a^{\frac{1}{2}}\, c^{\frac{3}{2}} \times -2a^{\frac{1}{2}}\, c^{\frac{3}{2}} \times -2a^{\frac{1}{2}}\, c^{\frac{3}{2}}, \text{ we should have}$$

$$3 \times 3\, a^4\, b^2, \text{ and } -8a^{\frac{3}{2}}\, c^{\frac{9}{2}}.$$

The signs being regulated by the number of factors when those factors are —.

This is a case of INVOLUTION.

---

is to be multiplied by $+c$, or $+c$ by $-a$: this is the same thing as taking $-a$ as many times as there are units in $+c$; and as the sum of any number of negative terms is negative, it follows that $-a \times +c$, or $+a \times -c$ make or produce $-ac$.

3. When $-a$ is to be multiplied by $-c$: here $-a$ is to be subtracted as often as there are units in c: but subtracting negatives is the same thing as adding affirmatives, by the demonstration of the rule for subtraction; consequently, the product is c times a, or $+ac$.

Otherwise. Since $a - a = 0$, therefore $(a - a) \times -c$ is also $= 0$, because 0 multiplied by any quantity, is still but 0; and since the first term of the product, or $a \times -c$ is $= -ac$, by the second case; therefore the last term of the product, or $-a \times -c$, must be ac, to make the sum $= 0$, or $-ac + ac = 0$; that is, $-a \times -c = +ac$.

Other demonstrations upon the principles of proportion, or by means of geometrical diagrams, have also been given; but the above is more natural, simple, and satisfactory.

EXAMPLES FOR PRACTICE.

1. Required the cube or 3d power of $3a^2\,(-1)$.
2. Required the 4th power of $2a^2bb^{-1}$.
3. Required the 3d power of $-4a^2b^{-2}$.
4. To find the biquadrate of $-\dfrac{a^2x}{2b^2}\,(\pm 1)$.
5. To find the 6th power of $\pm 2a^{\frac{1}{2}}\sqrt{\pm 1}$.
6. To find the 7th power of $(\pm 1)^3 \times (\pm a^2)\,(-a^{-3})$.

CASE II.

When one of the factors is a compound quantity.

Multiply every term of the multiplicand, or compound quantity, separately, by the multiplier, as in the former case; placing the products one after another, with the proper signs; and the result will be the whole product required.

EXAMPLES.

1.
$$\begin{array}{l} 5a - 3c \\ 2a \\ \hline 10a^2 - 6ac \\ \hline \end{array}$$

2.
$$\begin{array}{l} 3ac - 4b \\ -3a \\ \hline -9a^2c + 12ab \\ \hline \end{array}$$

3.
$$\begin{array}{l} 2a^2 - 3c + 5 \\ bc \\ \hline 2a^2bc - 3bc^2 + 5bc \\ \hline \end{array}$$

4.
$$\begin{array}{l} 12x - 2ac \\ 4a \\ \hline \\ \hline \end{array}$$

5.
$$\begin{array}{l} 25c - 7b \\ -2a \\ \hline \\ \hline \end{array}$$

6.
$$\begin{array}{l} 4x - b + 3ab \\ 2ab \\ \hline \\ \hline \end{array}$$

7.
$$\begin{array}{l} 3c^2 + x\sqrt{-1} \\ 4xy\sqrt{-1} \\ \hline \\ \hline \end{array}$$

8.
$$\begin{array}{l} 10x^2 - 3y^2 \\ -4x^2 \\ \hline \\ \hline \end{array}$$

9.
$$\begin{array}{l} 3a^2 - 2x^2 - 6b \\ 2ax^2 \\ \hline \\ \hline \end{array}$$

CASE III.

When both the factors are compound quantities.

Multiply every term of the multiplier by every term of the multiplicand separately; setting down the products one after or under another, with their proper signs; and add the several partial products together for the whole product required.

1.
$$\begin{array}{l} a + b \\ a + b \\ \hline a^2 + ab \\ \quad + ab + b^2 \\ \hline a^2 + 2ab + b^2 \\ \hline \end{array}$$

2.
$$\begin{array}{l} 3x + 2y \\ 4x - 5y \\ \hline 12x^2 + 8xy \\ \qquad - 15xy - 10y^2 \\ \hline 12x^2 - 7xy - 10y^2 \\ \hline \end{array}$$

3.
$$\begin{array}{l} 2x^2 + xy - 2y^2 \\ 3x - 3y \\ \hline 6x^3 + 3x^2y - 6xy^2 \\ \qquad - 6x^2y - 3xy^2 + 6y^3 \\ \hline 6x^3 - 3x^2y - 9xy^2 + 6y^3 \\ \hline \end{array}$$

$$\begin{array}{lll}
\text{4.} & \text{5.} & \text{6.} \\
a + b & x^2 + y & a^2 + ab + b^2 \\
a - b & x^2 + y & a - b \\
\hline
a^2 + ab & x^4 + yx^2 & a^3 + a^2b + ab^2 \\
\quad - ab - b^2 & \quad + yx^2 + y^2 & \quad - a^2b - ab^2 - b^3 \\
\hline
a^2 \quad * \quad - b^2 & x^4 + 2yx^2 + y^2 & a^3 \quad * \quad * \quad - b^3 \\
\hline
\end{array}$$

Note I. When the factors are all equal, and composed of two terms, as $a + x$, or $3a - 5x$, the operation is more readily performed by the BINOMIAL THEOREM, the rule of which may be referred to at once. For any purposes likely to occur in the earlier stages of Algebra, the result may be obtained by actual multiplication, as above. When there are more than two terms, there is a general theorem for finding the coefficients, called the MULTINOMIAL THEOREM: but as occasion for its use occurs comparatively seldom, it will not be given in this work. Reference may therefore be made to Young's Algebra, p. 262.

When the factors are all equal, as we have here supposed, the operation is called INVOLUTION.

Ex. Raise $a - x$ to the third power, and $2a - 3xy$ to the fourth power.

Note II. The following is a specimen of the method of disposing of the literal coefficients in vertical columns. It has not only the advantage of keeping an operation of considerable extent within the limits of the breadth of the page, but it dispenses with the collecting those coefficients together, after the multiplications are developed, on account of its readily disposing them in their places as we proceed.

Multiply together the binomials $x - a$, $x - b$, $x - c$.

$$\begin{array}{r|l|l}
x - a & & \\
x - b & & \\
\hline
x^2 - a & x + ab. & \\
- b & & \\
x - c & & \\
\hline
x^3 - a & x^2 + ab & x - abc. \\
- b & \quad + bc & \\
- c & \quad + ca & \\
\end{array}$$

which in the horizontal disposition of the terms of the coefficients would stand thus: $x^3 - (a + b + c)\,x^2 + (ab + bc + ca)\,x - abc.$

Had the number of factors been greater, the advantage would have been still more apparent.

In the case just considered, where the given quantities are numerically given, this disposition of them is the most favourable to their actual addition into one numerical sum.

MULTIPLICATION BY DETACHED COEFFICIENTS.

It is not necessary to write down the powers of the quantity according to which the work is arranged, till we have performed the whole of the arithmetical determination of the coefficients: since the same powers of that letter, if generated by the multiplication of factors in which none of the terms are wanting, will always, in the above arrangement, fall in the same vertical column. Also, since the indices of the powers of that letter in going from term to term, either

decrease by units or increase by units, according as we begin at the highest or lowest powers, we may write them in juxtaposition with the coefficients found as above indicated, without the slightest degree of attention beyond the most ordinary kind.

In finite expressions, it is most usual, though not essential, to begin with the highest, and in series, it is, from their interminable character, necessary to begin with the smallest index, whether positive or negative.

The following example will render the practice obvious: and the student is earnestly recommended to adopt it, not only on account of economy of time and room; but to familiarize his hand, his eye, and his thoughts, to the processes by which the roots of equations are (in the most improved state of that difficult problem) found or approximated to.

Ex. 1. Multiply $2a^3 - 4a + 15$ by $3a^2 + 4$.

Here the coefficients of a^2 in the first, and of a in the second factor, are each equal to *zero*. Hence the work will stand thus:

$$\begin{array}{l}
2 + 0 - 4 + 15 \\
3 + 0 + 4 \\
\hline
6 + 0 - 12 + 45 \\
\quad + 0^* + 8 + \ \ 0 - 16 + 60 \\
\hline
6 + 0 - 4 + 45 - 16 + 60
\end{array}$$

And the highest power of a being $a^3 \times a^2 = a^5$, we may write a^5, a^4, a^3, a^2, a^1, in juxtaposition with the above coefficients, which gives for the product

$$6a^5 + 0a^4 - 4a^3 + 45a^2 - 16a + 60.$$

The same result would have been obtained, but in an inverted order, by writing $(15 - 4a + 2a^3)\ (4 + 3a^2)$.

When the first term of the multiplier is also unity, the work may be shortened still more, by allowing the line of coefficients which constitute the multiplicand, as the first partial product of the work. Thus, were $x^3 - 6x^2 + 10x - 9$ given to be multiplied by $x^2 - 3x + 2$; the general method would require it to be executed as follows.

$$\begin{array}{ll}
(a) & 1 - 6 + 10 - \ \ 9 \\
 & 1 - 3 + \ \ 2 \\
\hline
(b) & 1 - 6 + 10 - \ \ 9 \\
 & \quad - 3 + 18 - 30 + 27 \\
 & \qquad\qquad 2 - 12 + 20 - 18 \\
\hline
 & 1 - 9 + 30 - 51 + 47 - 18
\end{array}$$

And $x^3 . x^2 = x^5$ is the highest power or first term. Whence, attaching the literal parts, we get as the product

$$x^5 - 9x^4 + 30x^3 - 51x^2 + 47x - 18.$$

But as the line (b) is the same with the line (a) we may put it thus:

$$\begin{array}{l}
1 - 6 + 10 - \ \ 9 \quad (- 3 + 2 \\
\quad - 3 + 18 - 30 + 27 \\
\qquad\qquad 2 - 12 + 20 - 18 \\
\hline
1 - 9 + 30 - 51 + 47 - 18
\end{array}$$

* Here, instead of a horizontal column of ciphers, the first term of the multiplication by 4 is made to commence under the 4, as in common arithmetical multiplication.

The coefficients of the remaining terms of the multiplier being placed either in curve to the right, as "$(-3+2$," or in any other way that may be thought convenient.

It may be further remarked, for the purpose of connecting the identity of arithmetic, in its usual form, with the practice of algebra, in the student's mind, that if all the signs were $+$ (for this is always so considered in arithmetic) and we make $a = 10$, the above process would be precisely the same as is practised in ordinary multiplication, except that the order is inverted.

Note I. In the multiplication of compound quantities, it is usually best to set them down in order, according to the powers and the letters of the alphabet. And in the actual operation, begin at the left-hand side, and multiply from the left-hand towards the right, in the manner that we write, which is contrary to the usual way, though analogous to the Indian, of multiplying numbers. But in setting down the several products, as they arise, in the second and following lines, range them under the like terms in the lines above, when there are such like quantities; which is the easiest way for adding them together.

In many cases, the multiplication of compound quantities need only be indicated by setting them down one after another, each within or under a vinculum; and either with a sign of multiplication between them, as $(a+b)\times(a-b)\times 3ab$, or in juxtaposition, $(a+b)(a-b)\,3ab$.

Note II. The operations in multiplication will often be greatly facilitated, by fixing the following rules and formulæ well in the recollection.

The square of any polynomial is equal to the sum of the squares of its terms $+$ twice the product of every two of its terms taken in all their different combinations.

$$\begin{aligned}\text{Thus, } &(a+b+c+d)(a+b+c+d)\\ &= a^2+b^2+c^2+d^2\\ &\quad + 2ab + 2ac + 2ad\\ &\qquad + 2bc + 2bd\\ &\qquad\quad + 2cd\end{aligned}$$

$$\begin{aligned}\text{and } &(a+b+c+d+e+f)(a+b+c+d+e+f)\\ &= a^2+b^2+c^2+d^2+e^2+f^2\\ &\quad + 2ab+2ac+2ad+2ae+2af\\ &\qquad + 2bc+2bd+2be+2bf\\ &\qquad\quad + 2cd+2ce+2cf\\ &\qquad\qquad + 2de+2df\\ &\qquad\qquad\quad + 2ef\end{aligned}$$

In all such cases the arrangement of the products is very simple, and the continuation of the process very obvious.

Note III. From the principle that the rectangle of the sum and difference of two quantities is equal to the difference of their squares, some useful theorems obviously flow: viz.

1. $a^2 - b^2 = (a+b)(a-b)$.
2. $a^4 - b^4 = (a^2+b^2)(a^2-b^2) = (a^2+b^2)(a+b)(a-b)$.
3. $a^8 - b^8 = (a^4+b^4)(a^4-b^4) = (a^4+b^4)(a^2+b^2)(a+b)(a-b)$.
4. $a^{16} - b^{16} = (a^8+b^8)(a^4+b^4)(a^2+b^2)(a+b)(a-b)$.
5. $a^3 - b^3 = (a^2+ab+b^2)(a-b)$.
6. $a^3 + b^3 = (a^2-ab+b^2)(a+b)$.
7. $(x+a)(x+b) = x^2+(a+b)x+ab$.
8. $(x-a)(x-b) = x^2-(a+b)x+ab$.
9. $(x+a)(x+b)(x+c) = x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$.
10. $(x-a)(x-b)(x-c) = x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$.

EXAMPLES FOR PRACTICE IN ALL THE CASES.

1. Multiply $10ac$ by $2a$. Ans. $20a^2c$.
2. Multiply $3a^2 - 2b$ by $3b$. Ans $9a^2b - 6b^2$.
3. Multiply $3a + 2b$ by $3a - 2b$. Ans. $9a^2 - 4b^2$.
4. Multiply $x^2 - xy + y^2$ by $x + y$. Ans. $x^3 + y^3$.
5. Multiply $a^3 + a^2b + ab^2 + b^3$ by $a - b$. Ans. $a^4 - b^4$.
6. Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$.
7. Multiply $3x^2 - 2xy + 5$ by $x^2 + 2xy - 6$.
8. Multiply $3a^2 - 2ax + 5x^2$ by $3a^2 - 4ax - 7x^2$.
9. Multiply $3x^3 + 2x^2y^2 + 3y^3$ by $2x^3 - 3x^2y^2 + 3y^3$.
10. Multiply $(a^2 + ab + b^2)\,y$ by $(a - 2b)\,x$.
11. Multiply $a^m + a^{m-1}x + a^{m-2}x^2 + \ldots\ldots a^2x^{m-2} + ax^{m-1} + x^m$ by $a - x$.
12. Multiply $ax + bx^2 + cx^3$ by $1 + x + x^2 + x^3$, and consider a, b, c, as coefficients of the powers of x: as in p. 120.
13. Develope $(x + a)(x + b)(x + c)(x + d)$ and also $(x - a)(x - b)(x - c)(x - d)$; and attach the combinations of a, b, c, d, to the powers of x as coefficients.
14. The student may take in another factor $(x + e)$, and it would be worth his while to attempt to discover inductively *some law* or *rule* by which he could form the terms *seriatim*, without the trouble of writing down the previous steps.
15. Multiply $x^2 + (a - b)\,x - ab$ by $x^2 + (c - d)\,x - cd$.
16. $(1 + r + r^2 + r^3 \ldots\ldots r^n) \times (1 - r) = \ldots\ldots$?

DIVISION.

Division in algebra, like that in numbers, is the converse of multiplication; and it is performed like that of numbers also, by beginning at the left-hand side, and dividing all the parts of the dividend by the divisor, when they can be so divided; or else by setting them down like a fraction, the dividend over the divisor, and then abbreviating the fraction as much as can be done, by cancelling any quantities which are common both to numerator and denominator. This may naturally be distinguished into the following particular cases.

CASE I.

When the divisor and dividend are both simple quantities.

Set the terms both down as in division of numbers, either the divisor before the dividend, or below it, like the denominator of a fraction. Then abbreviate these terms as much as can be done, by cancelling or striking out all the letters that are common to them both, and also dividing the one coefficient by the other, or abbreviating them after the manner of a vulgar fraction in arithmetic, by dividing them by their common measure.

It will, of course, be necessary to subtract the index of the less power (whether it be in the numerator or denominator of the fraction thus formed) from the

index of the greater, leaving the difference where the greater index previously was. If, however, on the contrary, any ulterior purposes render it advantageous (and this often happens in algebraic investigations) to keep the latter in that term of the fraction, from which it would thus be expunged, we may subtract the greater index from the less, and put the difference with the sign —. Thus $\frac{a^2 b^3}{a^5 b}$ may be written either $\frac{b^2}{a^3}$, or $\frac{b^2 a^{-3}}{1}$ (that is b^2a^{-3}), or $\frac{1}{a^3 b^{-2}}$ or $\frac{a^{-3}}{b^{-2}}$. And so with any other quantities which appear in the result of the indicated division.

Note. Like signs in the two factors make + in the quotient; and unlike signs make —; the same as in multiplication *.

EXAMPLES.

1. To divide $6ab$ by $3a$.

This may be written either $6ab \div 3a$, or $\frac{6ab}{3a}$, and the result is in either case $= 2b$.

2. Also $c \div c = \frac{c}{c} = 1$; and $abx \div bxy = \frac{abx}{bxy} = \frac{a}{y}$.
3. Divide $16x^2$ by $8x$. Ans. $2x$.
4. Divide $12a^2x^2$ by $-3a^2x$. Ans. $-4x$.
5. Divide $-15ay^2$ by $3ay$. Ans. $-5y$.
6. Divide $-18ax^2y$ by $-8axz$. Ans. $\frac{9xy}{4z}$.

CASE II.

When the dividend is a compound quantity, and the divisor a simple one.

Divide every term of the dividend by the divisor, as in the former case.

EXAMPLES.

1. $(ab + b^2) \div 2b$, or $\frac{ab + b^2}{2b} = \frac{a + b}{2} = \frac{1}{2}a + \frac{1}{2}b$.
2. $(10ab + 15ax) \div 5a$, or $\frac{10ab + 15ax}{5a} = 2b + 3x$.
3. $(30az - 48z) \div z$, or $\frac{30az - 48z}{z} = 30a - 48$.
4. Divide $6ab - 8ax + a$ by $2a$.
5. Divide $3x^2 - 15 + 6x + 6a$ by $3x$.
6. Divide $6abc + 12abx - 9a^2b$ by $3ab$.
7. Divide $10a^2x - 15x^2 - 25x$ by $5x$.

---

* Because the divisor multiplied by the quotient must produce the dividend. Therefore,

1. When both the terms are +, the quotient must be +; because + in the divisor multiplied by + in the quotient, produces + in the dividend.

2. When the terms are both —, the quotient is also +; because — in the divisor multiplied by — in the quotient, produces + in the dividend.

3. When one term is + and the other —, the quotient must be —; because + in the divisor multiplied by — in the quotient produces — in the dividend, or — in the divisor multiplied by + in the quotient gives — in the dividend.

So that the rule is general; viz. that like signs give +, and unlike signs give —, in the quotient.

8. Divide $15a^2bc - 15acx^2 + 5ad^2$ by $-5ac$.

9. Divide $15a + 3ay - 18y^2$ by $21a$.

10. Divide $-20d^2b^2 + 60ab^3$ by $-6ab$.

11. Divide $-\frac{60}{3}x^2 - \frac{25}{4}x$ by $\frac{\frac{4}{3}x^3}{(\frac{1}{2} - \frac{1}{3})x}$.

12. Find the quotient of $2x^{3n-5m} + y^{2n-3}$ by $6x^{1-5m}y^{-1}$.

13. Divide $\frac{14}{5}a^{-8} + \frac{16}{3}a^{-5} - a$ by $5a^3b^{-2}$.

14. Divide $\cdot 001x^2 + 1000x^3 + \cdot 01x^{-\cdot 01}$ by $\cdot 6x^{-\frac{2}{3}}$; and $10^5x^3 + 3.10^7y^2 + 5 \times 10^2xy$ by $\cdot 06 \times 10^5x^3$.

15. Divide $6(a+b)^{-9} + 3 \times 10^2(a+b)^{-11}$ by $-3(a+b)^4$; and $-6 \times (a+b+c)^3 - 4(a+b+c)^4$ by $-(a+b+c)^5$.

16. Divide $\cdot 00015 \times 10^5 + \cdot 03 \times 10^3x$ by $\cdot 0005^{-4}x^6$; and $\cdot 00015^{-2} \times 10^{-6}a^3 - \cdot 6^{-4} \times 10^{-3} \times \cdot 001^{-2}$ by 10^{-1}.

17. Divide $a^n - x^n$ by $a - x$, and by $a + x$, and also $a^n + x^n$ by $a + x$.

CASE III.

When the divisor and dividend are both compound quantities.

1. Set them down as in division of numbers, the divisor before the dividend, with a small curved line between them, and range the terms according to the powers of some one of the letters in both, the higher powers before the lower.

2. Divide the first term of the dividend by that of the divisor, as in the first case, and set the result in the quotient.

3. Multiply the whole divisor by the term thus found, and subtract the result from the dividend.

4. To this remainder bring down as many terms of the dividend as are requisite for the next operation, dividing as before; and so on to the end, as in common arithmetic.

Note I. If the divisor be not exactly contained in the dividend, the quantity which remains after the operation is finished may be placed over the divisor, like a vulgar fraction, and set it down at the end of the quotient, as in arithmetic.

EXAMPLES.

$$\begin{array}{l}
a - b)\ a^2 - 2ab + b^2\ (a - b \\
 \underline{a^2 - ab} \\
 - ab + b^2 \\
 \underline{- ab + b^2}
\end{array}$$

$$\begin{array}{l}
a - c)\ a^3 - 4a^2c + 4ac^2 - c^3\ (a^2 - 3ac + c^2 \\
 \underline{a^3 - a^2c} \\
 - 3a^2c + 4ac^2 \\
 \underline{- 3a^2c + 3ac^2} \\
 ac^2 - c^3 \\
 \underline{ac^2 - c^3}
\end{array}$$

$$
\begin{array}{l}
a + x)\ a^4 - 3x^4\ \left(a^3 - a^2x + ax^2 - x^3 - \dfrac{2x^4}{a+x}\right. \\
\qquad a^4 + a^3x \\
\qquad \overline{} \\
\qquad\quad - a^3x - 3x^4 \\
\qquad\quad - a^3x - a^2x^2 \\
\qquad\quad \overline{} \\
\qquad\qquad\qquad a^2x^2 - 3x^4 \\
\qquad\qquad\qquad a^2x^2 + ax^3 \\
\qquad\qquad\qquad \overline{} \\
\qquad\qquad\qquad\quad - ax^3 - 3x^4 \\
\qquad\qquad\qquad\quad - ax^3 - x^4 \\
\qquad\qquad\qquad\quad \overline{} \\
\qquad\qquad\qquad\qquad\qquad - 2x^4
\end{array}
$$

EXAMPLES FOR EXERCISE.

1. Divide $a^2 + 4ax + 4x^2$ by $a + 2x$. Ans. $a + 2x$.
2. Divide $a^3 - 3a^2z + 3az^2 - z^3$ by $a - z$. Ans. $a^2 - 2az + z^2$.
3. Divide 1 by $1 + a$. Ans. $1 - a + a^2 - a^3 + \ldots\ldots$
4. Divide $12x^4 - 192$ by $3x - 6$. Ans. $4x^3 + 8x^2 + 16x + 32$.
5. Divide $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ by $a^2 - 2ab + b^2$.
Ans. $a^3 - 3a^2b + 3ab^2 - b^3$.
6. Divide $48z^3 - 96az^2 - 64a^2z + 150a^3$ by $2z - 3a$.
7. Divide $b^6 - 3b^4x^2 + 3b^2x^4 - x^6$ by $b^3 - 3b^2x + 3bx^2 - x^3$.
8. Divide $a^7 - x^7$ by $a - x$.
9. Divide $a^3 + 5a^2x + 5ax^2 + x^3$ by $a + x$.
10. Divide $a^4 + 4a^2b^2 - 32b^4$ by $a + 2b$.
11. Divide $24a^4 - b^4$ by $3a - 2b$.
12. The quotient found from dividing $\frac{3}{2}x^3 - \frac{5}{4}x^2 - 8x + 9$ by $3x^2 + \frac{7}{2}x - 9$ is .....
13. Divide $\frac{9a^2b^2}{4c^2} - \frac{25f^2m^2}{9^2} + \frac{70dfm}{9} - 49d^2$ by $\frac{3ab}{2c} + \frac{5fm}{9} - 7d$.
14. Divide $\frac{a^2}{bc} - \frac{2a}{d} + \frac{ac}{be} + \frac{bc}{d^2} - \frac{c^2}{de}$ by $\frac{a}{b} - \frac{c}{d}$.

Note II. By observing that the ***number of terms*** in any remainder that takes place after all the terms are brought down from the dividend is always less than the number of terms in the divisor, it is clear that, however far the operation is carried, the work can never terminate. The remainder always occurring, the terms of the quotient may always be increased; and that without any assignable limit. The series of terms thus formed is, from its capability of unlimited extension, called an Infinite Series. By attending to the manner in which the successive terms are related to the preceding one or preceding ones, the law of the progression (in Infinite Series resulting from Division) may be *always* and very readily discovered: so that when a few of the first terms have been actually obtained by the prescribed process, the remaining ones may be written out to any extent we may choose or require, by merely attending to this law of observed dependence. Examples of these will be found under the head of Infinite Series in a future part of this volume.

Note III. Operations in division may often be facilitated by the formulæ given in the last note to multiplication, as well as by the following.

Remember that $2n$ denotes an even exponent, $2n + 1$ an odd exponent; then

$a^{2n} - b^{2n}$ is divisible by $a - b$, by $a + b$, or by $a^2 - b^2$.
$a^{2n} + b^{2n}$ is divisible neither by $a - b$, nor by $a + b$.
$a^n - b^n$ is always divisible by $a - b$.
$a^{2n+1} - b^{2n+1}$ is divisible by $a - b$, but not by $a + b$.
$a^{2n+1} + b^{2n+1}$ is divisible by $a + b$, but not by $a - b$.

Thus,

$$(a^4 - b^4) \div (a - b) = a^3 + a^2b + ab^2 + b^3,$$
$$(a^4 - b^4) \div (a + b) = a^3 - a^2b + ab^2 - b^3,$$
$$(a^5 - b^5) \div (a - b) = a^4 + a^3b + a^2b^2 + ab^3 + b^4,$$
$$(a^5 + b^5) \div (a + b) = a^4 - a^3b + a^2b^2 - ab^3 + b^4,$$
$$(a^5 - b^5) \div (a + b) = a^4 - a^3b + a^2b^2 - ab^3 + b^4 - \frac{2b^5}{a + b},$$

where the latter is evidently not a complete quotient.

These theorems will enable the student to effect important simplifications in the reduction of fractions, and of equations, and must therefore obtain sufficient attention before he proceeds further.

DIVISION BY DETACHED COEFFICIENTS.

As it was shown, in the note on multiplication, that the multiplication may be very conveniently carried on by means of the detached coefficients only, so it may be readily shown that the same can be done in division; and its practice is earnestly inculcated on the student for precisely the same reason as it was there done,—its economy of time and space, and especially as an introduction to the recent improvements made in the solution of numerical equations. Thus, for example, to divide $3x^3 - bx$ by $x^3 - bx - c$.

$$\begin{array}{l}
1 + 0 - b - c)\ 3 + 0 - b \ + 0\ (3 + 0 + 2b + 3c + 2b^2 + 5bc \ldots \\
\qquad\qquad\quad 3 + 0 - 3b - 3c \\
\qquad\qquad\quad \overline{\qquad\qquad\qquad} \\
\qquad\qquad\qquad 0 + 2b + 3c \\
\qquad\qquad\qquad 0 + 2b + 0 \ - 2b^2 - 2bc \\
\qquad\qquad\qquad \overline{\qquad\qquad\qquad\qquad} \\
\qquad\qquad\qquad\qquad 3c + 2b^2 + 2bc \\
\qquad\qquad\qquad\qquad 3c + 0 \ - 3bc - 3c^2 \\
\qquad\qquad\qquad\qquad \overline{\qquad\qquad\qquad\qquad} \\
\qquad\qquad\qquad\qquad\quad 2b^2 + 5bc + 3c^2 \\
\qquad\qquad\qquad\qquad\quad 2b^2 + 0 \ - 2b^3 - 2b^2c \\
\qquad\qquad\qquad\qquad\quad \overline{\qquad\qquad\qquad\qquad} \\
\qquad\qquad\qquad\qquad\qquad + 5bc + 3c^2 \ | + 2b^2c \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad + 2b^3 \ |
\end{array}$$

And since the highest power of x is $\frac{x^3}{x^3} = x^0$, we have for the result $3x^0 + 0x^{-1} + 2bx^{-2} + 3\,cx^{-3} + 2b^2x^{-4} + 5bcx^{-5} + \ldots$ ad inf.

This example has been adopted on account of its containing both literal and zero, as well as numeral, coefficients.

SYNTHETIC DIVISION.

When one algebraic function of a quantity is to be divided by another, the

coefficients of each being given in numbers, the following process, invented by the late Mr. Horner to subserve the solution of numerical equations, is of the utmost value.

1. Write down the coefficients of the dividend in a horizontal line with their proper signs, and where a term is wanting write 0 in the place of its coefficient.

2. Draw a vertical line before the first term, and to the left of this line put down the coefficients of the divisor, with the same precaution respecting absent terms, but the signs of these coefficients changed; and having them so disposed that the first coefficient* is in a line with the horizontal column spoken of in (1).

3. Bring down the first coefficient of the dividend: this will be the first term of the quotient.

4. To obtain the others in succession, multiply the immediately preceding term of the quotient by the remaining terms of the divisor, having their signs changed; and place them successively under the corresponding terms of the dividend in a diagonal column, beginning at the upper line. Add the results in the second column, which will give the second term of the quotient; and multiply the terms of x in the divisor by this result, placing the products in a diagonal series, as before. Add the next series of results, which will give the next coefficient of the dividend; and multiply x by this again, placing the products as before. This process, persevered in till the results become 0, or till the quotient is determined as far as necessary, will give the same series of terms as the common mode of division, or as the division by detached coefficients, in the last article, when carried to an equivalent extent.

Let us take as an example the division of $x^6 - 6x^5 + 20x^4 - 40x^3 + 50x^2 - 40x + 100$ by $x^3 - 2x^2 + 5x - 9$. Following the prescribed directions with respect to arrangement, we have the horizontal and vertical columns at once.

$$
\begin{array}{r|rrrrrrrrrrr}
1 & 1 & -6 & +20 & -40 & +50 & -40 & +100 & & & & \\
+2 & & +2 & -8 & +14 & +6 & -30 & -44 & +316 & +582 & & \\
-5 & & & -5 & +20 & -35 & -15 & +75 & +110 & -790 & -1455 & \\
+9 & & & & +9 & -36 & +63 & +27 & -135 & -198 & +1422 & +2619 \\
\hline
 & 1 & -4 & +7 & +3 & -15 & -22 & +158 & +291 & -406 & - \;.\;.\;.\;. &
\end{array}
$$

Multiply each term of the quotient in succession by all the terms of the divisor, (the first or 1 excepted, the upper line standing for the result of that step,) carrying the results to the places denoted by the corresponding powers of the quantity x. This will always be done when the deficient terms are supplied by zero, *to preserve the places as in arithmetic*, by carrying them out diagonally to the right, or moving one step to the right in making the commencement of each successive row. Thus we obtain the *diagonal* series $1 + 2 - 5 + 9$. Add the vertical column $-6 + 2$, and with the result -4, multiply all the terms of the divisor as before, giving the next diagonal series $-8 + 20 - 36$. Add the third column, and obtain the result $+7$; and by this obtain another diagonal column $+14 - 35 + 63$, and then another sum $+3$. Proceed in the same manner till the results either terminate in zeros, or have been carried far enough to answer the purpose in view. In the above work nine terms are obtained: to which the powers of x (the highest being $x^{6-3} = x^3$) may be attached as they stand, and the quotient is $x^3 - 4x^2 + 7x + 3 - 15x^{-1} - 22x^{-2} + 158x^{-3} + 291x^{-4} - 406x^{-5} - \ldots\ldots$ *ad infinitum.*

---

* It is to be understood that the coefficient of the leading term of the divisor is 1; and in cases where this does not occur, it can be made so, by dividing every coefficient of the divisor and dividend by that coefficient.

With the view of illustrating the operation, it will be advisable to work the same question in the usual way, employing, however, only the detached coefficients.

```
1—2+5—9) 1—6+20—40+50—40+100 (1—4+7+3—15—22+158+291—406
         1—2+ 5— 9
         ---------
          —4+15—31+50
          —4+ 8—20+36
          -----------
              7—11+14—40
              7—14+35—63
              ----------
                 3—21+23+100
                 3— 6+15— 27
                 -----------
                  —15+ 8+127
                  —15+30— 75+135
                  --------------
                      —22+202—135
                      —22+ 44—110+198
                      ---------------
                          158— 25—198
                          158—316+790—1422
                          ----------------
                              291—988+1422
                              291—582+1455—2619
                              -----------------
                                —406— 33+2619
                                —406+ 812—2030+3654
                                -------------------
                                    — 845+4649—3654
```

The connexion between this and the synthetic division will best appear by taking a form intermediate between the two: viz. by placing the subtrahends in order, *having their signs changed,* but still in the horizontal position which they occupy in the old method.

```
  Divisor.                Dividend.                    Quotient.
1—2+5—9)  1 —6 +20 —40 +50 —40 +100 (1—4+7+3—15—22+158+291—406
        —1*+2 — 5 + 9   :   :    :
          : +4*— 8 +20 —36  :    :
          :  :  — 7*+14 —35 +63  :
          :  :    : — 3*+ 6 —15 + 27
          :  :    :   : +15*—30 + 75 —135
          :  :    :   :   :  +22*— 44 +100 —198
          :  :    :   :   :    :  —158*+316 —790 +1422
          :  :    :   :   :    :    :  —291*+582 —1455+2619
          :  :    :   :   :    :    :    :  +406*— 812+2030—3654
Remainder 0  0    0   0   0    0    0    0    0  — 845+4649—3654
```

The relation of this to the common method is obvious.

Had we, however, left out the numbers marked with the asterisk in this work, the sums would severally have been the terms of the divisor; and hence, if we omit multiplying by — 1 (the first coefficient of the divisor with its sign changed) the line now marked as "remainder" might have been employed for the terms

of the quotient, which are the sums of the several columns. This is in accordance with the rule, which requires the first coefficient 1 to be omitted; and the change in the signs of all the other terms is effected by changing the remaining signs of the divisor before we begin to operate.

Further, to avoid bringing the work so far down the page, leaving so much space unoccupied on each side of the diagonal columns, the several products of the coefficients of the modified divisor by the successive quotient figures, may be themselves set down in diagonal columns: thus, instead of

$$\left.\begin{array}{l} 1 - 6 + 20 - 40 + 50 - 40 \ldots. \\ \quad 2 - \;5 + \;9 \\ \quad\;\; - \;8 + 20 - 36 \\ \qquad\qquad 14 - 35 + 63 \\ \qquad\qquad\quad .. \quad .. \quad .. \\ \qquad\qquad\qquad\quad .. \quad .. \end{array}\right\} \text{ write } \left\{\begin{array}{l} 1 - 6 + 20 - 40 + 50 - 40 \ldots. \\ \qquad 2 - \;8 + 14 \quad .. \quad .. \\ \qquad\quad - \;5 + 20 - 35 \quad .. \\ \qquad\qquad\; + \;9 - 36 + 63 \\ \hline 1 - 4 + \;7 + \;3 \;.\,\ldots\ldots \end{array}\right.$$

In comparing this mode of working with the preceding, we remark that:

1st. The coefficients which appear as subtrahends in the old method, appear as addends having their signs changed, in the new. The change of all the signs of the divisor except the first, in the new method secures this.

2nd. No coefficient is used till we arrive at the vertical column in which it appears, and which occurs immediately after that column is completed. This arises from only completing at each step the first term of what constitutes the remainder in the old method.

3rd. The work is contracted into a series of horizontal columns, in number equal to the terms of the divisor, without descending the page continually, as in the old method. This is effected by carrying the *first* term of each product to the upper line, and gradually descending in a diagonal line with the others.

4th. The work besides not descending on the page, does not extend across it so far as in the old method. This arises from the less breadth occupied by the divisor in its vertical than in its horizontal arrangement; and from the quotient falling beneath the work instead of being placed to the right, as in the ordinary method.

This last process completes the Algorithm of the method, and brings us to the rule as above laid down in every particular.

On the ground of economy of time alone, this method does not require half so much writing as the ordinary one; and the chances of mistake in the operation are lessened in a still greater degree *.

---

* In the example just given, if we compare any two corresponding columns, as that belonging to x^{-4} for instance in the two methods, they will stand thus:

$$\begin{array}{ccc} \text{In the synthetic method.} & \qquad & \text{In the old method.} \\ \begin{array}{r} +316 \\ +110 \\ -135 \\ \hline +291 \end{array} & & \begin{array}{r} +135x^{-4} \\ \hline -135x^{-4} \\ -110x^{-4} \\ \hline -\;\;25x^{-4} \\ -316x^{-4} \\ \hline 291x^{-4} \\ 291x^{-4} \\ \hline \end{array} \end{array}$$

If again we estimate the total saving, beside the compactness of its disposition and the fewer chances of error, we shall discover that

x occurs 67 times more in the old than in the new process;

[Indices

EXAMPLES.

1. Divide $x^6 - y^6$ by $x - y$; or, which is the same thing,

$$x^6\left\{1 - \left(\frac{y}{x}\right)^6\right\} \text{ by } x\left\{1 - \frac{y}{x}\right\}.$$

$$\begin{array}{r|l} \text{Here} & 1 + 0 + 0 + 0 + 0 + 0 - 1 \\ +1 & \quad + 1 + 1 + 1 + 1 + 1 + 1 \\ \hline & 1 + 1 + 1 + 1 + 1 + 1 \end{array}$$

which give the coefficients, and we have $x^{6-1} = x^5$ for the exterior term. Whence the quotient is

$$x^5\left\{1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3 + \left(\frac{y}{x}\right)^4 + \left(\frac{y}{x}\right)^5\right\}, \text{ or}$$

$$x^5 + x^4y^3 + xy^2 + x^2y^3 + xy^4 + y^5.$$

2. Prove that $\frac{2ab}{a+b} = 2b - \frac{2b^2}{a} + \frac{2b^3}{a^2} - \frac{2b^4}{a^3} + \ldots.$

3. Also that $\frac{1}{1 - 2x + x^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots.$

4. Find the series for $\frac{3 - x + x^2 - 5x^3}{1 - 3x + 3x^2 - 5x^3}$.

5. And those for $\frac{xy - x^2y^2}{x^2y^2 + xy}$, and $\frac{x^2y^2z^2 + x^3y^3z^3}{1 - xyz}$.

6. Expand $\frac{1}{1+1}$, $\left(\text{or } \frac{1}{2}\right)$ into an infinite series; and likewise $\frac{1}{1-1}$ $\left(\text{or } \frac{1}{0}\right)$ into a series, and show that $\frac{1}{0}$ is the symbol of an infinitely great quantity.

PROBLEMS AND THEOREMS ON THE FIRST FOUR RULES OF ALGEBRA.

1. Half the difference of two quantities added to half the sum gives the greater of them, and subtracted leaves the less.

[Let the student select his own symbols, and illustrate it with his own numbers.]

2. If $2s = a + b + c$, what are the values of $s - a$, $s - b$, and $s - c$? and what is half their sum equal to? Find also their product, and arrange its terms systematically.

3. The difference between the square of the sum of two numbers and the square of their difference is equal to four times their product; and the sum of the squares of their sum and difference is double the sum of their squares. Prove this.

4. The sum of two numbers multiplied by their difference is equal to ......

Indices without sign, (or + understood,) 32 times more;
Indices with sign —, 25 times more;

to which the system of detached coefficients has the advantage in common with the synthetic division.

In the detached system there occur 172 figures besides the answer; in the new only 72, or 100 less; and of the signs + and —, the number in the detached operation is 66, and in the other 32, or rather less than half the number.

5. If $2s = a + b + c + d$, what is the sum, and what the product of $s - a$, $s - b$, $s - c$, $s - d$? and what is the sum of their squares?

6. Let several binomial factors, of which x is the first term, and where $a, b, c, d, \ldots$ are the second terms of the several factors, be multiplied together: then describe the manner in which the coefficients of the several powers of x in the product are formed of the quantities $a, b, c, d, \ldots$

7. Prove that $a^2 - (b - c)^2 = (a - b + c)(a + b - c)$, and $-a^2 + (b + c)^2 = (a+b+c)(-a+b+c)$; and hence that $4a^2b^2 - (a^2+b^2-c^2)^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$.

8. Divide $x^5 + y^5$ by $x + y$, and $x^5 - y^5$ by $x - y$; and show that if any other odd whole number be substituted for 5 in these expressions, the division will terminate without a remainder.

9. Convert $(u + x + y + z)^2$ into the form $(u + x)^2 + (u + y)^2 + (u + z)^2 + (x + y)^2 + (x + z)^2 + (y + z)^2 - 2(u^2 + x^2 + y^2 + z^2)$;
and likewise into the form

$$\tfrac{1}{2}\{(u+x+y)^2 + (u+x+z)^2 + (u+y+z)^2 + (x+y+z)^2 - (u^2+x^2+y^2+z^2)\};$$

and again into

$$u^2 + (2u+x)x + \{2(u+x) + y\}y + \{2(u+x+y) + z\}z:$$

and show that in this last, u may also change its place with either of the other quantities, x, y, or z.

10. Multiply $a + b\sqrt{-1}$ by $a - b\sqrt{-1}$, and also by $c + d\sqrt{-1}$; and multiply together four factors, each equal to $a+b\sqrt{-1}$, and then four others, each of which is $a-b\sqrt{-1}$; and lastly, multiply the factors $a+b\sqrt{-1}, c+d\sqrt{-1}$, $e+f\sqrt{-1}$, $g+h\sqrt{-1}$, and $i+k\sqrt{-1}$ together.

11. If the term *rectangle of two lines*, in the first ten propositions of the second book of Euclid, be exchanged for the term *product of two numbers*, and *square on a line* for the *square of a number;* show that the propositions thus transformed are also true, whatever those numbers may be.

12. Divide a by $-1 + \sqrt{-3}$, and by $(-1 + \sqrt{-3})^2$: and show that the quotient in the latter case is the same as would be obtained if we divide $\frac{1}{2}a$ by $-1 - \sqrt{-3}$. Show also that $a(-1 - \sqrt{-3})^3 = a(-1 + \sqrt{-3})^3$.

13. Show that the sum of $x(x + y + z)$, $y(y + z + x)$, and $z(z + x + y)$ is $(x + y + z)^2$.

14. Simplify to the utmost degree the expressions,

$$x^2 + 2xy + y^2 - \{x^2+xy-y^2-(2xy-x^2-y^2)\};$$

$$a - \{a + b - \{a + b + c - (a + b + c + d)\}\};$$

$$\text{and } a^2 + (-b)^2 + (-c)^2 + 2a(-b) + 2a(-c) + 2(-b)(-c).$$

THE GREATEST COMMON MEASURE AND LEAST COMMON MULTIPLE OF TWO OR MORE POLYNOMIALS.

A *common measure* of two or more quantities, whether expressed algebraically or arithmetically, is any quantity which will divide them both without a remainder.

The *greatest common measure* is the greatest quantity which will divide them without a remainder.

A *common multiple* of any number of quantities is any quantity which is divisible by them without remainders.

The *least common multiple* is the least quantity that is divisible by them all without remainders.

Quantities are said to be *prime* to each other which have no common measure, except unity.

I. *To find the greatest common measure of two quantities.*

RULES.

1. If there be any visible factor of one of the terms, whether it be the numerator or denominator, which is not a factor of the other, it may be rejected as forming no part of the common measure.

This applies more especially to monomial factors, as it is not often easy, except in very simple cases, to detect binomial or higher factors.

On the same principle, if it will facilitate the future operation, any factor may be brought into either of the terms.

2. Range both expressions in ascending or descending (no matter which, but descending is most usual) powers of some one quantity concerned in the expressions. Divide the greater by the less and the less by the remainder; then the remainder by the previous one, and so on till the work terminates by giving no remainder. The last divisor is the greatest common measure of the fraction.

3. If more than two expressions be given, of which to find the greatest common measure, proceed as directed in the corresponding subject in arithmetic, pp. 42—4.

PROOFS AND REMARKS.

1. Denote them by X and X_1, and let the following series of operations be performed, where $Q, Q_1, Q_2 \ldots\ldots Q_m$ are the successive quotients, and $X_2, X_3, \ldots\ldots X_{m+2}$ the corresponding successive remainders. The first column indicates the operation according to the arithmetical type, and the second expresses continually that divisor $\times$ quotient $+$ remainder $=$ dividend.

$$\begin{array}{l|l}
X_1)\,X\,(Q & X = QX_1 + X_2 \\
\quad QX_1 & X_1 = Q_1 X_2 + X_3 \\
\quad\overline{} & X_2 = Q_2 X_3 + X_4 \\
\quad X_2)\,X_1\,(Q_1 & \ldots\ldots\ldots\ldots\ldots \\
\quad\quad Q_1 X_2 & \ldots\ldots\ldots\ldots\ldots \\
\quad\quad\overline{} & \ldots\ldots\ldots\ldots\ldots \\
\quad\quad X_3)\,X_2\,(Q_2 & \ldots\ldots\ldots\ldots\ldots \\
\quad\quad\quad Q_2 X_3 & \ldots\ldots\ldots\ldots\ldots \\
\quad\quad\quad\overline{} & \ldots\ldots\ldots\ldots\ldots \\
\quad\quad\quad X_4)\,X_3\,(Q_3 & \\
\quad\quad\quad\quad\vdots & \\
\quad\quad\quad X_{m+1})\,X_m\,(Q_m & X_{m-2} = Q_{m-2} X_{m-1} + X_m \\
\quad\quad\quad\quad Q_m X_{m+1} & X_{m-1} = Q_{m-1} X_m + X_{m+1} \\
\quad\quad\quad\quad\overline{} & X_m = Q_m X_{m+1} + X_{m+2} \\
\quad\quad\quad\quad X_{m+2} &
\end{array}$$

Now suppose X_{m+2} to be the remainder which becomes 0: then $X_m = Q_m X_{m+1}$, and X_{m+1} is the last remainder, which, therefore, divides X_m exactly.

Substitute this value of X_m in the preceding equation:

$$\text{then } X_{m-1} = Q_{m-1} Q_m X_{m+1} + X_{m+1} = X_{m+1} \{Q_{m-1} Q_m + 1\}$$

From which, as Q_{m-1} and Q_m are integral, we see that X_{m-1} is divisible by X_{m+1} exactly.

Next substitute the values of X_{m-1} and X_m in the third equation from the end, in terms of X_{m+1}, and we have

$$X_{m-2} = Q_{m-2}\{Q_{m-1}\,Q_m + 1\}\,X_{m+1} + Q_m\,X_{m+1}$$
$$= \Big\{Q_{m-2}\{Q_{m-1}\;Q_m + 1\} + Q_m\Big\}X_{m+1}$$

Whence as Q_m, Q_{m-1}, and Q_{m-2} are integral, X_{m-2} is divisible by X_{m+1}.

In the same way, we find that in succession all the preceding X_{m-3}, X_{m-4}, ... X_4, X_3, X_2, X_1, X, are divisible by X_{m+1}. Whence X_{m+1} is a common measure of the given expressions X and X_1; and it is likewise a common measure of all the subordinate quantities, or remainders, X_2, X_3 .... X_m.

2. In the next place, X_{m+1} is the greatest common measure. For, let Y be any other common measure, and put $X = mY$ and $X_1 = nY$. Then from the foregoing equations, and substituting these values of X and X_1,

$$\begin{aligned}
X_2 &= X \;-\; QX_1 \;\; = \{m - nQ\}\,Y &&= P_1Y,\\
X_3 &= X_1 - Q_1\,X_2 = \{n - mQ_1 + nQQ_1\}\,Y &&= P_2Y,\\
X_4 &= X_2 - Q_2\,X_3 = \{\ldots\ldots\ldots\ldots\ldots\}\,Y &&= P_3Y,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
X_{m+1} &= X_{m-1} - Q_{m-1}\,X_m = \{\ldots\ldots\ldots\}\,Y &&= P_mY,
\end{aligned}$$

where it is evident from the composition of the coefficients P_1, P_2, ... P_m, that they are all integers, since they are composed of the products, sums and differences of the integers m, n, Q, Q_1, ... Q_m, by hypothesis. Hence we have $P_mY = X_{m+1}$, and P_m integral; and therefore Y is less than X_{m+1}: that is, X_{m+1} is the *greatest common measure*.

3. It also follows from this, that any common measure Y of two terms is also a measure of their greatest common measure. For since $P_mY = X_{m+1}$ and P_m is an integer, X_{m+1} is divisible by Y without remainder.

4. The quotients $\dfrac{X}{X_{m+1}}$ and $\dfrac{X_1}{X_{m+1}}$ may be thus formed by a succession of operations; and each is a closer approximation than the preceding to the true value of the fraction.

Write down Q, Q_1, Q_2, ... Q_m, in a line as follows:—

	Q	Q_1	Q_2	Q_3	Q_4
for X;	1,	Q,	QQ_1+1,	$(QQ_1+1)\,Q_2+Q$,	$\{(QQ_1+1)\;Q_2+Q\}\;Q_3+QQ_1+1$
for X_1;	0,	1,	Q_1+0,	QQ_1+1 ,	$(QQ_1+1)\;Q_3\;+\;Q_1$

that is, multiplying each successive value by the quantity under which it stands, and adding the second preceding one to the quotient.

These expressions have several curious and interesting properties, which there is not room in this work to touch upon. One or two, however, will be stated under *Continued Fractions*, a little further on.

5. As a form of work we may adopt with advantage that given on the corresponding occasion in the arithmetic, p. 43; and, generally speaking, we may avail ourselves of the method of detached coefficients. Also, to avoid the introduction of fractional coefficients in the successive divisions, we may, by cross-multiplication of the coefficients of the highest terms of divisor and dividend, reduce the leading coefficients to identity.

6. When the given expressions can visibly be resolved into factors, it is always better to do so to the utmost possible extent. The factors which are common to both are common measures; and if the resolution has been complete, all the common measures will thus have been obtained; and their continued product will be the greatest common measure.

EXAMPLES.

Ex. 1. Find the greatest common measure of the expressions $x^5 - x^3 y^2 + x^2 y^3 - y^5$ and $x^5 + x^3 y^2 - x^2 y^3 - y^5$.

Apply art. (6): then we have

$$\begin{aligned} X = x^5 - x^3 y^2 + x^2 y^3 - y^5 &= (x^3 + y^3)(x^2 - y^2) \\ &= (x^2 - xy + y^2)(x + y)(x + y)(x - y) \text{ and} \\ X_1 = x^5 + x^3 y^2 - x^2 y^3 - y^5 &= (x^3 - y^2)(x^2 + y^2) \\ &= (x - y)(x^2 + xy + y^2); \end{aligned}$$

and we see that the only factor common to both terms is $x - y$; which is therefore the greatest common measure.

Ex. 2. To find the greatest common measure of $a^3 - ab^2$ and $a^2 + 2ab + b^2$.

$$\begin{array}{l} a^2 + 2ab + b^2)\ a^3 - ab^2\ (a \\ \qquad\qquad\quad\ \ \underline{a^3 + 2a^2b + ab^2} \\ \qquad\qquad\qquad - 2a^2b - 2ab^2)\ a^2 + 2ab + b^2\ (\end{array}$$

or dividing by $- 2ab$, which is not a divisor of the other quantity,

$$\begin{array}{l} a + b)\ a^2 + 2ab + b^2\ (a + b \\ \qquad\quad\ \underline{a^2 + ab} \\ \qquad\qquad\quad ab + b^2 \\ \qquad\qquad\quad \underline{ab + b^2} \end{array}$$

Therefore $a + b$ is the greatest common divisor.

But by detached coefficients, and under the indicated arrangement, (art. 5,) it would stand thus:

$$\begin{array}{r|r|r|l} 1 + 1 & 1 + 2 + 1 & 1 + 0 - 1 + 0 & 1 - 2 \\ & \underline{1 + 1} & \underline{1 + 2 + 1} & \\ & 1 + 1 & - 2 - 2 + 0 & \\ & 1 + 1 & - 2 - 4 - 2 & \\ \hline & . & \multicolumn{2}{l}{\quad 2 + 2, \text{ or dividing by } 2,} \\ & & \multicolumn{2}{l}{\quad 1 + 1} \end{array}$$

Hence $1a + 1b$, or $a + b$ is the greatest common measure.

Ex. 3. Find the greatest common measure of $x^6 + 3x^5 - 6x^4 - 6x^3 + 9x^2 + 3x - 4$ and $6x^5 + 15x^4 - 24x^3 - 18x^2 + 18x + 3$.

$$\begin{array}{r|l|r|l} & 6+15-24-18+18+3 & 1+3-6-6+9+3-4 & \\ & 2+5-8-6+6+1 & 4+12-24-24+36+12-16 & 2+1 \\ & \text{13 multiplier.} & 4+10-16-12+12+2 & \\ \hline 1 & 26+65-104-78+78+13 & 2-8-12+24+10-16 & \\ & 26+8-60-8+34 & 2+5-8-6+6+1 & \\ \hline & 57-44-70+44+13 & -13-4+30+4-17 & \\ & \text{26 multiplier} & -\text{ 2 multiplier} & \\ \hline 1 & 1482-1144-1820+1144+338 & 26+8-60-8+34 & \\ & 1482+456-3420-456+1938 & \text{57 multiplier} & \\ \hline - & -1600+1600+1600-1600 & 1482+456-3420-456-1938 & 1 \\ & \text{or dividing by } -1600 & \text{or dividing by } 114 & \\ & 1-1-1+1 & 13+4-30-4+17 & 13+17 \\ & & 13-13-13+13 & \\ \cline{3-3} & & 17-17-17+17 & \\ & & 17-17-17+17 & \\ \cline{3-3} \end{array}$$

And as there is no remainder, we have by restoring the letters $x^3 - x^2 - x + 1$ for the common measure of X and X_1.

Ex. 4. Find the greatest common measure of $a^2 - 4$ and $ab + 2b$.

Ex. 5. And of $a^5 - a^3b$ and $a^4 - b^4$.

Ex. 6. And of $a^3x + 2a^2x^2 + 2ax^3 + x^4$ and $5a^5 + 10a^4x + 5a^3x^2$.

Ex. 7. And of $6x^5 + 20x^4 - 12x^3 - 48x^2 + 22x + 12$ and $x^6 + 4x^5 - 3x^4 - 16x^3 + 11x^2 + 12x - 9$.

II. *The least common multiple.*

1. If the quantities be prime to each other, their least common multiple is their product.

2. If one of them be a multiple of all or any of the others, whatever is a multiple of this is a multiple of those others; and the least common multiple which takes in this greater quantity will be the least common multiple of all those others.

3. Let X, X_I, X_{II} .... be the quantities, no one of which is a multiple of any of the others; and let the greatest common measure of X and X_I be V, such that $X = mV$ and $X_I = nV$. Then m and n are prime to each other; and the least common multiple M of X and X_I is mnV, for it is the least quantity divisible by mV and nV, or by X and X_I. But

$$M = mnV = \frac{mV \cdot nV}{V} = \frac{XX_I}{V} = \frac{\text{product of the quantities.}}{\text{their greatest com. meas.}}$$

Again, the least common multiple of X, X_I, and X_{II}, is the least common multiple of M and X_{II}. Suppose V_I to be the greatest common measure of M and X_{II}, then the least common multiple of X, X_I, and X_{II}, is

$$M_I = \frac{MX_{II}}{V_I} = \frac{XX_I\,X_{II}}{VV_I} = \frac{\text{product of the quantities.}}{\text{product of their g. c. measures.}}$$

Proceeding in the same manner, we find for p quantities

$$M_{p-2} = \frac{XX_I\,X_{II} \ldots \; p \text{ factors}}{VV_I \ldots \; (p-1) \text{ factors}} = \frac{\text{product of the quantities}}{\text{product of their g. c. measures.}}$$

EXAMPLES.

Ex. 1. Required the least common multiple of $a^3 + a^2b$ and $a^2 - b^2$.

Here $a + b = V_I$ is their greatest common measure. Hence

$$\frac{(a^3 + a^2b)(a^2 - b^2)}{a + b} = \frac{a^2 (a + b)^2 (a - b)}{a + b} = a^2 (a+b)(a-b) = a^2 (a^2-b^2).$$

Ex. 2. Required the least common multiple of $x^3 + x^2 + x + 1$ and $x^3 - x^2 + x - 1$.

Here the greatest common measure is $x^2 + 1 = V$, and hence

$$M = \frac{(x^3 - x^2 + x - 1)(x^3 + x^2 + x + 1)}{x^2 + 1} = x^4 - 1.$$

Ex. 3. Required the least common multiple of a^2+ab, $a^4+a^2b^2$ and a^2-b^2.

Here $V = a$, and hence

$$M = \frac{XX_I}{V} = \frac{(a^2 + ab)(a^4 + a^2b^2)}{a} = a^2 (a + b)(a^2 + b^2).$$

Hence again V_I, the greatest common measure of M and X_{II}, or of $a^2 (a + b)(a^2 + b^2)$ and $a^2 - b^2$, is $a + b$; and therefore

$$M_I = \frac{MX_{II}}{V_I} = \frac{a^2 (a + b)(a^2 + b^2)(a^2 - b^2)}{a + b} = a^2 (a^4 - b^4).$$

Ex. 4. The least common multiple of $x^4 + ax^3 - 9a^2x^2 + 11a^3x - 4a^4$ and $x^4 - ax^3 - 3a^2x^2 + 5a^3x - 2a^4$ is $x^5 + 3ax^4 - 7a^2x^3 - 7a^3x^2 + 18a^4x - 8a^5$.

Ex. 5. The least common multiple of $x^3 - a^2x - ax^2 + a^3$, $x^4 - a^4$, and $ax^3 + a^3x - a^2x^2 - a^4$ is $ax^5 - a^2x^4 - a^5x + a^6$.

The arithmetical process at p. 48 is only a convenient method of putting down the work, where the divisors or common measures that will suit are found by inspection or by trials.

ALGEBRAIC FRACTIONS.

Algebraic Fractions have the same names and rules of operation, as numerical fractions in common arithmetic; as appears in the following Rules and Cases.

CASE I.

To reduce a mixed quantity to an improper fraction.

Multiply the integer by the denominator of the fraction, and to the product add the numerator, or connect it with its proper sign, + or −; then the denominator being set under this sum, will give the improper fraction required.

EXAMPLES.

1. Reduce $3\frac{4}{5}$, and $a - \dfrac{b}{x}$ to improper fractions.

First, $3\frac{4}{5} = \dfrac{(3 \times 5) + 4}{5} = \dfrac{15 + 4}{5} = \dfrac{19}{5}$ the answer.

And, $a - \dfrac{b}{x} = \dfrac{(a \times x) - b}{x} = \dfrac{ax - b}{x}$ the answer.

2. Reduce $a + \dfrac{a^2}{b}$ and $a - \dfrac{z^2 - a^2}{a}$ to improper fractions.

First, $a + \dfrac{a^2}{b} = \dfrac{(a \times b) + a^2}{b} = \dfrac{ab + a^2}{b}$ the answer.

And, $a - \dfrac{z^2 - a^2}{a} = \dfrac{a^2 - z^2 + a^2}{a} = \dfrac{2a^2 - z^2}{a}$ the answer.

3. Reduce $5\frac{3}{7}$ to an improper fraction. Ans. $\frac{38}{7}$.

4. Reduce $1 - \dfrac{3a}{x}$ to an improper fraction. Ans. $\dfrac{x - 3a}{x}$.

5. Reduce $2a - \dfrac{3ax + a}{4x}$ to an improper fraction.

6. Reduce $12 + \dfrac{4x - 18}{5x}$ to an improper fraction.

7. Reduce $x + \dfrac{1 - 3a - \frac{1}{2}c}{\frac{1}{2}c}$ to an improper fraction.

8. Reduce $\dfrac{4}{9} + \dfrac{2x}{3} - \dfrac{\dfrac{2x^3}{7} - \dfrac{3a}{5}}{\dfrac{5a}{8}}$ to an improper fraction.

9. Reduce $\cdot 4 - 2 \times 10^{-2}x + \dfrac{30x - 100 \times 10^{-2}}{\cdot 1x^2 - 40 \times 10^{-5}x}$ to an improper fraction.

10. Reduce $\cdot 001 - \frac{67x}{100} - \frac{x^2 - 4x}{100x - 5 \times 10^{-5}}$ to an improper fraction.

11. Reduce $1 + x + x^2 + x^3 + x^4 \ . \ . \ . \ . \ . \ . \ . \ x^n + \frac{x^{n+1}}{1 - x}$ to an improper fraction; and likewise $a^3 - a^2x + ax^2 - x^3 - \frac{2x^4}{a + x}$; and again the quantities $1 + \frac{b}{a} + \frac{b^2}{a^2} + \ . \ . \ . \ \frac{b^{n+1}}{a^{n+1}} \cdot \frac{a}{a - b}$, and $1 - 2x + 2x^2 - 2x^3 + 2x^4 - \ldots + \frac{2x^{10}}{1 + x}$ to improper fractions *.

CASE II.

To reduce an improper fraction to a whole or mixed quantity.

Divide the numerator by the denominator, for the integral part; and set the remainder, if any, over the denominator, for the fractional part; the two joined together will be the mixed quantity required.

EXAMPLES.

1. To reduce $\frac{16}{3}$ and $\frac{ab + a^2}{b}$ to mixed quantities.

First, $\frac{16}{3} = 16 \div 3 = 5\frac{1}{3}$, the answer required.

And, $\frac{ab + a^2}{b} = (ab + a^2) \div b = a + \frac{a^2}{b}$. Answer.

2. To reduce $\frac{2ac - 3a^2}{c}$ and $\frac{3ax + 4x^2}{a + x}$ to mixed quantities.

First, $\frac{2ac - 3a^2}{c} = (2ac - 3a^2) \div c = 2a - \frac{3a^2}{c}$. Answer.

And, $\frac{3ax + 4x^2}{a + x} = (3ax + 4x^2) \div (a + x) = 3x + \frac{x^2}{a + x}$. Ans.

3. Reduce $\frac{33}{5}$ and $\frac{2ax - 3x^2}{a}$ to mixed quantities. Ans. $6\frac{3}{5}$, and $2x - \frac{3x^2}{a}$.

4. Reduce $\frac{4a^2x}{2a}$ and $\frac{2a^2 + 2b}{a - b}$ to whole or mixed quantities.

5. Reduce $\frac{3x^2 - 3y^2}{x + y}$ and $\frac{2x^3 - 2y^3}{x - y}$ to whole or mixed quantities.

6. Reduce $\frac{10a^2 - 4a + 6}{5a}$ to a mixed quantity.

7. Reduce $\frac{15a^3 + 5a^2}{3a^3 + 2a^2 - 2a - 4}$ to a mixed quantity.

CASE III.

To reduce fractions to a common denominator.

Multiply each numerator by all the denominators except its own for a new

---

* In such examples as these the multiplication may be advantageously performed by detached coefficients.

numerator of the equivalent fraction; and all the denominators together for a common denominator to all the fractions equivalent to the given ones *.

It will be convenient in putting down the work to write all the numerators in succession in a vertical column, and commence the factors which follow by the denominators of those fractions which succeed them.

EXAMPLES.

1. Reduce $\frac{a}{x}$, $\frac{y}{b}$, and $-\frac{b}{c}$ to a common denominator.

$$\left.\begin{array}{r} a \times b \times c = abc \\ y \times c \times x = cxy \\ -b \times x \times b = -b^2x \end{array}\right\} = \text{new numerators.}$$

$$x \times b \times c = bcx = \text{common denominator.}$$

And the fractions are $\frac{abc}{bcx}$, $\frac{cxy}{bcx}$, and $-\frac{b^2x}{bcx}$.

2. Reduce $\frac{a-x}{a+x}$ and $\frac{a+x}{a-x}$ to a common denominator.

$$\left.\begin{array}{r} (a-x)(a-x) = a^2 - 2ax + x^2 \\ (a+x)(a+x) = a^2 + 2ax + x^2 \end{array}\right\} = \text{new numerators.}$$

$$(a+x)(a-x) = a^2 - x^2 = \text{new denominator.}$$

And the fractions are $\frac{a^2 - 2ax + x^2}{a^2 - x^2}$ and $\frac{a^2 + 2ax + x^2}{a^2 - x^2}$.

3. Reduce $\frac{2a}{x}$ and $\frac{3b}{2c}$ to a common denominator. Ans. $\frac{4ac}{2cx}$ and $\frac{3bx}{2cx}$.

4. Reduce $\frac{2a}{b}$ and $\frac{3a+2b}{2c}$ to a common denominator.

Ans. $\frac{4ac}{2bc}$ and $\frac{3ab+2b^2}{2bc}$.

5. Reduce $\frac{5a}{3x}$ and $\frac{3b}{2c}$, and $-4d$, to a common denominator.

Ans. $\frac{10ac}{6cx}$ and $\frac{9bx}{6cx}$ and $-\frac{24cdx}{6cx}$.

6. Reduce $-\frac{5}{6}$ and $-\frac{3a}{4}$ and $2b - \frac{3a}{b}$ to fractions having a common denominator. Ans. $-\frac{20b}{24b}$ and $-\frac{18ab}{24b}$ and $\frac{48b^2 - 72a}{24b}$.

7. Reduce $\frac{1}{3}$ and $\frac{2a^2}{4}$ and $\frac{2a^2+b^2}{a+b}$ to a common denominator.

8. Reduce $\frac{3b}{4a^2}$ and $-\frac{2c}{3a}$ and $\frac{d}{2a}$ to their least common denominator.

9. Reduce $\frac{x}{2a} - \frac{y}{4b} + \frac{z}{6c} = 1\frac{5}{9}$ and $\frac{2a}{3x} - \frac{4b}{6y} + \frac{6c}{9z} = 11\frac{5}{18}$, each to a common denominator.

10. Reduce $\cdot 3a - \cdot 03b$, and $\cdot 003c - 10^{-5}d$ to a common denominator.

11. Reduce $\frac{a+b}{a-b} + \frac{a-b}{a+b}$, and $\frac{1}{a} + \frac{1}{b} - \frac{b}{1} - \frac{a}{1}$ to common denominators.

---

* For this is only multiplying the numerator and denominator of each fraction by equal quantities, which does not alter its value.

CASE IV.

To reduce fractions to their least common denominator.

Find the least common multiple M of their denominators for a new denominator. Divide M by each of the denominators D, D_1, D_2 ...., and multiply the corresponding numerators N, N_1, N_2, by these quotients, for the corresponding new numerators.

EXAMPLES.

Ex. 1. Reduce $\frac{a}{bc}$, $\frac{b}{ac}$, $\frac{c}{ab}$, $\frac{ab}{c}$, $\frac{ac}{b}$, and $\frac{bc}{a}$ to a common denominator.

Here abc is the least common multiple of the denominators.

$$\left.\begin{array}{lll} \text{Therefore } \frac{abc}{bc} = a, & \text{and} & \frac{a}{bc} \times \frac{a}{a} = \frac{a^2}{abc} \\ \frac{abc}{ac} = b, & \text{and} & \frac{b}{ac} \times \frac{b}{b} = \frac{b^2}{abc} \\ \frac{abc}{ab} = c, & \text{and} & \frac{c}{ab} \times \frac{c}{c} = \frac{c^2}{abc} \\ \frac{abc}{c} = ab, & \text{and} & \frac{ab}{c} \times \frac{ab}{ab} = \frac{a^2b^2}{abc} \\ \frac{abc}{b} = ac, & \text{and} & \frac{ac}{b} \times \frac{ac}{ac} = \frac{a^2c^2}{abc} \\ \frac{abc}{a} = bc, & \text{and} & \frac{bc}{a} \times \frac{bc}{bc} = \frac{b^2c^2}{abc} \end{array}\right\} \begin{array}{c}\text{The proposed} \\ \text{fractions reduced} \\ \text{to the least} \\ \text{com. denom.} \\ abc.\end{array}$$

Exx. 2, 3, 4. Reduce Ex. 6, 8, 9, of the last case to their least common denominators.

CASE V.

To reduce a fraction to its lowest terms.

Find the greatest common measure of its numerator and denominator. Then divide both the terms of the fraction by the common measure thus found, and it will reduce it to its lowest terms at once, as was required. Or, divide the terms by any quantity which it may appear will divide them both, as explained in art. (6), p. 134.

EXAMPLES.

1. Reduce $\frac{ab + b}{ac^2 + bc^2}$ to its lowest terms.

Here $a + b$ is the greatest common measure, by which, dividing the numerator and denominator, we have $\frac{b}{c^2}$ for the fraction in its lowest terms.

2. To reduce $\frac{c^3 - b^2c}{c^2 + 2bc + b^2}$ to its least terms. Ans. $\frac{c^2 - bc}{c + b}$.

3. Reduce $\frac{c^3 - b^3}{c^4 - b^2c^2}$ to its lowest terms. Ans. $\frac{c^2 + bc + b^2}{c^3 + bc^2}$.

4. Reduce $\frac{a^2 - b^2}{a^4 - b^4}$ to its lowest terms. Ans. $\frac{1}{a^2 + b^2}$.

5. Reduce $\dfrac{a^4 - b^4}{a^3 - 3a^2b + 3ab^2 - b^3}$ to its lowest terms.

6. Reduce $\dfrac{3a^5 + 6a^4c + 3a^3c^2}{a^3c + 3a^2c^2 + 3ac^3 + c^4}$ to its lowest terms.

7. Simplify $\dfrac{20x^6 - 12x^5 + 16x^4 - 15x^3 - 14x^2 - 15x + 4}{15x^4 - 9x^3 + 47x^2 - 21x + 28}$.

CASE VI.

To add fractional quantities together.

If the fractions have a common denominator, add all the numerators together; then under their sum set the common denominator, and it is done.

If they have not a common denominator, reduce them to one (the *least*), and then add them as before.

EXAMPLES.

1. Let $\frac{a}{3}$ and $\frac{a}{4}$ be given, to find their sum.

Here $\frac{a}{3} + \frac{a}{4} = \frac{4a}{12} + \frac{3a}{12} = \frac{7a}{12}$ is the sum required.

2. Given $\frac{a}{b}$, $\frac{b}{c}$, and $\frac{c}{d}$, to find their sum.

Here $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} = \frac{acd}{bcd} + \frac{b^2d}{bcd} + \frac{bc^2}{bcd} = \frac{acd + b^2d + bc^2}{bcd}$ the sum required.

*3. Let $a - \frac{3x^2}{b}$ and $b + \frac{2ax}{c}$ be added together.

Here $a - \frac{3x^2}{b} + b + \frac{2ax}{c} = a - \frac{3cx^2}{bc} + b + \frac{2abx}{bc} = a + b + \frac{2abx - 3cx^2}{bc}$, the sum required.

4. Add $\frac{4x}{3a}$ and $\frac{2x}{5b}$ together. Ans. $\frac{20bx + 6ax}{15ab}$.

5. Add $\frac{a}{3}$, $\frac{a}{4}$, and $\frac{a}{5}$ together. Ans. $\frac{47}{60}a$.

6. Add $\frac{2a - 3}{4}$ and $\frac{5a}{8}$ together. Ans. $\frac{9a - 6}{8}$.

---

* In the addition of mixed quantities, it is best to bring the fractional parts only to a common denominator, and to annex their sum to the sum of the integers, with the proper sign. And the same rule may be observed for mixed quantities in subtraction also. If, however, in the sum of the fractions thus obtained there should happen to appear integer quantities, these will be better brought out and united with the integers of the given quantities.

Though, it must be further remarked, it is often the more convenient method to thus reduce the quantities to a mixed state, especially in the arithmetical part of a process; yet it also frequently happens (and this more particularly where the *formula* itself is the object of consideration) that the more elegant result is obtained by reducing the whole to the form of one fraction.

See also, the note to addition of fractions in the arithmetic, p. 50.

7. Add $2a - \frac{a+3}{5}$ to $4a - \frac{2a-5}{4}$. Ans. $6a - \frac{14a-13}{20}$.

8. Add $-6a$, and $-\frac{3a^2}{4b}$ and $\frac{a+b}{3b}$ together.

9. Add $\frac{5a}{4}$, and $\frac{6a}{5}$, and $-\frac{3a+2}{7}$ together.

10. Add the several results furnished by each of the questions in Case III., taking each of the examples in that case as an example in this.

11. Add $-2a$, and $\frac{3a}{8}$, and $-\left\{3 - \frac{a}{6}\right\}$ together.

12. Add $8a + \frac{3a}{4}$ and $-\left\{2a - \frac{5a}{8}\right\}$ together.

13. Find the sum of $\frac{x}{a+x}$ and $\frac{x}{a-x}$; and likewise of $-\left\{x - \frac{2xy}{x+y}\right\}$ and $-\left\{x + \frac{2xy}{x-y}\right\}$.

14. Ascertain the sum of $\frac{-1}{4a^3(a+x)}$, $\frac{1}{4a^3(x-a)}$, and $\frac{1}{2a^2 \times -(a^2+x^2)}$; and that of $\frac{x}{x^2-y^2}$, $\frac{y}{x+y}$, and $\frac{1}{x-y}$.

15. Express $\frac{p}{3my^2-x} \pm \frac{y-6mpy^2}{(3my^2-x)^2}$, in single fractions adapted to each of the cases where $+$ and $-$ are taken.

CASE VII.

To subtract one fractional quantity from another.

Reduce the fractions to a common denominator, (the *least* is the more elegant) if they have not a common denominator.

Subtract the numerators from each other, and under their difference set the common denominator.

EXAMPLES.

1. To find the difference of $\frac{3a}{4}$ and $\frac{4a}{7}$.

Here $\frac{3a}{4} - \frac{a}{7} = \frac{21a}{28} - \frac{16a}{28} = \frac{5a}{28}$ is the difference required.

2. To find the difference of $\frac{2a-b}{4c}$ and $\frac{3a-4b}{3b}$.

Here $\frac{2a-b}{4c} - \frac{3a-4b}{3b} = \frac{6ab-3b^2}{12bc} - \frac{12ac-16bc}{12bc} = \frac{6ab-3b^2-12ac+16bc}{12bc}$ is the difference required.

3. Subtract $\frac{4a}{7}$ from $\frac{3a}{4}$, and $\frac{3a}{4}$ from $\frac{4a}{7}$.

4. Required the difference of $\frac{10a}{9}$ and $\frac{4a}{7}$.

5. Take $\frac{2a-b}{4c}$ from $\frac{3a-4b}{3b}$, and $4a + \frac{2a}{c}$ from $2a - \frac{a-3b}{2c}$.

6. Subtract $\frac{5a}{4}$ from $\frac{2a}{3}$, and $6a$ from $\frac{3a}{4}$.

7. Subtract $\frac{2b}{c}$ from $\frac{3a+c}{b}$ and from $-\frac{3a+c}{b}$.

8. Take $\frac{2a+6}{9}$ from $\frac{4a+8}{5}$, and $\frac{3a+c}{b}$ from $\frac{2b}{c}$

9. Take $2a - \frac{a-3b}{c}$ from $4a + \frac{2a}{c}$.

10. Take $-\frac{9a+b}{4}$ from $-\frac{6a-4}{3}$, and $\frac{6}{7}$ from $-8+\frac{1}{7}$.

11. Subtract $-\frac{2a+6}{9}$ from $\frac{4a+8}{5}$, and likewise from $-\frac{4a+8}{5}$.

12. Which is the greater of the quantities $\frac{3a}{4}+6$ and $\frac{4a}{7}+6$, and what is their difference? Likewise when taken with sign of a changed, viz. $-\frac{3a}{4}+6$ and $-\frac{4a}{7}+6$?

13. Subtract $\frac{5x-3}{x+1}$ from $\frac{2x^2-13x+1}{x^2-1}$, and likewise from $\frac{3x+2}{x-1}$.

14. Subtract $\frac{1}{x-y}$ from $\frac{1}{x+y}$, and $\frac{1}{x^2-y^2}$ from $\frac{1}{x^2+y^2}$.

CASE VIII.

To multiply fractional quantities together.

Multiply the numerators together for a new numerator, and the denominators for a new denominator *.

EXAMPLES.

1. Required to find the product of $\frac{a}{8}$ and $\frac{2a}{5}$.

Here $\frac{a \times 2a}{8 \times 5} = \frac{2a^2}{40} = \frac{a^2}{20}$ the product required.

2. Required the product of $\frac{a}{3}$, $\frac{3a}{4}$, and $\frac{6a}{7}$.

$\frac{a \times 3a \times 6a}{3 \times 4 \times 7} = \frac{18a^3}{84} = \frac{3a^3}{14}$ the product required.

3. Required the product of $\frac{2a}{b}$ and $\frac{a+b}{2a+c}$.

---

* 1. When the numerator of one fraction, and the denominator of the other, can be divided by some quantity, which is common to both, the quotients may be used instead of them: or, in other words, the fractions may be reduced to their lowest terms *before* they are multiplied together.

2. When a fraction is to be multiplied by an integer, the product is found either by multiplying the numerator, or dividing the denominator by it; and if the integer be the same with the denominator, the numerator may be taken for the product. This may be readily deduced as a case of the general rule.

Here $\dfrac{2a \times (a+b)}{b \times (2a+c)} = \dfrac{2a^2+2ab}{2ab+bc}$ the product required.

4. Required the product of $\dfrac{4a}{3}$ and $\dfrac{6a}{5c}$.

5. Required the product of $\dfrac{3a}{4}$ and $\dfrac{4b^2}{3a}$; and of $-\dfrac{3a}{4}$ and $-\dfrac{4b^2}{3a}$.

6. To multiply $-\dfrac{3a}{b}$, and $\dfrac{8ac}{b}$, and $\dfrac{4ab}{3c}$ together.

7. Required the product of $2a+\dfrac{ab}{2c}$ and $\dfrac{3a^2}{b}$.

8. Required the product of $\dfrac{2a^2-2b^2}{3bc}$ and $\dfrac{4a^2+2b^2}{a+b}$.

9. Required the product of $3a$, and $\dfrac{2a+1}{a}$ and $\dfrac{2a-1}{2a+b}$.

10. Multiply $a+\dfrac{x}{2a}-\dfrac{x^2}{4a^2}$ by $x-\dfrac{a}{2x}+\dfrac{a^2}{4x^2}$.

11. Multiply together $\dfrac{3a-4b}{6a-8b}$, $-\dfrac{2a-\frac{1}{3}b}{a-\frac{4}{3}b}$, $\dfrac{8}{3}$ and $-\left(\dfrac{4x}{10}-\dfrac{15x}{5}\right)-\dfrac{12x}{18}$.

12. Find the product of $-\dfrac{9}{10}a$, $-10^{-5}a^{-5}$, $\dfrac{1}{100}a^3$, and $-\dfrac{6a}{10^{-5}} \times -\dfrac{3a^{-1}}{2^{-4}a^{-2}}$.

13. Multiply together $1-\dfrac{b^2+c^2-a^2}{2bc}$, $1-\dfrac{a^2+b^2-c^2}{2ab}$, and $1-\dfrac{a^2-b^2+c^2}{2ac}$; and in the result put $\dfrac{a+b+c}{2}=s$.

CASE IX.

To divide one fractional quantity by another.

Divide the numerators by each other, and the denominators by each other, if they will exactly divide: but, if not, then invert the terms of the divisor, and multiply by it as directed in multiplication, p. 143 *.

EXAMPLES.

1. Required to divide $\dfrac{a}{4}$ by $\dfrac{3a}{8}$.

Here $\dfrac{a}{4} \div \dfrac{3a}{8} = \dfrac{a}{4} \times \dfrac{8}{3a} = \dfrac{8a}{12a} = \dfrac{2}{3}$ the quotient.

---

* If the fractions to be divided have a common denominator, take the numerator of the dividend for a new numerator, and the numerator of the divisor for the new denominator. See 3. below.

2. When a fraction is to be divided by any quantity, the value is the same whether the numerator be divided by it, or the denominator multiplied by it.

3. When the two numerators, or the two denominators, can be divided by some common quantity, let that be done, and the quotients used instead of the fractions first proposed. This is obvious from the circumstance that if these quantities be suffered to remain, they constitute a common factor, or a common measure of the quantity which results from the division.

2. Required to divide $\frac{3a}{2b}$ by $\frac{5c}{4d}$.

Here $\frac{3a}{2b} \div \frac{5c}{4d} = \frac{3a}{2b} \times \frac{4d}{5c} = \frac{12ad}{10bc} = \frac{6ad}{5bc}$ the quotient.

3. To divide $\frac{2a+b}{3a-2b}$ by $\frac{3a+2b}{4a+b}$.

Here $\frac{2a+b}{3a-2b} \times \frac{4a+b}{3a+2b} = \frac{8a^2+6ab+b^2}{9a^2-4b^2}$ the quotient required.

4. To divide $\frac{3a^2}{a^3+b^3}$ by $\frac{a}{a+b}$.

Here $\frac{3a^2}{a^3+b^3} \times \frac{a+b}{a} = \frac{3a^2 \times (a+b)}{(a^3+b^3) \times a} = \frac{3a}{a^2-ab+b^2}$ is the quotient required.

5. To divide $\frac{3x}{4}$ by $\frac{11}{12}$, and $\frac{6x^2}{5}$ by $3x$ and $\frac{a+b}{a}$ by $-\frac{3a^3+3b^3}{3a^2}$.

6. To divide $\frac{3x+1}{9}$ by $\frac{4x}{3}$, and $\frac{4x}{2x-1}$ by $\frac{x}{3}$.

7. To divide $\frac{4x}{5}$ by $\frac{3a}{5b}$, and $\frac{2a-b}{4cd}$ by $\frac{5ac}{6d}$.

8. Divide $\frac{5a^4-5b^4}{2a^2-4ab+2b^2}$ by $\frac{6a^2+5ab}{4a-4b}$, and $-\frac{10^{-5}}{10^0}$ by $\frac{1^{-3}}{100}$.

9. Divide $a+\frac{2ax-1}{b}$ by $\frac{x-a}{ax+1}$, and 12 by $\frac{(a+x)^2}{x}-a$.

10. Suppose we divide x^3+y^3 by x^3-y^3, and then divide the result by the quotient of $x+y$ by $x-y$; what final quantity shall we get, provided $\frac{1}{2}x = -\frac{3}{4}y$?

11. If we add three-fourths of a number to one-half its square, and divide the sum by three-eighths of its cube; and moreover, if we subtract three-fourths of that number from half its square, and divide the result by *minus* one-eighth of the square; what is the quotient of the former by the latter result?

CASE X.

Continued fractions.

If we recur to the process for finding the greatest common measure, p. 133; and denote for simplicity the functions X, X_1, X_2 .... by a, b, c ..., and the quotients Q, Q_1, Q_2 .. by α, β, γ ...; and if we also denote the divisions on the left by fractions; then we shall have those two columns converted into the following:—

$$a + \cfrac{1}{\beta + \cfrac{1}{\gamma + \cfrac{1}{\delta + \dots}}}$$

or, as it is more conveniently written,

$$a + \frac{1}{\beta +}\,\frac{1}{\gamma +}\,\frac{1}{\delta +} \dots$$

$$\begin{aligned} a &= \alpha b + c, \\ b &= \beta c + d \\ c &= \gamma d + e \\ d &= \delta e + f \\ &\dots\dots\dots \end{aligned}$$

The expression (either form) on the left is called a *continued fraction*. It has many curious properties *, and in several inquiries is of great value. The only

* The idea of continued fractions was first started by Lord Brouncker, the first President of the Royal Society; but the method owes its present elegant form to Lagrange, who made ex-

one to which, however, the student of this course will have occasion to apply it, is to find a series of fractions converging towards the true value of the given fraction $\frac{a}{b}$, but having its terms expressed in smaller numbers than a and b.

The formation of the values of the terms of the converging fractions has been stated at p. 134, art. 4, where the upper line expresses the numerator, and the lower one the denominator of the converging fraction at the first, second, third, and successive steps.

EXAMPLES.

Ex. 1. Represent $\frac{365}{224}$ in the form of a continued fraction, and find the converging fractions.

	b	a	
$\beta = 1$	224	365	$1 = \alpha$
	141	224	
$\delta = 1$	83	141	$1 = \gamma$
	58	83	
$\zeta = 3$	25	58	$2 = \varepsilon$
	24	50	
	1	8	$8 = \eta$
		8	
		0	

Hence $$\frac{365}{224} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{8}}}}}}$$

Or, in Herschel's notation,

$$\frac{365}{224} = 1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{8}$$

And forming the converging fractions according to the rule, they are $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{13}{8}$, $\frac{44}{27}$, $\frac{365}{224}$, the last of which is the original fraction itself.

The properties of these fractions are:—

1. That they are alternately less and greater than the true value.
2. That each of them is in its lowest terms.
3. That each of them is a nearer value than any other that can be formed without taking higher values of the numerator and denominator.
4. That if $\frac{p_m}{q_m}$ and $\frac{p_{m+1}}{q_{m+1}}$ be two consecutive converging fractions, then $p_m\, q_{m+1} - p_{m+1}\, q_m = \pm 1$. See Hind's Algebra, pp. 284—306.

Ex. 2. Find the fractions converging to $\frac{314159}{100000}$.

---

tensive use of it, both in the solution of algebraical equations with numeral coefficients, and the solution of indeterminate equations.

The latest improvements in the use of the method as applied to the solution of algebraical equations, were made by the late Mr. Horner, and published in the Annals of Philosophy and Quarterly Journal. The second notation above given was proposed by Sir John Herschel, and for economy of space, both in writing and printing, is the most convenient. For a good view of the subject in an elementary form, the reader who wishes to go further into the subject may consult Hind's Algebra or Young's Equations:—both of them works deserving of cordial recommendation.

By the common measure we have the quotients 3, 7, 15, 1, ... ; and by the rule for the formation of the terms we have $\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots$

INVOLUTION AND EVOLUTION.

The term Involution has already been explained at pp. 118, 119, as the multiplication of several equal factors together; and it has been intimated that, for the case of the factors being binomial, a much more concise process will be given under the head of the *Binomial Theorem*. When there are more than two terms in the expression to be involved, it will, at least during elementary study, be better to have recourse to actual multiplication in the few cases that can arise, than to employ the *Multinomial Theorem*.

Evolution is the extraction of roots; that is, the inverse operation of *raising powers*, or of Involution. It constitutes but a very limited application of a general process, viz. that of the general solution of algebraic equations *. The rules usually given for Evolution are identical in substance with that here given for the square and cube roots; but in a form, though more briefly expressed, implying much more actual work in performing them. This rule is in fact only an expression of Horner's method of solving equations.

CASE I.

To extract any root of an expression composed of one single term.

Divide the indices of all its factors by the index of the root to be extracted, and extract by any arithmetical process the root of the numerical coefficient. The continued product of all these roots is the root sought.

Thus the n^{th} root of $pa^m b^r c^s$ is $p^{\frac{1}{n}} a^{\frac{m}{n}} b^{\frac{r}{n}} c^{\frac{s}{n}}$; and the cube root of $27a^3b^6c^9$ is $27^{\frac{1}{3}} a^{\frac{3}{3}} b^{\frac{6}{3}} c^{\frac{9}{3}}$ or $3ab^2c^3$.

Of the signs of the results, it is only necessary to recollect that all odd roots of a negative quantity are real and negative; of all positive roots, real and positive; that all even roots of a positive quantity are real, and either positive or negative; whilst all even roots of a negative quantity are impossible or imaginary. This, however, is to be understood as applying to one individual root in each single case: since, besides these, there may be several other roots, wholly imaginary; and in all instances above the square root there actually are such roots. This view of the subject, however, belongs to a more advanced stage of the study of algebra.

EXAMPLES.

1. The square root of $4a^2$, is $\pm 2a$.
2. The cube root of $8a^3$, is $2a^{\frac{3}{3}}$ or $2a$.
3. The square root of $\frac{5a^2b^2}{9c^2}$, is $\pm \frac{ab}{3c} \sqrt{5}$.

---

* An extension of the signification of the terms *Involution* and *Evolution* has been recently proposed by a distinguished mathematician, Professor De Morgan, viz. to the composition and resolution of algebraic equations. Involution and Evolution being particular cases of those general problems, the extension is perfectly justifiable; and in a treatise founded on this idea there would at least be the advantage of keeping those subjects together which were naturally connected with each other. It would, however, interfere too much with the existing arrangement of the work to adopt it here.

4. The cube root of $-\dfrac{16a^4b^6}{27c^3}$, is $-\dfrac{2ab^2}{3c}\sqrt[3]{2a}$.
5. The square root of $2a^2b^4$, is $\pm ab^2\sqrt{2}$.
6. The cube root of $-64a^3b^6$, is $-4ab^2$.
7. The square root of $\dfrac{8a^2b^2}{3c^3}$, is $\pm\dfrac{2ab}{c}\sqrt{\dfrac{2}{3c}}$.
8. The 4th root of $81a^4b^6$, is $\pm 3ab\sqrt{\pm b}$.
9. The 5th root of $-32a^5b^6$, is $-2ab\sqrt[5]{b}$.
10. The 6th root of $729a^6b^{12}$, is $\pm 3ab^2$.

CASE II.

To extract any root of a compound quantity.

The breadth of our page will not allow us to exhibit an example of a higher root than the third; but we shall enunciate the rule for all cases, and give the form of work for the square and cube roots.

1. Write the given quantity, arranged according to the powers of some one letter * in the place of the dividend, and the curve to the right for the root, as in the arithmetical square and cube roots, pp. 66—72.

2. Make N columns to the left of the given expression, numbering them backwards from that expression as columns (I), (II), (III), ... (N).

3. Extract the N[th] root of the first term of the given expression, and put that root in the column to the right of the curve. Denote, for the purpose of continuing the directions for working, this root by r, whatever that root may be.

4. Put 1 in column N; $1 \times r$ in col. $(N-1)$; $(1 \times r)\,r$ in column $(N-2)$; and continue the process till the last result falls under the given expression: then evidently this result will be equal to the first term of the given expression. Subtract this, and bring down N terms for a dividend.

5. Form a new horizontal line as follows. Multiply 1 by r, place it in column (N—1), and add it to the previous result in that column; multiply this sum by r, and add it to the next column; multiply this sum by r, and add it to the next; and so on till the result falls in column (I).

Then form a new horizontal line in the same manner, adding each product to the result above which it is written; but stop in each horizontal line one column sooner than in the preceding. We shall thus obtain a series of results, one in each column, preparatory to evolving the next term of the root.

6. With the expression in col. (I) as a divisor and the first term of the new dividend, find a new term of the quotient. This will be the second term of the root, which is to be used in forming the several columns, as before described.

Proceeding thus, we shall obtain the successive terms of the root, if it be an exact root, or as many of them as we desire, if it be not exact.

EXAMPLES.

1. Extract the square root $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.

Here the index of the root is 2, and the highest power of a is the fourth. Hence $a^{\frac{4}{2}} = a^2$, and the condition is fulfilled. Also the powers follow regularly, and we can work with detached coefficients; but to show the identity of the methods, the work is put down here both ways. Also as $n = 2$, there will be two columns, and we have

---

* The rule cannot be applied except the first term is of a power exactly divisible by the index of the root; that is, for the n^{th} root we must have the first term x^{mn} where m is an integer.

$$
\begin{array}{lll}
\text{(II)} & \text{(I)} & \qquad\qquad\text{(O)} \\
1 & a^2 & a^4 - 4a^3b + 6a^2b^2 - 4a^3b + b^4\ (a^2 - 2ab + b^2. \\
 & a^2 & a^4 \\
\hline
 & 2a^2 - 2ab & \quad - 4a^3b + 6a^2b^2 \\
 & \quad\ \ - 2ab & \quad - 4a^3b + 4a^2b^2 \\
\hline
 & 2a^2 - 4ab + b^2 & \qquad\quad 2a^2b^2 - 4a^3b + b^4 \\
 & & \qquad\quad 2a^2b^2 - 4a^3b + b^4, \\
\hline
\end{array}
$$

which is worked in strict accordance with the rule, and it is evidently identical with that given for the square root of numbers, p. 67.

The same example, by detached coefficients.

$$
\begin{array}{lll}
\text{(II)} & \text{(I)} & \qquad\quad\text{(O)} \\
1 & 1 & 1 - 4 + 6 - 4 + 1\ (1 - 2 + 1 \\
 & 1 & 1 \\
\hline
 & 2 - 2 & \ - 4 + 6 \\
 & \ \ \ - 2 & \ - 4 - 4 \\
\hline
 & 2 - 4 + 1 & \qquad 2 - 4 + 1 \\
 & & \qquad 2 - 4 + 1 \\
\hline
\end{array}
$$

Ex. 2. Extract the square root of $a^2 - x^2$.

Here $a^{\frac{2}{2}} = a$, and the condition of the applicability of the rule is fulfilled. Write it under the form $a^2\,(1 - \frac{x^2}{a^2})$: then we have to multiply the square root of $1 - \frac{x^2}{a^2}$ by that of a^2, or by $\pm\, a$. Considering $-\frac{x^2}{a^2}$ as the second term, the detached coefficients are $1 - 1$; and the process will be as follows:—

$$
\begin{array}{lll}
\text{(II)} & \text{(I)} & \qquad\text{(O)} \\
1 & 1 & 1 - 1\ (1 - \frac{1}{2} - \frac{1}{8} - \frac{1}{16} - \ldots\ldots \\
 & 1 & 1 \\
\hline
 & 2 - \frac{1}{2} & \ - 1 + 0 \\
 & \ \ \ - \frac{1}{2} & \ - 1 + \frac{1}{4} \\
\hline
 & 2 - 1 - \frac{1}{8} & \qquad - \frac{1}{4} \\
 & \qquad\ \ - \frac{1}{8} & \qquad - \frac{1}{4} + \frac{1}{8} + \frac{1}{64} \\
\hline
 & 2 - 1 - \frac{1}{4} - \frac{1}{16} & \qquad\qquad - \frac{1}{8} - \frac{1}{64} \\
 & & \qquad\qquad - \frac{1}{8} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} \\
\hline
 & & \qquad\qquad\qquad - \frac{5}{64} - \frac{1}{64} - \frac{1}{256}\,;
\end{array}
$$

in which the method of work is very simple and easy, and attaching the letters to the coefficients, we have $\pm\, a \left\{1 - \frac{x^2}{2a^2} - \frac{x^4}{8a^4} - \frac{x^6}{16a^6} - \ldots\ldots\right\}$, or multiplying out it becomes $\pm \left\{a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \ldots\ldots\right\}$.

Ex. 3. Extract the cube root of $a^6 - 6a^5 + 21a^4 - 44a^3 + 63a^2 - 54a + 27$.

Here there are three columns, and the condition is fulfilled for the application of the rule. Whence

$$\begin{array}{llll}
\text{(III)} & \text{(II)} & \text{(I)} & \text{(O)} \\
1 & 1 & 1 & 1-6+21-44+63-54+27\ (1-2+3 \\
 & 1 & 2 & 1 \\
 & 2 & 3 & -6+21-44 \\
 & 1 & -\ 6+\ 4 & -6+12-\ 8 \\
 & 3-2 & 3-\ 6+\ 4 & 9-36+63-54+27 \\
 & -2 & -\ 6+\ 8 & 9-36+63-54+27 \\
 & 3-4 & 3-12+12 & \\
 & -2 & \qquad\quad 9-18+9 & \\
 & 3-6+3 & 3-12+21-18+9, &
\end{array}$$

and attaching the powers, we have the root $a^2 - 2a + 3$.

The several terminated courses of operations are marked by dark lines immediately above their results.

4. The square root of $a^4 + 4a^3b + 10a^2b^2 + 12ab^3 + 9b^4$ is $a^2 + 2ab + 3b^2$.

5. To find the square root of $a^4 + 4a^3 + 6a^2 + 4a + 1$, and of $1 + 4a + 6a^2 + 4a^3 + a^4$. Ans. $a^2 + 2a + 1$, and $1 + 2a + a^2$.

6. Extract the square root of $a^4 - 2a^3 + 2a^2 - a + \frac{1}{4}$. Ans. $a^2 - a + \frac{1}{2}$.

7. The square root of $a^2 - ab$ is $a - \dfrac{b}{2} - \dfrac{b^2}{8a} - \dfrac{b^3}{16a^2} -$ &c.

8. Find the square root of $(a^2 + x^2)$ and of $(x^2 + a^2)$.

9. The square root of $a^2 - 2ab + 2ax + b^2 - 2bx + x^2$, is $a - b + x$.

10. The cube root of $a^6 - 3a^5 + 9a^4 - 13a^3 + 18a^2 - 12a + 8$, is $a^2 - a + 2$.

11. The 4th root of $81a^4 - 216a^3b + 216a^2b^2 - 96ab^3 + 16b^4$, is $3a - 2b$.

12. The 5th root of $a^5 - 10a^4 + 40a^3 - 80a^2 + 80a - 32$, is $a-2$.

13. Find the square root of $1 - y^2$ and the cube root of $1 - x^3$.

14. Expand $\sqrt{(a^2 - 2bx - x^2)}$ and $\sqrt{a^2 - x^2}$ into infinite series.

15. Expand $\sqrt{(1 + 1)} = \sqrt{2}$ into a series. Ans. $1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} + \ldots$

16. Expand $\sqrt{(1 - 1)}$ into an infinite series. Ans. $1 - \frac{1}{2} - \frac{1}{8} - \frac{1}{16} - \frac{5}{128} - \ldots$

17. Expand $\sqrt{(a^2 + x)}$ into an infinite series.

In all these cases, however, the *binomial theorem* will be more convenient and effective.

SURDS.

[The student is advised to defer this subject till he has read some portion of the simple and quadratic equations.]

Surds are such quantities as have no exact root, and are usually expressed by fractional indices, or by means of the radical sign $\sqrt{}$. Thus, $3^{\frac{1}{2}}$, or $\sqrt{3}$, denotes the square root of 3; and $2^{\frac{2}{3}}$, or $\sqrt[3]{2^2}$, or $\sqrt[3]{4}$, the cube root of the square of 2; where the numerator shows the power to which the quantity is to be raised, and the denominator its root. The index may be put also in the form of a decimal, and is often so used. As for $a^{\frac{1}{2}}$ writing $a^{.5}$, or for $a^{-\frac{1}{2}}\,b^{\frac{1}{3}}$, writing $a^{-.5}\,b^{.333\cdots}$ See *Definitions*, p. 107.

PROBLEM I.

To reduce a rational quantity to the form of a surd.

Raise the given quantity to the power denoted by the index of the surd; then over or before this new quantity set the radical sign, and it will be of the form required.

Note. When any radical quantity has a rational coefficient, this coefficient may be put under the irrational form, and the whole of the factors thereby brought under the symbol of radicality. Thus, instead of $3a\sqrt{b}$, we may put $\sqrt{3a \times 3a \times b}$, or $\sqrt{9a^2b}$; and so of others.

EXAMPLES.

1. To reduce 4 and -4 to the form of the square root.
   First, $4^2 = 4 \times 4 = 16$; then $+\sqrt{16}$ is the answer.
   Second, $(-4)^2 = -4 \times -4 = 16$, then $-\sqrt{16}$ is the answer.
2. To reduce $3a^2$ to the form of the cube root.
   First, $3a^2 \times 3a^2 \times 3a^2 = (3a^2)^3 = 27a^6$; then we have $\sqrt[3]{27a^6}$ or $(27a^6)^{\frac{1}{3}}$.
3. Reduce 6 to the form of the cube root. Ans. $(216)^{\frac{1}{3}}$ or $\sqrt[3]{216}$.
4. Reduce $-\frac{1}{2}ab$ to the form of the square root. Ans. $-\sqrt{\frac{1}{4}a^2b^2}$.
5. Reduce ± 2 to the form of the 4th root. Ans. $\pm (16)^{\frac{1}{4}}$.
6. Reduce $a^{\frac{1}{3}}$ to the form of the 5th root.
7. Reduce $a \pm x$ and $x \pm a$ to the form of the square root.
8. Reduce $a - x$ to the form of the cube root.
9. Reduce $(-4a^2 + b)\sqrt[3]{-a^2b}$ to the form of the sixth root; and likewise to the form of the square root.
10. Transform $(a-b)(a+b)\sqrt{a^2-b^2}$ into its simplest form, and likewise represent them as radicals of the 3d, 4th, 5th, and 6th degrees.
11. Reduce $(4-1) \times 3$ to the form of a cube root; and then reduce $\pm bc\sqrt{\pm 1} - bc\sqrt{\pm 1}$ to the simplest form it admits of.

PROBLEM II.

To reduce quantities to a common index.

1. Reduce the indices of the given quantities to a common denominator, and involve each of them to the power denoted by its numerator; then 1 set over the common denominator will form the common index. Or,

2. If the common index be given, divide the indices of the quantities by the given index, and the quotients will be the new indices for those quantities.—Then over the said quantities, with their new indices, set the given index, and they will make the equivalent quantities sought.

EXAMPLES.

1. Reduce $3^{\frac{1}{2}}$ and $5^{\frac{1}{5}}$ to a common index.

Here, $\frac{1}{2}$ and $\frac{1}{5} = \frac{5}{10}$ and $\frac{2}{10}$.

Therefore $3^{\frac{5}{10}}$ and $5^{\frac{2}{10}} = (3^5)^{\frac{1}{10}}$ and $(5^2)^{\frac{1}{10}} = \sqrt[10]{3^5}$ and $\sqrt[10]{5^2} = \sqrt[10]{243}$ and $\sqrt[10]{25}$.

2. Reduce a^3 and $b^{\frac{1}{3}}$ to the same common index $\frac{1}{2}$.

Here, $\frac{3}{1} \div \frac{1}{2} = \frac{3}{1} \times \frac{2}{1} = \frac{6}{1}$ the 1st index.

and $\frac{1}{3} \div \frac{1}{2} = \frac{1}{3} \times \frac{2}{1} = \frac{2}{3}$ the 2d index.

Therefore $(a^6)^{\frac{1}{2}}$ and $(b^{\frac{2}{3}})^{\frac{1}{2}}$ or $\sqrt{a^6}$ and $\sqrt{b^{\frac{2}{3}}}$ are the quantities.

3. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{2}}$ to the common index $\frac{1}{4}$, and $3^{\cdot 5}$ and $5^{-\cdot 4}$ to the common index $\cdot 2$, and then to the common index $-\cdot 2$.

Ans. $(256^{\frac{1}{3}})^{\frac{1}{4}}$ and $25^{\frac{1}{4}}$; $(3^{2\cdot 5})^{\cdot 2}$ and $(5^{-\cdot 2})^{\cdot 2}$; $(3^{-2\cdot 5})^{-\cdot 2}$ and $(5^{\cdot 2})^{-\cdot 2}$.

4. Reduce $a^{\frac{1}{3}}$ and $x^{\frac{1}{4}}$ to the common index $\frac{1}{6}$. Ans. $(a^2)^{\frac{1}{6}}$ and $(x^{\frac{3}{2}})^{\frac{1}{6}}$.

5. Reduce a^2 and x^3 to the same radical sign. Ans $\sqrt{a^4}$ and $\sqrt{x^6}$.

6. Reduce $(a + x)^{\frac{1}{3}}$ and $(a - x)^{\frac{1}{2}}$ to a common index.

7. Reduce $(a + b)^{\cdot 5}$ and $(a - b)^{-\cdot 25}$ to a common index.

8. Transform $a^{-\frac{1}{2}}b^{-\cdot 6}c^3d^{3\cdot 25}$ to another quantity whose index is $- 3\cdot 25$.

PROBLEM III.

To reduce surds to simpler forms.

Divide the surd, if possible, into two factors, one of which is a power of the kind that accords with the root sought; as a complete square, if it be a square root; a complete cube, if it be a cube root; and so on. Set the root of this complete power before the surd expression which indicates the root of the other factor; and the quantity is reduced as required.

If the surd be a fraction, the reduction is effected by multiplying both its numerator and denominator by some number that will transform the denominator into a complete square, cube, or other requisite power: its root will be the denominator to a fraction that will stand before the remaining part, or surd. See Ex. 3, below.

Recollect that $\sqrt{\frac{1}{n}} = \frac{1}{n}\sqrt{n}$: thus $\sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$, $\sqrt{\frac{1}{3}} = \frac{1}{3}\sqrt{3}$, and so on.

EXAMPLES.

1. To reduce $\sqrt{32}$ to simpler terms.

Here $\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4 \times \sqrt{2} = 4\sqrt{2}$.

2. To reduce $\sqrt[3]{(320)}$ to simpler terms.

$\sqrt[3]{320} = \sqrt[3]{(64 \times 5)} = \sqrt[3]{64} \times \sqrt[3]{5} = 4 \times \sqrt[3]{5} = 4\sqrt[3]{5}$.

3. Reduce $\sqrt{\frac{44}{75}}$ to simpler terms.

$$\sqrt{\frac{44}{75}} = \sqrt{\frac{44}{15.5}} = \sqrt{\left(\frac{44}{15.5}.\ \frac{5}{5}\right)} = \sqrt{\frac{4.11.5}{15.15}} = \sqrt[3]{\frac{2^2.55}{15^2}} = \frac{2}{15}\sqrt{55}.$$

4. Reduce $\sqrt{75}$ to its simplest terms. Ans. $5\sqrt{3}$.

5. Reduce $\sqrt[3]{-189}$ to its simplest terms. Ans. $-3\sqrt[3]{7}$.

6. Reduce $\sqrt[3]{\pm\frac{135}{32}}$ to its simplest terms. Ans. $\pm\frac{3}{4}\sqrt[3]{10}$.

7. Reduce $\sqrt{-75a^2b}$ to its simplest terms. Ans. $5a\sqrt{3b}\sqrt{-1}$.

8. Express the square root of $\pm a^2b^3c^4$ in the simplest form.

Note I. There are other cases of reducing algebraic surds to simpler forms, that are practised on several occasions; one of which, on account of its simplicity and usefulness, may be here noticed, viz. in fractional forms, having compound surds in the denominator, multiply both numerator and denominator

by the same terms of the denominator, but having one sign changed, from + to − or from − to +, which will reduce the fraction to a rational denominator.

Ex. 1. To reduce $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$, multiply it by $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$, and it becomes $\frac{16+4\sqrt{15}}{2}=8+2\sqrt{15}$.

Also, to reduce $\frac{3\sqrt{15}-4\sqrt{5}}{\sqrt{15}+\sqrt{5}}$, multiply it by $\frac{\sqrt{15}-\sqrt{5}}{\sqrt{15}-\sqrt{5}}$, and it becomes $\frac{65-7\sqrt{75}}{15-5}=\frac{65-35\sqrt{3}}{10}=\frac{13-7\sqrt{3}}{2}$.

And the same method may easily be applied to examples with three or more surds.

Ex. 2. Reduce the fractions $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{5}}\times\frac{\sqrt{18}-\sqrt{16}}{\sqrt{2}-4}$, and $\frac{-6+\sqrt{x^3}}{15-\sqrt{x^3}}$ to others having rational denominators: and then to such as have rational numerators.

Note II. In the same manner may any binomial surd be rendered rational in the denominator, whatever the degree of the radicals may be. If, for instance, the surd had been $\frac{c}{\sqrt[3]{a}\pm\sqrt[3]{b}}$, then the multiplier would be $\sqrt[3]{a^2}\mp\sqrt[3]{ab}+\sqrt[3]{b^2}$, and the surd itself become $\frac{c(\sqrt[3]{a^2}\mp\sqrt[3]{ab}+\sqrt[3]{b^2})}{a\pm b}$.

And generally since $\frac{a\pm b}{\sqrt[n]{a}\pm\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}\mp\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}\mp\ldots\ldots$ by actual division, the rule may be extended as we have stated above.

Ex. Rationalize the denominator of $\frac{2}{\sqrt[3]{7}-\sqrt[3]{5}}$; and of $\frac{45}{2\pm\sqrt[3]{3}}$; and of $\frac{3}{\sqrt[3]{5}-\sqrt[3]{2}}$; and of $\frac{\sqrt[3]{3}+\sqrt[3]{4}}{\sqrt[4]{3}+\sqrt[4]{4}}$.

PROBLEM IV.

To add surd quantities together.

1. Bring all fractions to a common denominator, and reduce the quantities to their simplest terms, as in the last problem. 2. Reduce also such quantities as have unlike indices to other equivalent ones, having a common index. 3. Then if the surd part be the same in them all, annex it to the sum of the rational parts, with the sign of multiplication, and it will give the total sum required.

But if the surd part be not the same in all the quantities, the addition can only be indicated by the signs + and −.

EXAMPLES.

1. Required to add $\sqrt{18}$ and $\sqrt{32}$ together.

First, $\sqrt{18}=\sqrt{9\times 2}=3\sqrt{2}$; and $\sqrt{32}=\sqrt{16\times 2}=4\sqrt{2}$.
Then, $3\sqrt{2}+4\sqrt{2}=(3+4)\sqrt{2}=7\sqrt{2}=$ sum required.

2. It is required to add $\sqrt[3]{375}$ and $\sqrt[3]{192}$ together.

First, $\sqrt[3]{375}=\sqrt[3]{125\times 3}=5\sqrt[3]{3}$; and $\sqrt[3]{192}=\sqrt[3]{64\times 3}=4\sqrt[3]{3}$.
Then, $5\sqrt[3]{3}+4\sqrt[3]{3}=(5+4)\sqrt[3]{3}=9\sqrt[3]{3}=$ sum required.

3. Required the sum of $\sqrt{27}$ and $\sqrt{48}$. Ans. $7\sqrt{3}$.
4. Required the sum of $-\sqrt{50}$ and $\sqrt{72}$. Ans. $\sqrt{2}$.
5. Required the sum of $-\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{1}{15}}$. Ans. $-\frac{2}{15}\sqrt{15}$.

6. Required the sum of $\sqrt[3]{56}$ and $\sqrt[3]{-189}$. Ans. $-\sqrt[3]{7}$.

7. Required the sum of $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[3]{\frac{1}{32}}$. Ans. $\frac{3}{4}\sqrt[3]{2}$.

8. Required the sum of $3\sqrt{a^2b}$ and $5\sqrt{16a^4b}$. Ans. $(3a+20a^2)\sqrt{b}$.

9. Find the sum of $\sqrt[3]{-27a^3}$ and $\pm\sqrt{64a^2}$. Ans. $5a$ or $-11a$.

10. Find the sum of $-\dfrac{\sqrt{8}-\sqrt{2}}{\sqrt{8}+\sqrt{2}}$ and $\dfrac{\sqrt{5}-\sqrt{15}}{\sqrt{5}+\sqrt{15}}$: also of $a\sqrt{8}$, $b\sqrt{18}$, $a\sqrt{27}$, $-b\sqrt{45}$, $b\sqrt{125}$, and $a\sqrt{147}$; and again of $-2\sqrt{32}$, $9.243^{\cdot 5}$, $5.68^{\cdot 5}$, $17 \times 54^{\frac{1}{3}}$, $3\sqrt[3]{432}$, $\sqrt[3]{128}$, $\sqrt{1452}$, $\sqrt[3]{1458}$, $363^{\cdot 5}$, and $11\sqrt{1331}$.

11. What is the sum of $-48^{\cdot 5}a - 25^{\cdot 2}b$, and $3^{-\cdot 5}a - 6^{-\cdot 5}b$?

PROBLEM V.

To find the difference of surd quantities.

PREPARE the quantities the same way as in the last rule; then subtract the rational parts, and to the remainder annex the common surd, for the difference of the surds required.

But if the quantities have no common surd, the subtraction can only be indicated by means of the sign $-$.

EXAMPLES.

1. To find the difference between $\sqrt{320}$ and $\sqrt{80}$.

First, $\sqrt{320} = \sqrt{64 \times 5} = 8\sqrt{5}$; and $\sqrt{80} = \sqrt{16 \times 5} = 4\sqrt{5}$.

Then, $8\sqrt{5} - 4\sqrt{5} = 4\sqrt{5}$ the difference sought.

2. To find the difference between $\sqrt[3]{128}$ and $\sqrt[3]{54}$.

First, $\sqrt[3]{128} = \sqrt[3]{64 \times 2} = 4\sqrt[3]{2}$; and $\sqrt[3]{54} = \sqrt[3]{27 \times 2} = 3\sqrt[3]{2}$.

Then, $4\sqrt[3]{2} - 3\sqrt[3]{2} = \sqrt[3]{2}$, the difference required.

3. Required the difference of $\sqrt{75}$ and $\sqrt{48}$. Ans. $\sqrt{3}$.

4. Required the difference of $\sqrt[3]{256}$ and $\sqrt[3]{32}$. Ans. $2\sqrt[3]{4}$.

5. Required the difference of $\sqrt{\frac{3}{4}}$ and $\sqrt{\frac{1}{3}}$. Ans. $\frac{1}{6}\sqrt{3}$.

6. Find the difference of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{3}{8}}$. Ans. $\frac{1}{12}\sqrt{6}$.

7. Required the difference of $\sqrt[3]{\frac{3}{5}}$ and $\sqrt[3]{\frac{25}{9}}$. Ans. $\frac{2}{15}\sqrt[3]{75}$.

8. Find the difference of $\sqrt{24a^2b^2}$ and $\sqrt{54b^4}$. Ans. $(3b^2-2ab)\sqrt{6}$.

9. Subtract $-9^{\cdot 5}$ from $64^{-\frac{1}{3}}$ and add the sum to the difference of $\sqrt{\frac{3}{4}}$ and $3^{-\frac{1}{2}}$.

10. From the half of $-\sqrt{\frac{3}{4}}$ take a third of $\sqrt{\frac{1}{3}}$.

11. Find the difference of $\sqrt{8} - \sqrt{2}$ and $\sqrt{252} - \sqrt{175}$. Ans. $1{\cdot}2315377$.

12. Subtract $-\dfrac{\sqrt{8}-\sqrt{2}}{\sqrt{8}+\sqrt{2}}$ from $\dfrac{\sqrt{252}+\sqrt{175}}{\sqrt{252}-\sqrt{175}}$, and $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{20}+\sqrt{12}}$ from $\dfrac{\sqrt{15}-\sqrt{5}}{\sqrt{15}+\sqrt{5}}$. Ans. $\dfrac{34}{3}$, and $-\dfrac{1}{2}\sqrt{3}\{2-\sqrt{5}\}$.

13. Take $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$ from $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$, and $\dfrac{\sqrt{5}-\sqrt{1}}{\sqrt{5}+\sqrt{1}}$ from $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$.

PROBLEM VI.

To multiply surd quantities together.

REDUCE the surds to the same index, if necessary; next multiply the rational quantities together, and the surds together; then annex the one product to the

other for the whole product required; which may be reduced to more simple terms if necessary.

EXAMPLES.

1. Required to find the product of $4\sqrt{12}$ and $3\sqrt{2}$.

Here $4 \times 3 \times \sqrt{12} \times \sqrt{2} = 12\sqrt{12 \times 2} = 12\sqrt{24} = 12\sqrt{4 \times 6} = 12 \times 2 \times \sqrt{6} = 24\sqrt{6}$, the product required.

2. Required to multiply $\frac{1}{4}\sqrt[3]{\frac{3}{4}}$ by $\frac{1}{3}\sqrt[3]{\frac{3}{8}}$.

Here $\frac{1}{4} \times \frac{1}{3}\sqrt[3]{\frac{3}{4}} \times \sqrt[3]{\frac{3}{8}} = \frac{1}{12} \times \sqrt[3]{\frac{9}{32}} = \frac{1}{12} \times \sqrt[3]{\frac{18}{64}} = \frac{1}{12} \times \frac{1}{4} \times \sqrt[3]{18} = \frac{1}{48}\sqrt[3]{18}$, the product required.

3. Required the product of $3\sqrt{2}$ and $2\sqrt{8}$. Ans. 24.
4. Required the product of $\frac{1}{3}\sqrt[3]{4}$ and $\frac{3}{4}\sqrt[3]{12}$. Ans. $\frac{1}{2}\sqrt[3]{6}$.
5. To find the product of $-\frac{5}{3}\sqrt{\frac{3}{8}}$ and $\frac{9}{10}\sqrt{\frac{2}{5}}$. Ans. $-\frac{3}{20}\sqrt{15}$.
6. Required the product of $2\sqrt[3]{-14}$ and $3\sqrt[3]{4}$. Ans. $-12\sqrt[3]{7}$.
7. Required the product of $2a^{\frac{2}{3}}$ and $a^{\frac{4}{3}}$. Ans. $2a^2$.
8. Required the product of $(a+b)^{\frac{1}{n}}$ and $(a+b)^{\frac{3}{4}}$.
9. Required the product of $2x + \sqrt{b}$ and $2x - \sqrt{b}$.
10. Required the product of $-(a + 2\sqrt{b})^{\frac{1}{2}}$, and $(a - 2\sqrt{b})^{\frac{1}{2}}$.
11. Required the product of $-2x^{\frac{1}{n}}$ and $-3x^{\frac{1}{n}}$
12. Required the product of $4x^{\frac{1}{n}}$ and $2y^{\frac{1}{n}}$.
13. Multiply $\frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{3}}$ by $\frac{3\sqrt{2}-\sqrt{3}}{5}$, and $\sqrt{3}\,.\,(\sqrt{5}-1)$ by $\sqrt{27}\,(\sqrt{5}+1)$;

and add the resulting products together.

14. Required the continued product of $\frac{4(a-b)}{7+3\sqrt{5}}$, $\frac{\sqrt{2}\,.\,b^{\frac{1}{2}}}{1+\sqrt{3}}$, $\frac{7-3\sqrt{5}}{2}\,.\,(a+b)$ and $-\frac{2}{\sqrt{6}+\sqrt{2}}(a^2-b^2)^{-1}b^{-5}$.

15. Reduce $(\pm a)^{2} \times (\pm a)^{-5} \times (\pm a)^{7} \times (\pm a)^{4}$ to a single term.

PROBLEM VII.

To divide one surd quantity by another.

Reduce the surds to the same index, if necessary; then take the quotient of the rational quantities, and annex it to the quotient of the surds, and it will give the whole quotient required; when the result can be reduced to more simple terms, it should then be done.

EXAMPLES.

1. Required to divide $6\sqrt{96}$ by $3\sqrt{8}$.

Here $6 \div 3\,.\,\sqrt{(96 \div 8)} = 2\sqrt{12} = 2\sqrt{(4 \times 3)} = 2 \times 2\sqrt{3} = 4\sqrt{3}$, the quotient required.

2. Required to divide $12\sqrt[3]{280}$ by $3\sqrt[3]{5}$.

Here $12 \div 3 = 4$, and $\sqrt[3]{56} = \sqrt[3]{8} \times \sqrt[3]{7} = 2\sqrt[3]{7}$;

therefore $4 \times 2 \times \sqrt[3]{7} = 8\sqrt[3]{7}$, is the quotient required.

3. Let $4\sqrt{50}$ be divided by $2\sqrt{5}$. Ans. $2\sqrt{10}$.
4. Let $6\sqrt[3]{100}$ be divided by $3\sqrt[3]{-5}$. Ans. $-2\sqrt[3]{20}$.

5. Let $\frac{5}{6}\sqrt{\frac{1}{30}}$ be divided by $\frac{3}{4}\sqrt{\frac{2}{5}}$. Ans. $\frac{1}{9}\sqrt{5}$.

6. Let $\frac{3}{4}\sqrt[3]{\frac{3}{16}}$ be divided by $\frac{3}{5}\sqrt[3]{\frac{2}{5}}$. Ans. $\frac{5}{16}\sqrt[3]{30}$.

7. Let $\frac{4}{5}\sqrt{a}$, or $\frac{4}{5}a^{\frac{1}{2}}$, be divided by $\frac{2}{3}a^{\frac{1}{3}}$. Ans. $\frac{6}{5}a^{\frac{1}{6}}$.

8. Let $-a^{\frac{4}{3}}$ be divided by $4a^{\frac{2}{3}}$, and $5a^{-2}$ by $-a^{\frac{1}{4}}$.

9. To divide $3a^{\frac{1}{n}}$ by $4a^{\frac{1}{m}}$, and $3a^{-\frac{1}{n}}$ by $4a^{-\frac{1}{m}}$.

PROBLEM VIII.

To involve or raise surd quantities to any power.

Raise both the rational part and the surd part. Or multiply the index of the quantity by the index of the power to which it is to be raised, and to the result annex the power of the rational parts, which will give the power required.

EXAMPLES.

1. Required to find the square of $\frac{3}{4}a^{\frac{1}{2}}$.

First, $(\frac{3}{4})^2 = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$, and $(a^{\frac{1}{2}})^2 = a^{\frac{1}{2} \times 2} = a^{\frac{2}{2}} = a$;

therefore $(\frac{3}{4}a^{\frac{1}{2}})^2 = \frac{9}{16}a$, is the square required.

2. Required to find the square of $\frac{1}{2}a^{\frac{2}{3}}$.

First, $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, and $(a^{\frac{2}{3}})^2 = a^{\frac{4}{3}} = a^3\sqrt{a}$;

therefore $(\frac{1}{2}a^{\frac{2}{3}})^2 = \frac{1}{4}a^3\sqrt{a}$, is the square required.

3. Required to find the cube of $\frac{2}{3}\sqrt{6}$ or $\frac{2}{3} \times 6^{\frac{1}{2}}$.

First, $(\frac{2}{3})^3 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$, and $(6^{\frac{1}{2}})^3 = 6^{\frac{3}{2}} = 6\sqrt{6}$;

therefore $(\frac{2}{3}\sqrt{6})^3 = \frac{8}{27} \times 6\sqrt{6} = \frac{16}{9}\sqrt{6}$, the cube required.

4. Required to find the square of $2\sqrt[3]{2}$. Ans. $4\sqrt[3]{4}$.

5. Required the cube of $3^{\frac{1}{2}}$, or $\sqrt{3}$. Ans. $3\sqrt{3}$.

6. Required the 3d power of $-\frac{1}{3}\sqrt{3}$. Ans. $-\frac{1}{9}\sqrt{3}$.

7. Required to find the 4th power of $\frac{1}{2}\sqrt{2}$. Ans. $\frac{1}{4}$.

8. Required to find the $-m$th power of $-a^{-\frac{1}{n}}$.

9. Required to find the square of $2 \pm \sqrt{3}$.

PROBLEM IX.

To evolve or extract the roots of surd quantities.

I. When the given expression contains but one term, extract both the rational part and the surd part. Or divide the index of the given quantity by the index of the root to be extracted; then to the result annex the root of the rational part, which will give the root required.

EXAMPLES.

1. Required to find the square root of $16\sqrt{6}$.

First, $\sqrt{16} = 4$, and $(6^{\frac{1}{2}})^{\frac{1}{2}} = 6^{\frac{1}{2} \div 2} = 6^{\frac{1}{4}}$;

therefore $(16\sqrt{6})^{\frac{1}{2}} = 4 . 6^{\frac{1}{4}} = \pm 4\sqrt[4]{6}$, is the square root required.

2. Required to find the cube root of $\frac{1}{27}\sqrt{3}$.

First, $\sqrt[3]{\frac{1}{27}} = \frac{1}{3}$, and $(\sqrt{3})^{\frac{1}{3}} = 3^{\frac{1}{2} \div 3} = 3^{\frac{1}{6}}$;

therefore $(\frac{1}{27}\sqrt{3})^{\frac{1}{3}} = \frac{1}{3} . 3^{\frac{1}{6}} = \frac{1}{3}\sqrt[6]{3}$, is the cube root required.

3. Required the square root of 6^3. Ans. $\pm 6\sqrt{6}$.
4. Required the cube root of $\frac{1}{8}a^3b$. Ans. $\frac{1}{2}a\sqrt[3]{b}$.
5. Required the 4th root of $16a^2$. Ans. $\pm 2\sqrt{\pm a}$.
6. Required to find the $-m$th root of $x^{\frac{1}{n}}$.
7. Required the square root of $a^2 - 6a\sqrt{b} + 9b$.

II. To extract the square root of a binomial quadratic surd as of $a \pm \sqrt{b}$.

Its root is $\sqrt{\dfrac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\dfrac{a \mp \sqrt{a^2 - b}}{2}}$.

(1.) The product of two quadratic surds will be a surd, except when one of them is some rational multiple (integer or fractional) of the other.

Let $\sqrt{x} \cdot \sqrt{y} = m$: then

$$\sqrt{y} = \frac{m}{\sqrt{x}} = \frac{m}{x}\sqrt{x}$$

Or thus:—

Let $y = mx$ then $\sqrt{y} = \sqrt{mx}$,
and $\sqrt{xy} = \sqrt{mx^2} = x\sqrt{m}$.

where, except $\sqrt{m}$ be rational, the result is irrational.

Hence, if m be rational, $\sqrt{y}$ is a rational multiple of $\sqrt{x}$; if not, not. Whence the proposition is true.

(2). One quadratic surd cannot be the sum or difference either of a rational quantity and a quadratic surd, or two quadratic surds.

For, first, assume $\sqrt{z} = x \pm \sqrt{y}$; then squaring, $z = x^2 \pm 2x\sqrt{y} + y$,

$$\text{or } \sqrt{y} = \pm \frac{z - x^2 - y}{2x}.$$

Whence, if we suppose $\sqrt{z} = x \pm \sqrt{y}$, we shall have a surd equal to a rational quantity, which is contrary to the definition of a surd.

Again, secondly, assume $\sqrt{z} = \sqrt{x} \pm \sqrt{y}$: then squaring, we have

$$z = x \pm 2\sqrt{x} + y, \text{ or } \sqrt{xy} = \pm \tfrac{1}{2}(z - x - y), \text{ or again,}$$

a surd equivalent to a rational quantity, which is contrary to definition.

This demonstration may be objected to as incomplete, inasmuch as $\sqrt{x}$ and $\sqrt{y}$ may be the one a rational multiple of the other, in which case $\sqrt{xy}$ will be rational. Then, however, the expression would take the form $\sqrt{z} = (1 \pm m)\sqrt{x}$; and multiplying both by $\sqrt{z}$, we have $z = (1 \pm m)\sqrt{xz}$, and the equation cannot hold good, except $\sqrt{z}$ be a rational multiple of $\sqrt{x}$, or the converse. In this case we have then simply $\sqrt{z} = p\sqrt{x} = q\sqrt{y}$, and the equation would take the form $\sqrt{z} = \left(\frac{1}{p} \pm \frac{1}{q}\right)\sqrt{z}$, or the surd factor $\sqrt{z}$ is simply a multiplier of every term of the equation, and should be rejected, leaving the expression entirely rational.

It has now been proved that the equations

$$\sqrt{z} = x \pm \sqrt{y} \text{ and } \sqrt{z} = \sqrt{x} \pm \sqrt{y}$$

cannot have a real existence; and, therefore, that whenever they occur they are the result of incompatible conditions amongst the data of the inquiry. They constitute in fact one of the many forms of the imaginary symbol.

It is to be understood that x, y, z, are to be perfectly general in their nature, and not restricted to special numbers.

It readily follows from this, that in the equation

$$a \pm \sqrt{b} = x \pm \sqrt{y},$$

we must have $x = a$, and $\sqrt{y} = \sqrt{b}$.

Assume $a = x + \alpha$: then

$x + \alpha \pm \sqrt{b} = x \pm \sqrt{y}$, and hence,

$\pm \sqrt{y} = \alpha \pm \sqrt{b}$, which has been proved impossible, except $\alpha = 0$: and then $x = a$, and $\pm \sqrt{y} = \pm \sqrt{b}$.

(3.) To extract the square root of a binomial surd of the form $a \pm \sqrt{b}$.

Assume $\sqrt{a \pm \sqrt{b}} = u \pm \sqrt{v}$, or squaring we have

$a \pm \sqrt{b} = u + v \pm 2\sqrt{uv}$; and hence,

$u + v = a$, and $\pm 2\sqrt{uv} = \pm \sqrt{b}$.

In resolving these equations, we shall have successively,

$$\left.\begin{aligned} u^2 + 2uv + v^2 &= a^2 \\ 4uv \quad\quad &= b \end{aligned}\right\}\text{ or subtracting}$$

$$u^2 - 2uv + v^2 = a^2 - b$$

$$\left.\begin{aligned} \text{and } u - v &= \pm \sqrt{a^2 - b} \\ u + v &= a \end{aligned}\right\}\text{ hence,}$$

$$u = \frac{a \pm \sqrt{a^2 - b}}{2}, \text{ or } \sqrt{u} = \sqrt{\frac{a \pm \sqrt{a^2 - b}}{2}}$$

$$v = \frac{a \mp \sqrt{a^2 - b}}{2}, \text{ or } \sqrt{v} = \sqrt{\frac{a \mp \sqrt{a^2 - b}}{2}}$$

$$\text{Whence, } \sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a \pm \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a \mp \sqrt{a^2 - b}}{2}}.$$

It is quite obvious that, except $\sqrt{a^2 - b}$ be a rational quantity, the formula, on the right side of the equation here obtained, is far less convenient than that on the left. If this criterion be fulfilled, the solution will be more simple, and the method will be of advantage; but if not, it is better to calculate $a \pm \sqrt{b}$, and then extract its square root. As the criterion is always easy to apply, it is desirable to do so in the outset, and then be guided by its result in the choice of the method to be employed in the actual numerical valuation.

EXAMPLES.

1. What is the square root of $8 + \sqrt{39}$; of $10 - \sqrt{96}$; of $6 + \sqrt{20}$; of $6 - 2\sqrt{5}$; of $7 - 2\sqrt{10}$; and of $42 + 3\sqrt{174\frac{2}{9}}$?

2. What is the sum of $\sqrt{27}$, $\sqrt{48}$, $-4\sqrt{147}$, and $3\sqrt{75}$?

3. Rationalize the denominators of the fractions $\dfrac{a}{\sqrt[3]{x} + \sqrt[3]{y}}\ \dfrac{\cdot 5}{\sqrt[4]{3} + \sqrt[4]{4}}$.

4. Divide $a + b - c + 2\sqrt{ab}$ by $\sqrt{a} + \sqrt{b} + \sqrt{c}$, and by $\sqrt{a} + \sqrt{b} - \sqrt{c}$.

5. Multiply $x + y - \sqrt{xy}$ by $\sqrt{x} + \sqrt{y}$; and $a^{2\cdot 5} + a^2 b^{\frac{1}{3}} + a^{1\cdot 5} b^{\frac{2}{3}} + ab + a^{\cdot 5} b^{\frac{4}{3}} + b^{\frac{5}{3}}$ by $a^{\cdot 5} - b^{\frac{1}{3}}$.

6. Write the following expressions with fractional and (where possible) decimal indices: $\sqrt{a}$, $\sqrt[3]{(a + x)^2}$, $\sqrt[m]{(a^2 - bx)^n}$, $\sqrt{\dfrac{ax}{by}}$, and $\dfrac{\sqrt{a^2 - x^2}}{\sqrt[4]{2by - y^2}}$: and express, by means of radicals, the following quantities: $-a^{1\cdot 5}$; $\left\{(a-x)^{-1}\right\}^{\frac{1}{2}}$; $\pm (a^2 + x^2)^{\cdot 333\ldots}$; $(a^m)^{-\frac{1}{n}}$; $\left(\dfrac{1}{a^{-m}}\right)^{\frac{1}{m}}$, and $\left(\dfrac{1}{a^{-m}}\right)^{-\frac{1}{m}}$.

7. Ascertain whether any of the following expressions are rational:

$\left\{\dfrac{1^{1\cdot 5}\, b^{1\cdot 5}\, c^{-1\cdot 5}}{(a + b + x)^{\cdot 5}}\right\}^{\frac{2}{3}}$; $\pm \sqrt{a^2 + b^2 + x^2 + 2ab \mp 2ax \mp 2bx}$; $\sqrt{(a + bx)^4\, xy}$;

$\sqrt[3]{(a+x)^2b^2}$; $\sqrt[m]{\dfrac{(cx-x^2)^m\ (a+x)}{b+x}}$; and $\left\{\dfrac{3\ (a^2-x^2)\ \{a^2-4\ ax+x^2\}}{5b^2\ (a+x)\ (a-x)^3\ c^5}\right\}^{\frac{1}{2}}$.

8. Reduce $\sqrt{a^2-x^2}$ and $\sqrt[4]{a^4+x^4}$ to others having a common index $\frac{1}{8}$; and $(a^{-1}-x^{-1})^{-\cdot 25}$ and $(a-x)^{\cdot 25}$ to others having the common index $-\cdot 5$.

9. Divide a^3-b^3 by $a^{\cdot 75}-b^{\cdot 75}$, and by $a^{\frac{3}{4}}-b^{\frac{3}{4}}$.

10. Find the square root of $a-2\sqrt{abx}+bx$.

11. Express the mth root of the nth root of $a^{\frac{1}{p}}$ in as many ways as possible; and show that $\sqrt[m]{\sqrt[n]{\sqrt[p]{\ldots\ldots\ldots a}}} = a^{\frac{1}{mnp\ldots\ldots}}$.

12. Extract the square root of the expressions $x-2\sqrt{x-1}$, of $2-2(1-x)\sqrt{1+2x+x^2}$; of $28+5\sqrt{12}$; and of $\sqrt{32}-\sqrt{24}$.

Scholium.

Amongst the examples already given in this work, the symbol $\pm\sqrt{-1}$ as a co-efficient has appeared. This is called the imaginary symbol, and expressions into which it enters are called imaginary quantities.

This symbol is one which indicates an operation that cannot be performed in real numbers, positive or negative. For since $(+1)^2$ and $(-1)^2$ are alike unity, -1 cannot be the result of squaring either of them; whilst the direction to extract the square root of -1 implies that -1 has been produced by squaring some quantity, and which quantity it is required to assign. All, then, that can be done is to prefix the radical symbol, giving the general form $\pm\sqrt{-1}$.

In all problems where this symbol appears in the result, there has been some incongruity in the conditions of the question; or in other words, the alleged conditions would not co-exist.

The calculus by means of these is precisely similar to those laid down for quadratic surds, and no remark seems necessary, except to caution the student to use due care in the *signs* of his reduced quantities. The laws, however, are precisely the same as already laid down for products and quotients.

ARITHMETICAL PROGRESSION, OR PROGRESSION OF DIFFERENCE.

[Though in accordance with the original arrangement of the *Course*, the subject of progressions is retained in this place, it is very desirable that the study of it should be deferred till the simple and quadratic equations are thoroughly understood. The investigations at least, as they depend on the solution of simple equations, will be unintelligible to the student who has read no farther than the preceding pages.]

An Arithmetical Progression is a series of quantities which either increase or decrease by the same common difference.

Thus, 1, 3, 5, 7, 9, 11, ... and a, $a\pm b$, $a\pm 2b$, $a\pm 3b$, $a\pm 4b$, ... are series in arithmetical progression, whose common differences are 2 and $\pm b$ respectively.

When the quantity b is affected by sign $+$, the progression is said to be *increasing;* and when by the sign $-$, it is said to be a *decreasing progression.* These signs are therefore the indications of the *kind* of progression.

The most useful part of arithmetical progression has been given in the arithmetic. The same may be exhibited algebraically, thus:

Let a denote the least term,
z the greatest term,
d the common difference,
n the number of the terms,
and s or s_n the sum of n terms;

then the principal relations are expressed by these equations, viz.

1. $z = a + d\,(n - 1)$;
2. $a = z - d\,(n - 1)$;
3. $s = (a + z)\,\dfrac{n}{2}$;
4. $s = \{z - \tfrac{1}{2}d\,(n - 1)\}n$;
5. $s = \{a + \tfrac{1}{2}d\,(n - 1)\}n$ *.
6. $d = \dfrac{z - a}{n - 1}$;
7. $d = \dfrac{2\,(s - an)}{n(n - 1)}$;
8. $d = \dfrac{2(nz - s)}{n(n - 1)}$.

EXAMPLES.

1. The first term of a series of quantities in arithmetical progression is 1, their difference 2, and the number of terms 21: what is the sum?

Here $a = 1$; $d = +2$; and $n = 21$: and hence, by formula (5), we have

$$s_{21} = \frac{1 + 41}{2} \times 21 = 441.$$

2. A *decreasing* arithmetical series has its first term 199, its difference -3, and its number of terms 67: what is its sum?

Here $z = 199$, $d = -3$, and $n = 67$; hence $a = z - (n - 1)\ d = 199 - 66 \times 3 = 1$, and $\dfrac{199 + 1}{2} \times 67 = 6700$, the answer required.

3. To find the sum of 100 terms of the natural numbers 1, 2, 3, 4, 5, 6, &c.; and likewise of ·1, ·2.... Ans. 5050, and 505.

4. Required the sum of 99 terms of the negative odd numbers -1, -3, -5, -7, -9.... Ans. -9801.

5. The first term of an arithmetical series is 10, the common difference $-\frac{1}{3}$, and the number of terms 21: required the sum of the series. Ans. 140.

6. One hundred stones being placed on the ground, in a straight line, at the distance of two yards from each other: how far will a person travel, who shall bring them one by one to a basket, which is placed 2 yards from the first stone? Ans. 11 miles and 840 yards.

7. The first term is $a^2 - 2ax + x^2$, the last is $a^2 + 2ax + x^2$, and the number

---

* For, writing the given series first in a direct line and then in an inverted order, we have

$$\{a\} + \{a + d\} + \{a + 2d\} + \ldots\ldots + \{a + (n - d)d\}$$
$$\{a + (n - 1)\,d\} + \{a + (n - 2)d\} + \{a + (n - 3)d\} + \ldots\ldots + \{a\}$$

and adding up the columns vertically, we have double the sum of the series equal to

$$\{2a + (n - 1)\,d\} + \{2a + (n - 1)\,d\} + \{2a + (n - 1)\,d\} + \ldots + \{2a + (n - 1)\,d\}$$

which is equal to $n\{2a + (n - 1)\,d\}$, there being n such terms;

and hence $s_n = \{a + \frac{1}{2}(n - 1)d\}n$. This is the formula marked (5) above.

That marked (1) is obvious from the formation of the successive terms: the (2) is obtained from it by transposition. Also from the method of proceeding in the proof of (5) that marked (3) is evident; and substituting the value of a in (5) the result (4) is obtained.

From those expressions which involve values of n and d, formulæ for finding these may be derived by the common operations of algebra: but it is unnecessary to annex the work here.

of terms is $a^2 - x^2$: what is the sum of the series, and the common difference of its terms? Ans. Sum $= a^4 - x^4$, com. dif. $= \frac{+4ax}{a^2 - x^2 - 1}$.

8. The first term is $-a$, and the last term is nine times the first, and the number of terms one-fifth of the sum of the first and last terms: what is the sum and common difference? Ans. Sum $= 10a^2$, com. dif. $= \frac{8a}{2a+1}$.

9. The sum of the numbers, 1, 2, 3, ... n is $\frac{n(n+1)}{2}$; the sum of the n numbers, 1, 3, 5, ... $(2n-1)$ is n^2; and the sum of n terms each equal to n is n^2. Investigate these theorems.

10. The first term is 16·5, the last is $-$ 6·5, and the sum is 100. Find the number of terms and the common difference.

11. The tenth term of an arithmetical series is 17·5, and the fiftieth is $1587\frac{1}{2}$. What are the separate sums of the first twenty, and of the last thirty terms?—Find also the common difference; and the 11th, 21st, 31st, and 41st terms.

12. Sum the series $-7, -5{\cdot}5, -4, - \ldots$ to 10 terms,
$16, \frac{46}{3}, 14\frac{2}{3}, \ldots$ to 49 terms,
$17, +\frac{49}{3}, +15\frac{2}{3}, + \ldots$ to 20 terms,
and find the 7th term from each end of each of these series.

13. The common difference is ·001; the number of terms is one million, and the greatest term is 0. What is the least? Find also the sum of the 100th, 200th, 300th, ... terms of which the series is composed.

14. The first term is 1, the common difference is successively taken 1, 2, 3, ... write ten terms of the first six series, and express the sum of each series to n terms.

15. Find the difference between the sum of n terms of the odd numbers 1, 3, 5, ... and n terms of the series of even numbers 0, 2, 4, ...

16. If the first term be a, and the common difference be $2a\,(1+a+a^2+\ldots a^{m-2})$, show that the sum of the series of a terms is equal to a^m. And apply this to the square and cube as values of m.

17. Let the first term be $-a$, the common difference d, and the number of terms $-n$ *, what is the expression for the value of the $\pm$ nth term? Also, in numbers where $a = \pm 5$, $d = \frac{3}{2}$, and $n = 10$. Ans. $z = -a \pm \frac{\pm n - 1}{2} d$.

18. A triangular battalion † consists of thirty ranks, in which the first rank is formed of one man only, the second of 3, the third of 5, and so on: what is the strength of such a triangular battalion? Ans. 900 men.

19. A detachment having 12 successive days to march, with orders to advance

---

* As n denotes the number of terms to be taken in the direction indicated by the sign of the common difference, so also $-n$ denotes the term from which we must have started to obtain $-a$ (the difference still being d) as the nth term of the series.

† By triangular battalion, is to be understood, a body of troops ranged in the form of a triangle, in which the ranks exceed each other by an equal number of men: if the first rank consist of one man only, and the difference between the ranks be also 1, then its form is that of an equilateral triangle; and when the difference between the ranks is more than one, its form may then be an isosceles or scalene triangle. The practice of forming troops in this order, which is now laid aside, was formerly held in greater esteem than forming them in a solid square, as admitting of a greater front, especially when the troops were to make simply a stand on all sides.

the first day only 2 leagues, the second $3\frac{1}{2}$, and so on, increasing $1\frac{1}{2}$ league each day's march: find the length of the whole march, and the last day's march.

Ans. the last day's march is $18\frac{1}{2}$ leagues, and the whole march, 123 leagues.

20. A brigade of sappers * having carried on 15 yards of sap the first night, the second only 13 yards, and so on, decreasing 2 yards every night, till at last they carried on in one night only 3 yards: what is the number of nights they were employed; and what is the whole length of the sap?

Ans. they were employed 7 nights, and the whole sap was 63 yards.

21. A number of gabions † being given to be placed in six ranks, one above the other, in such a manner as that each rank exceeding one another equally, the first may consist of 4 gabions, and the last of 9: what is the number of gabions in the six ranks; and what is the difference between each rank?

Ans. the difference between the ranks is 1, and the number of gabions is 39.

22. Two detachments, distant from each other 37 leagues, and both designing to occupy an advantageous post equi-distant from each other's camp, set out at different times; the first detachment increasing every day's march 1 league and a half, and the second detachment increasing each day's march 2 leagues: both the detachments arrive at the same time; the first after 5 days' march, and the second after 4 days' march? What is the number of leagues marched by each detachment each day?

Ans. the first marches $\frac{7}{10}$, $2\frac{2}{10}$, $3\frac{7}{10}$, $5\frac{2}{10}$, $6\frac{7}{10}$, leagues on the successive days, and the second $1\frac{5}{8}$, $3\frac{5}{8}$, $5\frac{5}{8}$, $7\frac{5}{8}$, leagues.

23. A triangular course of shot of n in each side is composed of $n, n-1, n-2, \ldots. 3, 2, 1$ shot in succession. Find the number of shot, c_n, in the whole course. Ans. $c_n = \frac{n(n+1)}{2}$.

24. A rectangular course of shot has n shot in its shorter side, and $m+n$ in its longer one. How many shot are there in the course? Ans. $c_n = (m+n)n$.

GEOMETRICAL PROGRESSION, OR PROGRESSION BY RATIO.

If a series of terms (three at least) be so taken that each is the product of the

---

* A brigade of sappers consists generally of 8 men, divided equally into two parties. While one of these parties is advancing the sap, the other is furnishing the gabions, fascines, and other necessary implements: and when the first party is tired, the second takes its place, and so on, till each man in turn has been at the head of the sap. A sap is a small ditch, between 3 and 4 feet in breadth and depth; and is distinguished from the trench by its breadth only, the trench having between 10 and 15 feet breadth. As an encouragement to sappers, the pay for all the work carried on by the whole brigade is given to the survivors.

† Gabions are baskets, open at both ends, made of ozier twigs, and of a cylindrical form: those made use of at the trenches are 2 feet wide, and about 3 feet high; which, being filled with earth, serve as a shelter from the enemy's fire: and those made use of to construct batteries are generally higher and broader. There is another sort of gabion, made use of to raise a low parapet: its height is from 1 to 2 feet, and 1 foot wide at top, but somewhat less at bottom, to give room for placing the muzzle of a firelock between them: these gabions serve instead of sand bags. A sand bag is generally made to contain about a cubical foot of earth.

preceding one of the series by some constant factor *, these terms are said to form a geometrical series. Thus, a, ar, ar^2, ar^3, ... ar^{n-1} form a series of terms in geometrical progression; as likewise do 2, 6, 18, 54, ... and 2, —6, 18, —54, ... of which the constant factors are respectively r, 3, and —3. These factors are called the *ratios* of the series.

The following notation is generally employed:—

a for the first, and z for the last term;
r for the common multiplier or ratio;
n for the number of terms; and
s or s_n for the sum of n terms of the series.

FORMULÆ OF SOLUTION.

From the formation of the terms, $z = ar^{n-1}$ .... (1)

$$s^n = a(r^{n-1} + r^{n-2} + \ldots . r^2 + r + 1) \cdot \frac{r-1}{r-1} = \frac{a(r^n - 1)}{r-1} \quad \ldots\ldots (2)$$

From (1, 2) we have $a = \frac{z}{r^n - 1}$, and $a = \frac{(r-1)s_n}{r^n - 1}$ .... (3, 4)

From (1, 2), $ar^n = rz$ and $ar^n - a = (r-1)s_n$;
whence, by subtraction, $a = rz - (r-1)s_n$ .... (5)

From (1, 4), $z = ar^{n-1}$, and $a = \frac{(r-1)s_n}{r^n - 1}$; whence $z = \frac{s_n(r-1)r^{n-1}}{r^n - 1}$ .. (6)

From (2), $(r-1)s_n = ar^n - a$; whence $a + (r-1)s_n = ar^n = rz$; whence

$$z = \frac{a + (r-1)s_n}{r} = s_n - \frac{s_n - a}{r} \quad \ldots\ldots\ldots\ldots (7)$$

From (1), $r^{n-1} = \frac{z}{a}$; whence, extracting, $r = \left(\frac{z}{a}\right)^{\frac{1}{n-1}}$ .... (8)

From (1, 2), $rz = ar^n$ and $rs_n - s_n = ar^n - a$; hence $r = \frac{s_n - a}{s_n - z}$.... (9)

From (2), $s^n = \frac{a(r^n - 1)}{r-1} \cdot \frac{r^{n-1}}{r^{n-1}} = \frac{(r^n - 1)\, ar^{n-1}}{(r-1)\, r^{n-1}} = \frac{(r^n - 1)\, z}{(r-1)\, r^{n-1}}$ ...... (10)

From (8), $r = \frac{z^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}}$; hence $r^n - 1 = \frac{z^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{a^{\frac{n}{n-1}}}$, and $r - 1 = \frac{z^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}}$;

whence $s_n = \frac{a(r^n - 1)}{r-1} = \frac{z^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{z^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}} \cdot \frac{a \cdot a^{\frac{1}{n-1}}}{a^{\frac{n}{n-1}}} = \frac{z^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{z^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}$ .... (11)

Reducing (2) we have $r^n - \left(\frac{s_n}{a}\right) r + \frac{s_n}{a} = 1$ .... (12)

And from (6) we have $r^n - \frac{s_n}{s_n - z}\, r^{n-1} + \frac{z}{s_n - z} = 0$ .... (13)

From (1, 9) we get $r^{n-1} = \frac{z}{a}$ and $r = \frac{s_n - a}{s_n - z}$; whence $r^{n-1} = \left(\frac{s_n - a}{s_n - z}\right)^{n-1} =$
$\frac{z}{a}$ .... (14)

* The numerical or algebraical character of this factor is of no consequence; as it may be positive, negative, or imaginary, integer, fractional, or irrational. Its constancy throughout the series is the only condition essential to it.

The only general method of finding n is by logarithms: but when n is known, r may be found by the solution of the equations 13, 14. When logarithms are used, equation (14) is a convenient form for n.

When the series runs out *ad infinitum*, if the ratio be greater than 1, the sum is necessarily infinite, as r^n will become infinitely great. When $r = 1$, the sum of the series takes an indeterminate form $s_{\frac{1}{0}} = \frac{0}{0} \cdot a$: but other obvious considerations show that it is infinite. When r is less than 1, r^n continually decreases as the value of n increases, till $r^{\frac{1}{0}} = 0$. In this case $s_{\frac{1}{0}} = \frac{r^{\frac{1}{0}} - 1}{r - 1}\, a =$

$$\frac{1 - r^{\frac{1}{0}}}{1 - r} \cdot a = \frac{a}{1 - r} \qquad (15)$$

The doctrine of geometrical progression finds its application in almost every department of mathematical inquiry; but one of the most elementary and most frequently required cases, is that of

CIRCULATING DECIMALS.

It has already been seen in the arithmetic, p. 60, that when certain vulgar fractions are converted into decimals, the terms of that decimal form a series of circulating periods of figures, always recurring in the same order. If we convert these periods into separate terms with the proper denominators indicated by their places in the decimal scale, we shall find them to constitute a geometrical series.

Thus, $\cdot 333 \ldots$ ad inf. $= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} \ldots = \frac{3}{10}\left\{1 + \frac{1}{10} + \frac{1}{10^2} + \ldots\right\}$

and $\cdot 215215 \ldots$ ad inf. $= \frac{215}{10^3} + \frac{215}{(10^3)^2} + \frac{215}{(10^3)^3} \ldots = \frac{215}{10^3}\left\{1 + \frac{1}{10^3} + \frac{1}{10^6} + \ldots\right\}$

which, again, may be conveniently written in the following manner:—

$$3(\cdot 1 + \cdot 1^2 + \cdot 1^3 + \ldots) \text{ and } 215(\cdot 1^3 + \cdot 1^6 + \cdot 1^9 + \ldots).$$

Employing the formula (15) for a decreasing infinite series,

$$s_{\frac{1}{0}} = \frac{3.\cdot 1.10}{10 - 1} = \frac{1}{3} \text{ for the first decimal } \cdot 333 \ldots$$

$$s_{\frac{1}{0}} = \frac{215.\ \cdot 1^3.\ 10^3}{10^3 - 1} = \frac{215}{999} \text{ for the second } \cdot 215215 \ldots$$

To take a general view of the subject, let us suppose the decimal is composed of a series of circulating periods preceded by a finite decimal. Let the circulating period be composed of n figures, which, taken integrally, denote the number N, and the preceding or finite part of the decimal be composed of m figures, which, together with the integers, all taken integrally, denote the number M. Then the entire decimal will be represented by

$$\frac{M}{10^m} + \frac{N}{10^{m+n}} + \frac{N}{10^{m+2n}} + \frac{N}{10^{m+3n}} + \ldots \text{ ad inf., which may be written}$$

$$\frac{M}{10^m} + \frac{N}{10^{m+n}}\left\{\cdot 1 + \cdot 1^n + \cdot 1^{2n} + \cdot 1^{3n} + \ldots\right\} = \frac{M}{10^m} + \frac{N}{10^{m+n}} \cdot \frac{10^n}{10^n - 1}$$

$$= \cdot 1^m\left\{M + \frac{N}{99 \ldots (n-1) \text{ places}}\right\}.$$

When $m = 0$, $\cdot 1^m = 1$, and the circulating period commences at the decimal point, and is represented by the fraction $\frac{N}{999 \ldots (n-1) \text{ places}}$.

EXAMPLES IN GEOMETRICAL PROGRESSION.

1. The first term is 1, the ratio 2, and the number of terms 12: find the last term and sum of the series.

$$z = ar^{n-1} = 1 \,.\, 2^{11} = 2048, \text{ the last term; and}$$

$$s_{12} = \frac{(r^n - 1)\,a}{r - 1} = \frac{(4096 - 1)\,.\,1}{1} = 4095, \text{ the sum of the series.}$$

2. Given the first term $\frac{1}{3}$ and the ratio $-\frac{1}{2}$ to find the eighth term and the sum of eight terms of the geometrical series.

$$z = ar^n = \tfrac{1}{3}\,(-\tfrac{1}{2})^7 = -\tfrac{1}{384} \text{ the eighth term; and}$$

$$s_8 = \frac{r^8 - 1}{r - 1}\,.\,a = \frac{\frac{1}{256} - 1}{-\frac{1}{2} - 1}\,.\,\tfrac{1}{3} = \frac{85}{384}, \text{ the sum of eight terms.}$$

3. Required the sum of 12 terms of the series, 1, 3, 9, 27, 81, ...; and of 7 terms of the series $1 - 3 + 9 - 27 + \ldots.$ Ans. 265720, and . . .

4. Required the sum of 12 terms of the series, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, ... ; and 10 of the series $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots$ Ans. $\frac{265720}{177147}$, and . . .

5. Given $r = 2$, $n = 6$, $s = 189$. Required a and z. Ans. $a = 3$, $z = 96$.

6. Given $a = -4$, $n = 6$, $z = -12500$. Required r and s.
Ans. $r = 5$, $s = -15624$.

7. Find the tenth terms and the sum of the first ten in each of the following series:

$$\frac{16}{729} + \frac{8}{243} + \frac{4}{81} + \ldots. \text{ and } \frac{16}{729} - \frac{8}{243} + \frac{4}{81} - \ldots.$$

and find the 11th and 12th terms of each series; and then sum each series to 6 and to 7 terms.

8. Find the sum of ten terms, and the difference between that and the sum to infinity of the series $\frac{3}{4} + \frac{1}{2} + \frac{1}{3} + \ldots$ and likewise of $-\frac{3}{4} + \frac{1}{2} - \frac{1}{3} + \ldots$ Also of $8 \pm 4 - 2 \pm 1 - \ldots$ to six terms, and to infinity.

9. Find the values of ·333 . . . ad inf.; and of $\cdot 25 + \cdot 25^2 + \cdot 25^3 + \ldots$ to 5 terms; and assign the 9th and 10th terms. Likewise the sum of both to infinity.

10. What is the sum of the infinite series $-\cdot 1^2 + \cdot 1^4 - \cdot 1^6 + \cdot 1^8 - \ldots.$ and of $\cdot 1^3 a + \cdot 1^6 a^2 + \cdot 1^9 a^3 + \ldots.$

11. Required the sum of $1 + \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \ldots.$ to infinity, and likewise of $1 + \dfrac{1}{x+1} + \dfrac{1}{(x+1)^2} + \ldots$ Ans. $\dfrac{x}{x-1}$ and $\dfrac{x+1}{x}$.

12. The first and last terms are $\frac{1}{2}$ and $\frac{2}{9}$, and the number of terms is 5: what are the three intermediate terms? Also insert two and three geometrical means respectively, between 4·5 and 2.

13. Suppose a body to move for ever in this manner, viz. 20 miles the first minute, 19 miles the second, 18 05 the third, and so on in geometrical progression: required the utmost distance it can reach.

14. Suppose the elastic power of a ball which falls from a height of 100 feet to be such as to cause it to rise ·9375 of the height from which it fell; and to continue in this way diminishing the height to which it will rise in geometrical progression, till it comes to rest: how far will it have moved?

GEOMETRICAL PROPORTION.

If there be four quantities, a, b, c, d, such that $ad = bc$, they are said to be geometrical proportionals. The statement is usually written $a : b :: c : d$, the

factors of one side of the equation being taken as the extreme, and those of the other side as the mean terms. In this a and c are called the *antecedents*, and b and d the *consequents* of the ratios $a : b$ and $c : d$. *Homologous terms* are either both the antecedents or both the consequents. With this understanding we have at once the four following forms of the statement:—

$$a : b :: c : d \ldots\ldots\ldots\ldots (1)$$
$$b : a :: d : c \ldots\ldots\ldots\ldots (2)$$
$$a : c :: b : d \ldots\ldots\ldots\ldots (3)$$
$$c : a :: d : b \ldots\ldots\ldots\ldots (4)$$

Again, by division of the original or defining equation,

$$\frac{a}{b} = \frac{c}{d} \text{ or } \frac{a}{c} = \frac{b}{d} \ldots\ldots\ldots\ldots (5,\ 6)$$

From (5, 6) $\frac{a}{b} \pm 1 = \frac{c}{d} \pm 1$, or $\frac{a}{c} \pm 1 = \frac{b}{d} \pm 1$; whence,

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}, \text{ or } a+b : a-b :: c+d : c-d \ldots\ldots (7)$$
$$\frac{a+c}{a-c} = \frac{b+d}{b-d}, \text{ or } a+c : a-c :: b+d : b-d \ldots\ldots (8)$$

and these two will each admit of four varieties of form analogous to those of (1, 2, 3, 4).

Add bd to both sides of the defining equation: then $ad + bd = bc + bd$, or $(a + b)\,d = (c + d)\,b$; and $b : d :: a + b : c + d$.

By this and analogous processes we shall obtain

$$a : c : a \pm b : c \pm d \ldots\ldots\ldots\ldots (9)$$
$$a : b :: a \pm c : b \pm d \ldots\ldots\ldots\ldots (10)$$
$$c : d :: a \pm b : c \pm d \ldots\ldots\ldots\ldots (11)$$
$$c : d :: a \pm c : b \pm d \ldots\ldots\ldots\ldots (12)^*$$

Again, if we have several ranks of proportionals, the products of all their corresponding terms will be proportional quantities. Let them be,

$$\begin{array}{lll} a : b :: c : d; & \text{then} & a\,d = b\,c \\ a_1 : b_1 :: c_1 : d_1; & & a_1 d_1 = b_1 c_1 \\ a_2 : b_2 :: c_2 : d_2; & & a_2 d_2 = b_2 c_2 \\ \vdots & & \vdots \\ a_m : b_m :: c_m : d_m & & a_m d_m = b_m c_m \end{array}$$

and hence $aa_1a_2 \ldots a_m \times dd_1d_2 \ldots d_m = bb_1b_2 \ldots b_m \times cc_1c_2 \ldots c_m$, and we have

$$aa_1a_2 \ldots a_m : bb_1b_2 \ldots b_m :: cc_1c_2 \ldots c_m : dd_1d_2 \ldots d_m \ldots\ldots (13)$$

If there be n of these factors, and all equal, we have

$$a^n : b^n :: c^n : d^n \ldots\ldots\ldots\ldots (14)$$

If four quantities be proportional, their n^{th} roots are also proportional. For if $ad = bc$, then $a^{\frac{1}{n}}\, d^{\frac{1}{n}} = b^{\frac{1}{n}}\, c^{\frac{1}{n}}$, and $a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}} \ldots\ldots\ldots\ldots (14)$

Also $a^{\frac{m}{n}}\, d^{\frac{m}{n}} = b^{\frac{m}{n}}\, d^{\frac{m}{n}}$, and $a^{\frac{m}{n}} : b^{\frac{m}{n}} :: c^{\frac{m}{n}} : d^{\frac{m}{n}} \ldots\ldots\ldots\ldots (15)$

When the second and third terms are equal, we have

$$a : b :: b : c, \text{ and } b^2 = ac.$$

In this case b is called a *mean proportional* between a and c, and c a *third proportional* to a and b.

In a geometrical series we have the terms continually proportional: viz.

$$\left.\begin{array}{l} a : ra :: ra : r^2a \\ ra : ra^2 :: ra^2 : r^3a \\ r^2a : r^3a :: r^3a : r^4a \end{array}\right\} \text{since these fulfil the defining condition of proportionals.}$$

* Several technical terms are used by geometers to distinguish these different modes of

HARMONICAL PROGRESSION AND PROPORTION.

When the reciprocals of a series of numbers form an arithmetical progression, the numbers themselves constitute an *harmonical progression*. Thus, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ $\frac{3}{5}, \frac{3}{7}, \frac{3}{9}, \frac{3}{11}, \ldots$, or $\frac{c}{a}, \frac{c}{a+b}, \frac{c}{a+2b}, \frac{c}{a+3b}, \ldots$ constitute an *harmonical series*, or an *harmonical progression*.

If, therefore, we are required to form an harmonical progression, or to find the law from a sufficient number of terms given in any part of it, we have only to form the reciprocals, and thence the arithmetical progression, and finally to take the reciprocals of these. The resulting series is the one sought.

Thus, to find a fourth harmonical to the given terms $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, we have only to take 2, 3, 4, and find the fourth arithmetical proportional to 2, 3, 4, viz. 5. Then the reciprocal of this, viz. $\frac{1}{5}$, is the fourth term sought. It is, however, sufficient to give two of the terms immediately preceding that sought, as the arithmetical progression of reciprocals follows from these two, without employing any of the more remotely precedent ones of the series *.

It will be obvious, that as an arithmetical series does not admit of indefinite extension in one direction, without employing negative numbers; so also the harmonical series does not admit of indefinite extension in the other direction, without also employing negative numbers. Employing, however, as is always done in Algebra, both positive and negative numbers, all three series admit of indefinite extension both ways. It is with this explanation that the remark made by writers on these classes of series is to be understood, when they say that the arithmetical series admits of indefinite extension *only* upwards, the harmonical *only* downwards, and the geometrical, in both directions, both increasing and decreasing.

The following properties furnish a specimen of those which belong to numbers in harmonical progression.

1. If there be three terms in harmonical progression, then, the first is to the third as the difference of the first and second is to the difference of the second and third.

For, by the definition, if a, b, c be three numbers in arithmetical progression, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ will be in harmonical progression. Also, $\frac{b-c}{abc} = \frac{a-b}{abc}$, in virtue of the arithmetical progression, and by proportion we obtain from this $\frac{1}{a} : \frac{1}{c} :: \frac{a-b}{ab} : \frac{b-c}{bc} :: \frac{1}{b} - \frac{1}{a} : \frac{1}{c} - \frac{1}{b}$.

The converse of this is obviously true: that if three given numbers fulfil this proportion they are harmonicals.

2. The harmonical mean between two numbers is equal to twice the product of those numbers divided by their sum. For in the preceding proportion we have $\frac{1}{ac} - \frac{1}{ab} = \frac{1}{bc} - \frac{1}{ac}$, from which $\frac{2}{ac} = \frac{1}{b}\left(\frac{1}{a} + \frac{1}{c}\right)$;

---

* An extension of the definition, so as to render the fourth term dependent on the three immediately precedent ones, has been given by several authors, though that definition has no reference whatever to musical intervals, nor do the terms of it form the reciprocals of an arithmetical series.

$$\text{or, } \frac{1}{b} = \frac{\frac{2}{ac}}{\frac{1}{a} + \frac{1}{c}} = \frac{2 \cdot \frac{1}{a} \cdot \frac{1}{c}}{\frac{1}{a} + \frac{1}{c}};$$

where a, b, c are the arithmetical reciprocals of the three harmonical terms.

3. A third harmonical to two given terms is equal to the product of those terms divided by the difference between twice the first and the second terms. For from the same we have

$$\frac{1}{c} = \frac{\frac{1}{ab}}{\frac{2}{a} - \frac{1}{b}}$$

4. In an *harmonical series*, any three terms, the extremes of which are equidistant from the extremes, are in *harmonical progression.* For their reciprocals are in arithmetical progression *.

5. Let $a, b, c, \ldots . h, k$, be an harmonical series: then

The product of any two adjacent terms is to the product of any other two adjacent terms, as the difference of the first pair is to the difference of the second pair.

$$\text{For } \frac{1}{a} - \frac{1}{b} = \frac{1}{h} - \frac{1}{k} \text{ by definition.}$$

$$\text{Hence, } \frac{b-a}{ab} = \frac{k-h}{hk}, \text{ or } \frac{hk}{ab} = \frac{k-h}{b-a} = \frac{h-k}{a-b}.$$

6. When the first two terms a and b are given, the n^{th} may be thus expressed:

$$z = \frac{ab}{(n-1)\,a - (n-2)\,b}$$

$$\text{For } d = \frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab}; \text{ and}$$

$$\frac{1}{z} = \frac{1}{a} + (n-1)\frac{a-b}{ab} = \frac{(n-1)\,a - (n-2)\,b}{ab}$$

$$\text{or } z = \frac{ab}{(n-1)\,a - (n-2)\,b}.$$

7. We may insert n harmonic means between a, b.

$$\text{For } \frac{1}{b} = \frac{1}{a} + (n-1)\,d, \text{ or } d = \frac{a-b}{(n-1)\,ab}.$$

---

* As the doctrine of geometrical proportion, which essentially involves four terms, has been applied to three only, by taking the second and third as identical (thereby constituting continued proportionals), so in the harmonical progression continued (each step of which essentially contains three terms), the second term has been supposed to be replaced by a different one, which stands as the third, whilst that which originally stood as the third, thereby becomes the fourth. In this case the *four numbers* have been said to be in *harmonical proportion*, when the first is to the fourth as the difference of the first and second is to the difference of the third and fourth: thus, if a, b, c, d fulfil the condition $a : d :: a \backsim b : c \backsim d$, the four quantities a, b, c, d are harmonicals.

In this case, if c be made $= b$, and $d = c$, the harmonicals previously defined will result: but the definition of harmonicals there given does not apply to any of the other numbers which fulfil this condition. Some writers also speak of *contra-harmonicals*. Into the study of each of these kinds of proportion, the reader who desires to enter will find ample information in *Malcolm's Arithmetic*, pp. 297—313, 1730. Further notice of them here would be incompatible with the plan and objects of this Course of Mathematics.

from which the arithmetical progression, and thence the harmonical, is readily found.

EXAMPLES.

1. Find the fifth term of an harmonical series whose first and second terms are 3 and 4, and likewise of that whose first and second terms are 4 and 3.

In the first case, $\frac{1}{3}$ and $\frac{1}{4}$ are the first and second terms of an arithmetical progression descending, since the harmonical progression ascends. Hence $\frac{1}{3} = z$, is the greatest term; and we have $\frac{1}{3} - \frac{1}{4} = \frac{1}{12} =$ common difference $= d$: and $z = \frac{1}{3}$, and $n = 5$. Therefore the term a, or least term, is $\frac{1}{3} - \frac{4}{12} = \frac{1}{3} - \frac{1}{3} = 0$, and the fifth term is therefore $\frac{1}{0} =$ infinite.

In the second case the harmonical series descends, and hence the arithmetical one ascends; and, therefore, as before, $d = \frac{1}{12}$, and $n = 5$, and the least term is $\frac{1}{12}$: the fifth term is, therefore, $\frac{1}{3} + \frac{4}{12} = \frac{9}{12} = \frac{3}{4}$ The fifth term of the harmonical series, 4, 3, &c. is therefore $\frac{4}{3}$.

2. Find an harmonical mean between 3 and 4, and six harmonical means between 1 and 2.

3. An harmonical series consists of fifteen terms, and the greatest and least terms are x and y: what is the middle term?

4. A line, whose length is 10 inches, is divided harmonically, so that the first section (from the origin) is 3 inches: how far distant is the second point of section from the first?

5. Four terms are in harmonical *proportion;* the first and last are 6 and 10: what is the relation between the second and third?

Ans. $10x + 6y = 120$; where x is the second, and y the third term.

6. The first and second terms of an harmonical *proportion* are 4, 5: and the first, second, and third terms of an harmonic *progression* are also 4, 5, 6. Find the fourth term of the *proportion*, and the fourth term of the *progression*.

7. Ten terms are in harmonical progression, and the last two are $\frac{1}{10}$ and $\frac{1}{11}$: what are the terms, and what is their sum?

EQUATIONS.

An *equation* is the algebraic expression of the equality of two assemblages of quantities to one another; and consists in writing $=$, the sign of equality, between them *. Thus $10 - 4 = 6$ is an equation expressing the equality of $10 - 4$ to 6; and $4x + b = c - d$ is an equation expressing that $4x + b$ is equal to $c - d$.

Equations are designated by different names, according to the manner of their composition, and the highest power of the unknown quantity which enters into them. When the highest power is the first, the equation is called a *simple equation,* or *an equation of the first degree:* when it is of the second, the equation is

---

* The mark here used was introduced into algebra by the first English author on the subject, Robert Recorde, in his "Whetstone of Witte," (sig. Ff. 1b,) 1557. He gives his reason in his own quaint manner in the following words: "And to avoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus: $=$, bicause noe 2 thynges can be more equalle."

For a long period afterwards, the Continental mathematicians employed the symbol $\propto$, which was, doubtless, a rapid formation of the diphthong *æ*, the initial of the phrase *æquale est.*

called a *quadratic equation*, or an *equation of the second degree:* when it is of the third, the equation is called a *cubic equation*, or *an equation of the third degree:* and so on.

The *known*, or *given quantities*, are represented by the earlier letters of the alphabet, $a, b, c, \ldots$ and the *unknown*, or *quantities whose values are sought*, by the later ones, $z, y, x, w, \ldots$.

It often happens that equations arise which are composed of only *two terms*, in which the power of the unknown is of a higher degree than the first. These are called *binomial equations*, as $x^5 = -100$; and otherwise *pure equations*, to distinguish them from *adfected* (or *affected*) equations. These are, for the purposes of solution, considered as *simple equations*, the operations that are requisite for completing the solution being purely of the arithmetical kind. Examples in which these occur are therefore classed amongst those of simple equations.

When there are several equations given, the unknown of which in one is capable of such multiplications or divisions by that in some of the others, or when they admit of a ready combination with one another, so as to form results that are known to be powers of some binomial or trinomial expression, they are frequently classed also amongst the exercises on *simple equations*. Such a method of classification is, evidently, very arbitrary; and hence there are several questions in the following series which are by some authors distributed under a different denomination: though in general this classification accords with the most common practice of algebraical writers.

The quantities which precede the mark of equality, are often called together *the first member or the first side of the equation; those which follow it, the second member or the second side.*

The resolution of equations, is the finding the value of the unknown quantity, or in disengaging that quantity from the known ones; and this consists in so transforming the equation, that the unknown letter or quantity may stand alone on one side of the equation, without a coefficient; and all the rest, or the known quantities, on the other side.

SIMPLE EQUATIONS, WITH ONE UNKNOWN.

In these, the unknown quantity, when properly transformed, is of the first degree, as $ax = b$, and its solution in this state is obvious: but as they seldom so occur, we must lay down the principles of transformation so as to disengage x from all other quantities on one side of the equation.

In general, the unknown quantity is disengaged from the known ones, by performing always the reverse operations. That is, if the known quantities are connected with it by +, or addition, they must be subtracted; if by minus, —, or subtraction, they must be added; if by multiplication, we must divide by them; if by division, we must multiply; when it is in any power, we must extract the corresponding root; and when in any radical, we must raise it to the corresponding power. The following special rules are founded on this general principle, viz.: that when equivalent operations are performed on equal quantities, the results must still be equal; whether by equal additions, subtractions, multiplications, divisions, extractions, or involutions.

I. When known quantities are connected with the unknown by + or —; transpose them to the other side or member of the equation, and change their signs. Which is only adding or subtracting the same quantities on both sides,

in order to get all the unknown terms on one side of the equation, and all the known ones on the other *.

The same rule applies whether the known quantities be given in numbers or in symbols.

Thus, if $x + 5 = 8$; then transposing 5, gives $x = 8 - 5 = 3$.

If $x - a + b = cd$, then by transposing a and b, it is $x = a - b + cd$.

II. When the unknown term is multiplied by any quantity; divide all the terms of the equation by it.

---

* Here it is *earnestly recommended* that the pupil be accustomed, at every line or step in the reduction of the equations, to name the *particular operation to be performed* on the preceding equation, in order to produce the next form or state of the equation, in applying each of these rules, according as the particular form of the equation may require: applying them according to the order in which they are here placed, and always allotting a single line for each operation and its description, and ranging the equations under each other, in the several lines, as they are successively produced. *The master, indeed, never ought to receive a solution from his pupil in writing in which this rule is not complied with, and as much attention given to the proper concatenation of the verbal descriptions as to the mere work set down in the algebra.* Due regard being had to this point would prevent algebra from becoming a mere piece of ingenious mechanism, as it now too often does become.

The procedure here enforced differs in no respect from that employed by the earlier algebraical writers, as may be seen by reference to Wallis, Ronayne, Kersey, Ward, and others. It was also a useful custom, and one which has been recently revived, to number the several successive steps of the process, and to quote the equation by means of the number attached to it. The older writers ruled a column down the middle of the page in which to put the ordinal numbers, and kept the written description of the process on the left, and the work itself on the right of this column. However, in the extended equations, to which modern physical science gives rise, the great inequality in the length of the lines renders it more convenient to write the ordinal numbers, (1), (2), (3) ... at the margin of the page. The mode of taking the ordinal column down the middle is better, however, for the learner, as his *work* is thereby kept in one vertical column to the right of it, and is therefore much more easily inspected by himself as well as by the master. On this account its adoption is advised in the earlier stages of study, even though it may ultimately be laid aside when good and regular habits are formed. Thus, if the equations $x^2 - y^2 = a^2$, and $x + y = b$, had been given, we should have had

Given	1	$x^2 - y^2 = a^2$
	2	$x + y = b$
Divide (1) by (2) then	3	$x - y = \frac{a^2}{b}$
Add (2), (3) together,	4	$2x = \frac{a^2}{b} + b$
or reducing to common den.	5	$2x = \frac{a^2 + b^2}{b}$
and dividing (5) by 2	6	$x = \frac{a^2 + b^2}{2b}$.
Again, subtracting (3) from (2)	7	$2y = b - \frac{a^2}{b}$
and reducing (7) to a common denom.	8	$2y = \frac{b^2 - a^2}{b}$
and dividing (8) by 2.	9	$y = \frac{b^2 - a^2}{2b}$.

In this notation a figure enclosed in a parenthesis, as (2) or (3) indicates the words "the equation marked two," whilst in the case of no parenthesis, it signifies the number 2.

This subject is illustrated and enforced very elegantly in Butler's Course of Mathematics, vol. ii. p. 17. The author correctly traces the first proposal of the practice to Dr. Pell, an eminent analyst of the early English school.

Thus, if $ax = ab - 4a$; then dividing by a, gives $x = b - 4$.

In like manner, if $ax + 3ab = 4c^2$; then by dividing by a, it is $x + 3b = \frac{4c^2}{a}$; and then transposing $3b$, gives $x = \frac{4c^2}{a} - 3b$.

III. When the unknown term is divided by any quantity, we must then multiply all the terms of the equation by that divisor; which takes it away.

[*Note.* If there be several terms in which fractions appear, it is often best to multiply the numerator of every term of the equation by the *least common multiple* of all the denominators, and divide by the corresponding denominator. All the terms are thus cleared of fractions at once, whether known or unknown.]

Thus, if $\frac{x}{4} = 3 + 2$: then mult. by 4, gives $x = 12 + 8 = 20$.

And if $\frac{x}{a} = 3b + 2c - d$: then mult. by a, it gives $x = 3ab + 2ac - ad$.

IV. When the unknown quantity is included under any root or surd expression: transpose the rest of the terms, if there be any, by rule 1; then raise each side to such a power as is denoted by the index of the surd; viz. square each side when it is the square root; cube each side when it is the cube root; and so on, which removes that radical from the equation.

Thus, if $\sqrt{x} - 3 = 4$: then transposing 3, gives $\sqrt{x} = 7$; and squaring both sides gives $x = 49$.

Also, if $\sqrt[3]{3x + 4} + 3 = 6$: then by transposing 3, it is $\sqrt[3]{3x + 4} = 3$; and by cubing, it is $3x + 4 = 27$; and by rules I. II. $x = 7\frac{2}{3}$.

V. When that side or member of the equation which contains the unknown quantity is a complete power, or can easily be reduced to one, by rule I. II. or 3; then extract the root of the said power on both sides of the equation; that is, extract the square root when it is a square power, or the cube root when it is a cube, and so on.

Thus, if $x^2 + 8x + 16 = 36$, or $(x + 4)^2 = 36$; then by extracting the root, it is $x + 4 = 6$; whence $x = 2$.

Also, if $\frac{3}{4}x^2 - 6 = 24$: then transposing 6, gives $\frac{3}{4}x^2 = 30$; and multiplying by 4, gives $3x^2 = 120$; then dividing by 3, gives $x^2 = 40$; and, lastly, extracting the root, gives $x = \sqrt{40} = 6{\cdot}324555$.

VI. When there is any analogy or proportion, it is to be changed into an equation, by multiplying the two extreme terms together, and the two means together, and making the one product equal to the other.

Thus, if $2x : 9 :: 3 : 5$, gives $10x = 27$; and by rule II. $x = 2\frac{7}{10}$.

And if $\frac{3}{4}x : a :: 5b : 2c$; then $\frac{3}{2}cx = 5ab$: hence by rules II. III. $x = \frac{10ab}{3c}$.

VII. When the same quantity is found on both sides of an equation, with the same sign, either plus or minus, it may be cancelled or left out of both; and when every term in an equation is either multiplied or divided by the same quantity, that quantity may be struck out of them all.

Thus, if $\frac{2}{3}x - \frac{7}{3} = \frac{10}{3} - \frac{7}{3}$; then, cancelling $\frac{7}{3}$, it becomes $\frac{2}{3}x = \frac{10}{3}$; and multiplying by 3, it is $2x = 10$; or $x = 5$.

The following example furnishes specimens of all the rules just laid down:

Let $2x + 2\sqrt{a^2 + x^2} = \frac{5a^2}{\sqrt{a^2 + x^2}}$ be given; to find x.

Multiplying by $\sqrt{a^2 + x^2}$, gives $2x\sqrt{a^2 + x^2} + 2a^2 + 2x^2 = 5a^2$; transposing $2a^2 + 2x^2$, gives $2x\sqrt{a^2 + x^2} = 3a^2 - 2x^2$; squaring both sides, $4x^2 \times (a^2 + x^2) = (3a^2 - 2x^2)^2$; that is, $4a^2x^2 + 4x^4 = 9a^4 - 12a^2x^2 + 4x^4$; can-

celling $4x^4$ from both sides, we have $4a^2x^2 = 9a^4 - 12a^2x^2$; transposing $12a^2x^2$, gives $16a^2x^2 = 9a^4$; dividing by a^2 gives $16x^2 = 9a^2$; dividing by 16, gives $x^2 = \frac{9}{16}a^2$; and lastly, extracting the square root, gives $x = \frac{3}{4}a$.

EXAMPLES FOR PRACTICE.

1. Given $2x - 5 + 16 = 21$; to find x. Ans. $x = 5$.
2. Given $6x - 15 = x + 6$; to find x. Ans. $x = 4\frac{1}{5}$.
3. Given $8 - 3x + 12 = 30 - 5x + 4$; to find x. Ans. $x = 7$.
4. Given $x + \frac{1}{3}x - \frac{1}{4}x = 13$; to find x. Ans. $x = 12$.
5. Given $-3x - \frac{1}{2}x - 2 = -5x + 4$; to find x. Ans. $x = +4$.
6. Given $4ax + \frac{1}{3}a - 2 = ax - bx$; to find x. Ans. $x = \dfrac{6-a}{9a+3b}$.
7. Given $\frac{1}{3}x - \frac{1}{4}x + \frac{1}{5}x = \frac{1}{2}$; to find x. Ans. $x = \frac{30}{17}$.
8. Given $\sqrt{4+x} = 4 - \sqrt{x}$: to find x. Ans. $x = 2\cdot25$.
9. Given $4a + x = \dfrac{x^2}{4a+x}$; to determine x. Ans. $x = -2a$.
10. Given $\sqrt{4a^2 + x^2} = \sqrt[4]{4b^4 + x^4}$; to find x. Ans. $x = \sqrt{\dfrac{b^4 - 4a^4}{2a^2}}$.
11. Given $\sqrt{x} + \sqrt{2a+x} = \dfrac{4a}{\sqrt{2a+x}}$; to find x. Ans. $x = \frac{2}{3}a$.
12. Given $\dfrac{a}{1+2x} + \dfrac{a}{1-2x} = 2b$; to find x^2. Ans. $x^2 = \dfrac{b-a}{4b}$.
13. Given $a + x = \sqrt{a^2 + x\sqrt{4b^2 + x^2}}$; to find x. Ans. $x = \dfrac{b^2 - a^2}{a}$.
14. Given $\sqrt{4 + \sqrt{x^4 - x^2}} = x - 2$. Ans. $x = 2\cdot125$.
15. Given $(a+x)(b+x) - a(b+c) = \dfrac{a^2c}{b} + x^2$. Ans. $x = \dfrac{ac}{b}$.
16. In $\sqrt{a+x} - \sqrt{\dfrac{a}{a+x}} = \sqrt{2a+x}$, $x = \dfrac{1 - 2\sqrt{a} - a}{2 + \sqrt{a}}\cdot\sqrt{a}$.
17. Find x in $\dfrac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = \sqrt{b}$. Ans. $x = \dfrac{2a\sqrt{b}}{1+b}$.
18. Find x in $\cdot15x + \cdot2 - \cdot875x + \cdot375 = \cdot0625x - 1$. Ans. $x = 2$.
19. Given $\dfrac{2x + 3\cdot5}{9} - \dfrac{13x - 22}{17x - 32} + \dfrac{x}{3} = \dfrac{7x}{12} - \dfrac{x+16}{36}$. Ans. $x = 4$.
20. Find x from $\sqrt[3]{a + \sqrt{x}} + \sqrt[3]{a - \sqrt{x}} = \sqrt[3]{b}$.

    Ans. $x = \dfrac{8a^3 + 15a^2b + 6ab^2 - b^3}{27b}$.
21. In $3\cdot25x - 5\cdot007 - x = \cdot2 - \cdot34x$, what is x? Ans. $2\cdot010424\ldots$
22. Given $(2+x)^{\frac{1}{2}} + x^{\frac{1}{2}} = 4\ (2+x)^{-\frac{1}{2}}$ to find x. Ans. $x = \frac{2}{3}$.
23. Given $\dfrac{\sqrt{a^2 - y^2}}{\sqrt{a - y}} + y = a + 2y$ to find y. Ans. $y = 1 - a$.
24. Given $x + \sqrt{a - x} = a\,(a - x)^{-\frac{1}{2}}$ to find x. Ans. $x = a - 1$.
25. Given $\dfrac{(x + x^2)^{\frac{1}{2}} - \cdot3333\ldots \text{ ad inf.}}{x} = 1$, to find x. Ans. $x = \frac{1}{3}$.
26. Given $x + \dfrac{a}{b}x + \dfrac{c}{b}x = m$ to find x. Ans. $x = \dfrac{bm}{a+b+c}$.

27. Given $\sqrt{\sqrt[3]{\sqrt[4]{a+x}}} = b$; to find x; and $(z - \frac{3}{4})^3 = 100$ to find z^2.
Ans. $x = b^{24} - a$, and $z^2 = 29{\cdot}06923 \ldots$

28. When $\frac{-\cdot 5}{x^2} = \frac{x^3}{\frac{1}{3}}$, what is the value of $x^{-\cdot 5}$?

29. If the recurring decimal $\cdot 082082 \ldots$ be multiplied by x^2, and the square of the result divided by x^3, gives the same value, $\cdot 082 \ldots$; what is the value of x?
Ans. $\frac{999}{82}$.

30. Resolve $\frac{(a+b)(x-b)}{a-b} - 3a = \frac{4ab - b^2}{a+b} - 2x + \frac{a^2 - bx}{b}$.

Ans. $x = \frac{a^4 + 3a^3b + 4a^2b^2 - 6ab^3 + 2b^4}{2b(a+b)(2a-b)}$.

31. Given $\frac{2v + v^{\frac{1}{2}}}{2v - v^{\frac{1}{2}}} + \frac{2v - v^{\frac{1}{2}}}{2v + v^{\frac{1}{2}}} = 2\frac{4}{15}$; to find v. Ans. $v = 4$.

SIMULTANEOUS OR COEXISTING EQUATIONS.

When an equation contains two or more unknown quantities, it is obviously insufficient for the determination of the value of any one of them. The methods hitherto laid down enable us to obtain the value of any one symbol which is involved in the equation in terms of the remaining ones, whether they be numeral or literal, known or unknown; but if there be more than one unknown quantity, the expression of the value of any one of them that we may select will involve the remaining unknowns, and such value will therefore be indeterminable till such other conditions are added as shall enable us to assign the values of all these last-named unknowns.

If now a second equation, different in its composition from the former, but involving the same unknowns, be given; then also the value of the selected one can be obtained as in the preceding case; and if this second equation also express a second condition to which the relation of the unknowns is subjected, and which, therefore, must exist *simultaneously* with the former, the two values of the unknown must be identical. There may hence be formed a third equation, which will also be true simultaneously with the two former. This equation will involve one unknown less than either of the others.

If there remain more than one unknown in this equation, it is still incapable of furnishing the value of either of them, as before; and there must hence be still other relations given to render the problem determinate. Suppose then a third equation to express a third condition, which is simultaneous with the other two. Then from this also we can obtain a value of the same unknown that we at first selected, in terms of the remaining ones, and this value equated to either of the other values, will furnish a second equation, containing only the remaining unknowns. Having thus two equations containing the remaining unknowns, we can find two values of a second unknown, and equate them; which will give us one equation which is freed from both the forementioned unknowns. If this yet contain more than one unknown, we shall still want other conditions, and must proceed in the same manner to *eliminate* them one by one from each pair of equations that is either given, or which results from the previous elimina-

tions; till, at last, we arrive at an equation which contains only one unknown, the value of which must be determined as has been already explained and practised.

It will be quite apparent from this reasoning, that there must be as many equations simultaneously given as there are unknown quantities involved in all of them together; and that though some of the equations should not contain all the unknowns, yet they may be conceived to do so by considering 0 as the coefficient of any one that is absent.

Other processes besides that explained above can sometimes be used more advantageously; and as facility in the practice of elimination is best attained by exercise upon the simple cases, the following rules have been adapted to the case of two unknowns. The extension of the same kind of processes to three or more simultaneous equations will then become easy and obvious.

It is, however, to be understood, that any involutions, transpositions, multiplications, or divisions by which the equations can be reduced to simpler forms than the given ones, must be executed previous to the application of any of the special rules here laid down.

TWO SIMULTANEOUS EQUATIONS.

To exterminate or eliminate one of the two unknown quantities from two simultaneous equations; that is, to reduce the two simple equations containing them, to a single one.

I. Find the value of one of the unknown letters, in terms of the other quantities, in each of the equations, by the methods already explained. Then put those two values equal to each other for the new equation, involving only one unknown. The value of this is to be found as before.

It is evidently best to begin by determining the values of that letter which is easiest to be found from the two proposed equations.

EXAMPLES.

1. Given $2x + 3y = 17$, and $5x - 2y = 14$, to find x and y.

From (1) we have $x = \frac{17 - 3y}{2}$, and from (2), $x = \frac{14 + 2y}{5}$.

Equating these gives $\frac{14 + 2y}{5} = \frac{17 - 3y}{2}$, or $y = 3$.

Also $x = \frac{17 - 3y}{2} = 4$.

Or, again, by finding two values of y.

From (1) we have $y = \frac{17 - 2x}{3}$, and from (2), $y = \frac{5x - 14}{2}$.

Equating these, $\frac{17 - 2x}{3} = \frac{5x - 14}{2}$, or $x = 4$.

Also $y = \frac{17 - 2x}{3} = 3$.

2. Given $\frac{1}{2} x + 2y = a$, and $\frac{1}{2} x - 2y = b$; to find x and y.

Ans. $x = a + b$, and $y = \frac{1}{4}a - \frac{1}{4}b$.

3. In $3x + y = 22$, and $3y + x = 18$, find x and y. Ans. $x = 6$, and $y = 4$.

4. In $\frac{1}{2}x + \frac{1}{3}y = 4$, and $\frac{1}{3}x + \frac{1}{2}y = 3\frac{1}{2}$; $x = 6$, $y = 3$.

5. Given $\frac{2x}{3} + \frac{3y}{5} = \frac{22}{5}$, and $\frac{3x}{5} + \frac{2y}{3} = \frac{67}{15}$. Ans. $x = 3$, and $y = 4$.

6. In $x + 2y = s$, and $x^2 - 4y^2 = d^2$; $x = \frac{s^2 + d^2}{2s}$, and $y = \frac{s^2 - d^2}{4s}$.

7. In $x - 2y = d$, and $x : y :: a : b$; $x = \frac{ad}{a - 2b}$, and $y = \frac{bd}{a - 2b}$.

8. Given $b(x + y) = a(x - y)$, and $x^2 - y^2 = c^2$; to find x and y.

Ans. $x = \frac{(a + b)c}{2\sqrt{ab}}$, and $y = \frac{(a - b)c}{2\sqrt{ab}}$.

9. Given $a_{,}x + b_{,}y = c_{,}^2$, and $a_{,,}x + b_{,,}y = c_{,,}^2$; to find x and y.

Ans. $x = \frac{b_{,,}c_{,}^2 - b_{,}c_{,,}^2}{a_{,}b_{,,} - a_{,,}b_{,}}$; $y = \frac{a_{,}c_{,,}^2 - a_{,,}c_{,}^2}{a_{,}b_{,,} - a_{,,}b_{,}}$.

10. Given $\cdot 3x^{\frac{1}{2}} - \cdot 03y^{\frac{1}{2}} = 300$, and $300y^{\frac{1}{2}} + 30x^{\frac{1}{2}} = 30000$; to find x^{-2} and $y^{\frac{3}{2}}$. Ans. $y^{\frac{3}{2}} = 0$, $x^{-2} = 10^{-12}$.

II. Find the value of one of the unknown letters, in one of the equations, as in the former rule, and substitute this value instead of that unknown quantity in the other equation: then there will arise a new equation, with only one unknown quantity, whose value is to be found as before.

It is evidently best to begin with that letter whose value is most easily found in either of the given equations.

EXAMPLES.

1. Given $2x + 3y = 17$, and $5x - 2y = 14$, to find x and y.

From (1), $x = \frac{17 - 3y}{2}$, which substituted in (2) gives $\frac{85 - 15y}{2} - 2y = 14$, or $y = 3$; and $x = 4$.

Or, finding x from the second equation.

Here $x = \frac{14 + 2y}{5}$, which, substituted in (1) gives $\frac{28 + 4y}{5} + 3y = 17$, or $y = 3$; and hence $x = 4$.

In a similar way we may begin to operate by finding y from either of the equations and substituting its value in the other.

2. In $2x + 3y = 29$, and $3x - 2y = 11$, we have $x = 7$, and $y = 5$.

3. In $x + y = 14$, and $x - y = 2$, we have $x = 8$, and $y = 6$.

4. In $x : y :: 3 : 2$, and $x^2 - y^2 = 20$, we have $x = 6$, and $y = 4$.

5. In $\frac{x}{3} + 3y = 21$, and $\frac{y}{3} + 3x = 29$; $x = 9$, and $y = 6$.

6. Given $10 - \frac{x}{2} = \frac{y}{3} + 4$, and $\frac{x - y}{2} + \frac{x}{4} - 2 = \frac{3y - x}{5} - 1$; to find x and y. Ans. $x = 8$, $y = 6$.

7. Given $x : y :: 4 : 3$, and $x^3 - y^3 = 37$; to find the product of x^2 and y^3, and the difference of x and y. Ans. $x^2 y^3 = 432$, $x - y = 1$.

8. From $x + y = a(x - y)$, and $x^2 + y^2 = b^2$; find x^2 and y^2.

Ans. $x^2 = \frac{(a + 1)^2 b^2}{2(a^2 + 1)}$; $y^2 = \frac{(a - 1)^2 b^2}{2(a^2 + 1)}$.

9. In $bx - cy = 0$, and $x^3 - y^3 = a^3$; what are the values of $x^{\frac{1}{6}}$ and $y^{\frac{1}{6}}$?

Ans $x^{\frac{1}{6}} = \dfrac{a^{\frac{1}{6}} c^{\frac{1}{6}}}{(c^3 - b^3)^{\frac{1}{18}}}, y^{\frac{1}{6}} = \dfrac{a^{\frac{1}{6}} b^{\frac{1}{6}}}{(c^3 - b^3)^{\frac{1}{18}}}$.

10. Given $7x + \frac{5}{2}y = 411\frac{1}{2}$, and $39x - 14y = -935\frac{9}{10}$; to find x and y.

Ans. $x = 17\frac{1}{2}$, and $y = 115\frac{3}{5}$.

11. Given $(x + 5)(y + 7) = (x + 1)(y - 9) + 112$, and $2x + 10 = 3y + 1$; to find x and y. Ans. 5 and 3: which is x?

12. From the equations $\dfrac{a}{b + y} = \dfrac{b}{3a + x}$, and $ax + 2by = d$; to find x and y.

Ans. $y = \dfrac{3a^2 - b^2 + d}{3b}$, and $x = \dfrac{2b^2 - 6a^2 + d}{3a}$.

III. Let the given equations be so multiplied, or divided, by such numbers or quantities, as will make the terms which contain one of the unknown quantities the same in both equations.

Then by adding or subtracting the equations, according as the signs may require, there will result a new equation, with only one unknown quantity, as before: that is, add the two equations when the signs are unlike, but subtract them when the signs are alike, to cancel that common term.

The best multipliers generally are those of the selected term in the alternate equations; as in the example, $ax + by = c^2$, and $a_{,}x + b_{,}y = c_{,}^2$, where the first equation being multiplied by $a_{,}$, and the second by a, we get the co-efficients of x equal.

Again, it will often happen that a and $a_{,}$ have a common measure; and in this case, instead of a and $a_{,}$, take the quotients of them by that common measure for the cross-multipliers. This will always, when it can be done, lessen the arithmetical labour.

Let us take as a numerical example $4x + 6y = 2$, and $10x - 3y = 59$.

Here in eliminating x, the co-efficients 4 and 10 have the common measure 2, and hence 5 and 2 are the multipliers. Hence we get $20x + 30y = 10$, and $20x - 6y = 118$.

Subtracting, we have $36y = -108$, or $y = -3$; and hence $x = 5$.

Or again, multiply (2) by 2, and (1) by 1, then $4x + 6y = 2$, and $-6y + 20x = 118$; and adding $24x = 120$, or $x = 5$; and hence again $y = -3$.

EXAMPLES.

1. In $\dfrac{x + 8}{4} + 6y = 21$, and $\dfrac{y + 6}{3} + 5x = 23$; $x = 4$, and $y = 3$.

2. In $\dfrac{3x - y}{4} + 10 = 13$, and $\dfrac{3y + x}{2} + 5 = 12$; $x^{-1} = \frac{1}{3}$, and $y^{-1} = \frac{1}{3}$.

3. In $\dfrac{3x^3 + 4y^2}{5} + \dfrac{x^3}{4} = 10$, and $\dfrac{6x^3 - 2y^2}{3} + \dfrac{y^2}{6} = 14$; $x = 2$, and $y = 2$.

4. In $3x - 4y = 38$, and $4x + 3y = 9$; $x = 6$, and $y = -5$.

5. In $\dfrac{4x - 2y + 3}{3} - \dfrac{18 - x + 5y}{7} = \dfrac{x}{4} - \dfrac{y}{5} - \dfrac{1}{7} - 7\dfrac{7}{10}$, and $\left\{2x - y + 15\right\} \left\{\dfrac{y}{4} - \dfrac{x}{3} + \dfrac{1}{12}\right\} = \left\{y - 2x + 15\right\} \left\{\dfrac{x}{3} - \dfrac{y}{4} + \dfrac{3}{4}\right\}$; $\frac{1}{3}x = 6$, $\dfrac{y}{4} = 6$.

6. Given $(x + \frac{1}{2})(y + \cdot7) = (x + \frac{1}{10})(y - \cdot9) + 11\frac{1}{3}$, and $\dfrac{x}{5} + 1 = \dfrac{3y}{10} + \cdot1$; to find x and y. Ans. $x = 5\frac{17}{110}$, and $y = 6\frac{29}{70}$.

7. From $\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c}$, and $\frac{y}{a}+\frac{x}{b}=1+\frac{y}{c}$; find the values of x and y.

Ans. $x=\frac{abc\,(ac+ab-bc)}{a^2\,b^2+a^2\,c^2-b^2\,c^2}$, and $y=\frac{abc\,(ac-ab-bc)}{a^2\,b^2+a^2\,c^2-b^2\,c^2}$.

8. In $x\,(bc-xy)=y\,(xy-ac)$, and $xy\;(ay+bx-xy)=abc\,(x+y-c)$; find $x^{\frac{1}{2}}$ and $y^{\cdot 5}$. Ans. $x^{\frac{1}{2}}=\pm\sqrt{\pm\sqrt{\pm\,ac}}$, and $y^{\cdot 5}=\pm\sqrt{\pm\sqrt{\pm\,bc}}$.

In the examples given under the different rules for elimination, those have not always been chosen which are most simply solved by the rule under which they are given. This was purposely done for the sake of leading the student to examine the different questions by other of the rules, so as to afford him the means of judging in some degree from the appearance of any given equations, which rule will be most applicable to their solution. The improvement in his judgment will amply repay him for the trouble of solving each of them by all the methods.

THREE OR MORE SIMULTANEOUS EQUATIONS.

These are in their nature and mode of solution precisely similar to those already treated of: they are, however, generally longer, and often the particular process to be employed is less obvious.

When it happens that the co-efficients of several of the equations are alike with the same or opposite signs, it will be found advantageous to add the equations together, and divide the sum by the number of them; then subtracting each equation from this quotient will give the value of each of the unknowns in succession. Sometimes when they are combined in factors, it will be advantageous to multiply or to divide only the others, and thus get equations in a simpler form. These, however, are special rules, and can only be acquired by observation and practice.

Ex. 1. Given $x+y+z=9$, $x+2y+3z=16$, and $x+3y+4z=21$.

By the first method we have three values of x, viz.:

$$x=9-y-z,\ x=16-2y-3z,\ \text{and}\ x=21-3y-4z.$$

Equating the second value with each of the others, we have

$$9-\ y-\ z=16-2y-3z,\ \text{or}\ y=7-2z,\ \text{and}$$
$$16-2y-3z=21-3y-4z,\ \text{or}\ y=5-z.$$

Equating these values of y, we get $5-z=7-2z$, or $z=2$.

Hence $y=5-z=3$, and $x=9-y-z=4$.

Or, by the second method, we have from the first equation $x=9-y-z$; and substituted in the two others gives

$$9+y+2z=16,\ \text{or}\ y=7-2z;\ \text{and}\ 9+2y+3z=21.$$

Substitute the former of these in the latter; then we get $23-z=21$, or $z=2$; and x and y will be found as in the last case.

Again, by the third method, the co-efficients of x are already equal; hence the equations are prepared for subtraction. Let (1) be taken from (2), and (2) from (3), then $y+2z=7$, and $y+z=5$.

The coefficients of y are here equal, hence subtracting, $z=2$; and hence x and y may be found.

Ex. 2. Given $x+y=10$, $y+z=23$, and $z+x=19$.

Add all three together and divide by 2: then we get

$$x+y+z=26.$$

Subtract each of the given equations from this, and we find

$$x = 3,\ y = 7, \text{ and } z = 16.$$

The student should also solve this by the other methods.

Ex. 3. Given $xy = a^2$, $yz = b^2$, and $zx = c^2$.

Multiply all three together: then $x^2y^2z^2 = a^2b^2c^2$, or $xyz = \pm\, abc$.

Divide this by each of the given equations: then there will result

$$x = \pm\frac{ac}{b},\ y = \pm\frac{ab}{c}, \text{ and } z = \pm\frac{bc}{a}.$$

This example is an instance of the remark on classification at p. 170; and would, in strict theory, like some that have gone before, be classed as a quadratic.

Ex. 4. There are given the three following equations for solution:

$$ax + by + cz = d^2 \ \ldots\ldots\ (1)$$
$$a_{,}x + b_{,}y + c_{,}z = d_{,}^2 \ \ldots\ (2)$$
$$a_{,,}x + b_{,}y + c_{,,}z = d_{,,}^2 \ \ldots\ (3)$$

Multiply (1) by $a_{,}$ and (2) by a, and subtract: then we get

$$(a_{,}b - ab_{,})\,y + (a_{,}c - ac_{,})\,z = a_{,}d^2 - ad_{,}^2 \ \ldots\ (4)$$

Multiply (2) by $a_{,,}$ and (3) by $a_{,}$, and subtract: then we have

$$(a_{,,}b_{,} - a_{,}b_{,,})\,y + (a_{,,}c_{,} - a_{,}c_{,,})\,z = a_{,,}d_{,}^2 - a_{,}d_{,,}^2 \ \ldots\ (5)$$

Multiply (4) by $a_{,,}b - a_{,}b_{,,}$, and (5) by $a_{,}b - ab_{,}$, and subtract: then resolving for z, and substituting in the values of x and y, we have

$$z = \frac{d^2(a_{,,}b_{,} - a_{,}b_{,,}) + d_{,}^2(ab_{,,} - a_{,,}b) + d_{,,}^2(a_{,}b - ab_{,})}{c(a_{,,}b_{,} - a_{,}b_{,,}) + c_{,}(ab_{,,} - a_{,,}b) + c_{,,}(a_{,}b - ab_{,})}$$

$$y = \frac{d^2(a_{,,}c_{,} - a_{,}c_{,,}) + d_{,}^2(ac_{,,} - a_{,,}c) + d_{,}^2(a_{,}c - ac_{,})}{b(a_{,,}c_{,} - a_{,}c_{,,}) + b_{,}(ac_{,,} - a_{,,}c) + b_{,,}(a_{,}c - ac_{,})}$$

$$x = \frac{d^2(b_{,,}c_{,} - b_{,}c_{,,}) + d_{,}^2(bc_{,,} - b_{,,}c) + d_{,,}^2(b_{,}c - bc_{,})}{a(b_{,,}c_{,} - b_{,}c_{,,}) + a_{,}(bc_{,,} - b_{,,}c) + a_{,,}(b_{,}c - bc_{,})}.$$

This is the *general solution* for three unknowns, and by substituting any given numbers for the co-efficients in these, the corresponding values of x, y, z, would be obtained.

Ex. 5. Equations of the following forms are of very frequent occurrence in the subsequent parts of algebra, and hence it may be desirable to indicate the best mode of resolving them.

$$\ldots + 1^3u + 1^2x + 1^1y + z = a_1$$
$$\ldots + 2^3u + 2^2x + 2^1y + z = a_2$$
$$\ldots + 3^3u + 3^2x + 3^1y + z = a_3$$
$$\ldots + 4^3u + 4^2x + 4^1y + z = a_4$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

where there are as many such equations as there are unknowns; and the second side of each equation given.

Subtract the first from the second, the second from the third, the third from the fourth, and so on to the end. This will give $n - 1$ equations clear of z. Pursue the same course with these $n - 1$ equations, and we shall obtain $n - 2$ equations clear of z and y. Pursue again the same course, and $n - 3$ equations will be obtained clear of z, y, and x. Proceeding thus, we shall at last obtain a single equation involving only the letter to the left, whose value is thus found. Substitute this value in either of the two results obtained by the above process immediately before the last, and we obtain the value of the second letter. Substitute these two in either of the three next preceding results, and we shall get the value next unknown. We may thus obtain the whole very simply and conveniently.

The following example, adapted to four unknowns, may serve to illustrate the process and mode of writing the successive steps of the work.

The given equations.	First differences.	Second diffs.	Third diff.
$u + x + y + z = 2$	$7u + 3x + y = 22$	$12u + 2x = 62$	$6u = 66$
$8u + 4x + 2y + z = 24$	$19u + 5x + y = 84$	$18u + 2x = 128$	
$27u + 9x + 3y + z = 108$	$37u + 7x + y = 212$		
$64u + 16x + 4y + z = 320$			

Whence $u = 11$; which substituted in either of the second differences (that with the least co-efficient will of course be most convenient) gives $x = 31 - 6u = -35$; and these values of u and x in the first difference gives $y = 22 - 3x - 7u = 22 + 105 - 77 = 50$; and lastly, $z = 2 - y - x - u = 2 - 50 + 35 - 11 = -24$.

The student may solve the following examples.

$$\begin{aligned} u + x + y + z &= 1 \\ 8u + 4x + 2y + z &= 4 \\ 27u + 9x + 3y + z &= 10 \\ 64u + 16x + 4y + z &= 20 \end{aligned}$$

Ans. $u = \frac{1}{6}$, $x = \frac{1}{2}$, $y = \frac{1}{3}$, $z = 0$

$$\begin{aligned} u + x + y + z &= 1 \\ 8u + 4x + 2y + z &= 5 \\ 27u + 9x + 3y + z &= 14 \\ 64u + 16x + 4y + z &= 30 \end{aligned}$$

Ans. $u = \frac{1}{3}$, $x = \frac{1}{2}$, $y = \frac{1}{6}$, $z = 0$.

EXAMPLES FOR PRACTICE.

1. When $\left\{\begin{aligned} x + y + z &= 18 \\ x + 3y + 2z &= 38 \\ x + \tfrac{1}{3}y + \tfrac{1}{2}z &= 10 \end{aligned}\right\}$; then $x = 4$, $y = 6$, $z = 8$.

2. If $\left\{\begin{aligned} x + \tfrac{1}{2}y + \tfrac{1}{3}z &= 27 \\ x + \tfrac{1}{3}y + \tfrac{1}{4}z &= 20 \\ x + \tfrac{1}{4}y + \tfrac{1}{5}z &= 16 \end{aligned}\right\}$; then $x = 1$, $y = 12$, $z = 60$.

3. If $x - y = 2$, $x - z = 3$, and $y + z = 9$; then $x = 7$, $y = 5$, $z = 4$.

4. When $\left\{\begin{aligned} x\,(x + y + z) &= 45 \\ y\,(x + y + z) &= 75 \\ z\,(x + y + z) &= 105 \end{aligned}\right\}$; then $x = \pm 3$, $y = \pm 5$, and $z = \pm 7$.

5. Given $uxyz = a$, $tuxy = b$, $tuxz = c$, $tuyz = d$, $txyz = e$; to find t, u, x, y, z.

Ans. $t = \dfrac{\sqrt[4]{abcde}}{a}$, $u = \dfrac{\sqrt[4]{abcde}}{b}$, $x = \dfrac{\sqrt[4]{abcde}}{c}$, $y = \dfrac{\sqrt[4]{abcde}}{d}$, and $z = \dfrac{\sqrt[4]{abcde}}{e}$

6. Given $x + y + z = a$, $my = nx$, and $pz = qx$; to find x, y, z.

Ans. $x = \dfrac{amp}{mp + np + mq}$, $y = \dfrac{anp}{mp + np + mq}$, and $z = \dfrac{amq}{mp + np + mq}$.

7. Given $\dfrac{xy}{ay + bx} = l$, $\dfrac{yz}{cz + dy} = m$, and $\dfrac{xz}{ez + fx} = n$; to find x, y, z.

Ans. $x = \dfrac{lmn\,(bde + acf)}{cfmn - bfln + bdlm}$, $y = \dfrac{lmn\,(bde + acf)}{afln + demn - adlm}$, $z = \dfrac{lmn\,(bde + acf)}{beln - cemn + aclm}$.

8. Given $\left\{\begin{aligned} x + y + z + t + u &= a \\ x + y + z + u + w &= b \\ x + y + z + t + w &= c \\ x + y + u + t + w &= d \\ x + z + u + t + w &= e \\ y + z + u + t + w &= f \end{aligned}\right.$

Ans. put $s = \dfrac{a + b + c + d + e + f}{5}$; then $w = s - a$, $t = s - b$, $u = s - c$, $z = s - d$, $y = s - e$, $x = s - f$.

9. Given $x\,(y + z) = a^2$, $y\,(x + z) = b^2$, and $z\,(x + y) = c^2$; to find the unknowns.

10. Given $x - ay + a^2z - a^3 = 0$, $x - by + b^2z - b^3 = 0$, and $x - cy + c^2z - c^3 = 0$.

11. Given $xy = a(x + y)$, $xz = b(x + z)$, and $yz = c(y + z)$; to find the reciprocals of x, y, z.

12. Given $\left\{\begin{array}{l} u + x + y + z = 4 \\ ux + uy + uz + xy + xz + yz = 6 \\ uxy + uxz + uyz + xyz = 4 \\ uxyz = 1 \end{array}\right\}$; to find u, x, y, z.

A COLLECTION OF QUESTIONS PRODUCING SIMPLE EQUATIONS.

QUEST. 1. To find two numbers, such, that their sum shall be 10, and their difference 6.

Let x denote the greater number, and y the less *.

Then the first and second conditions are $x + y = 10$, and $x - y = 6$, $x = 10 - y$; whence $x = 8$, $y = 2$.

QUEST. 2. Divide 100*l* among A, B, C, so that A may have 20*l* more than B, and B 10*l* more than C.

Let A's share $= x$, B's $= y$, and C's $= z$.

Then $x + y + z = 100$, $x = y + 20$, and $y = z + 10$.

From which $x = 50$, $y = 30$, and $z = 20$.

QUEST. 3. A prize of 500*l* is to be divided between two persons, so that their shares may be in proportion as 7 to 8; required the share of each.

Put x and y for the two shares; then the conditions are

$7 : 8 :: x : y$, or $7y = 8x$, and $x + y = 500$; hence $x = 233\frac{1}{3}$ and $y = 266\frac{2}{3}$.

QUEST. 4. What fraction is that, to the numerator of which if 1 be added, the value will be $\frac{1}{2}$; but if 1 be added to the denominator, its value will be $\frac{1}{3}$?

Denote the fraction by $\frac{x}{y}$: then $\frac{x+1}{y} = \frac{1}{2}$, and $\frac{x}{y+1} = \frac{1}{3}$.

From which $x = 3$, $y = 8$, and $\frac{x}{y} = \frac{3}{8}$.

QUEST. 5. A labourer engaged to serve for 30 days on these conditions: that for every day he worked, he was to receive 20*d*, but for every day he played, or was absent, he was to forfeit 10*d*. Now at the end of the time he had to receive just 20 shillings, or 240 pence. It is required to find how many days he worked, and how many he was idle?

Let x be the days worked, and y the days idled.

Then $20x$ are the pence earned, and $10y$ the forfeits;

Hence, by the question $x + y = 30$, and $20x - 10y = 240$;

Whence $x = 18$, the days worked; and $y = 12$, the days idled.

QUEST. 6. Out of a cask of wine which had leaked away $\frac{1}{4}$, 30 gallons were drawn, and then, being gauged, it appeared to be half full: how much did it hold?

Suppose it held x gallons; then it leaked $\frac{1}{4}x$ gallons.

Hence there had been $\frac{1}{4}x + 30$ gallons taken away, and by the question, $\frac{1}{2}x = \frac{1}{4}x + 30$; and $x = 120$, the gallons it held.

---

* In these solutions, as many unknown letters are always used as there are unknown numbers to be found, purposely for exercise in the modes of reducing the equations: avoiding the short ways of notation, which, though they may give neater solutions, afford less exercise in practising several rules in reducing equations. It is also considered unnecessary to carry out the solutions to their completion, as the steps are so familiar to the student from the exercise in reduction which has preceded. The examples, indeed, are given principally with a view to practice in the translation of the verbal conditions of a question into the symbolical language of algebra.

QUEST. 7. To divide 20 into two such parts, that 3 times the one part added to 5 times the other may make 76.

Let x and y denote the two parts.

Then, by the question, $x + y = 20$, and $3x + 5y = 76$.

From which $x = 12$, and $y = 8$.

QUEST. 8. A market woman bought in a certain number of eggs at 2 a penny, and as many more at 3 a penny, and sold them all out again at the rate of 5 for two-pence, and by so doing, contrary to expectation, found she lost three-pence; what number of eggs did she buy?

Suppose she bought x eggs of each kind: then the cost of the first lot was $\frac{1}{2}x$, and that of the second lot was $\frac{1}{3}x$. Also in selling $2x$ eggs at 5 for two-pence, she received $\frac{2}{5} . 2x$ pence: and by the question, this was three-pence less than she gave for them. Hence $\frac{x}{2} + \frac{x}{3} - 3 = \frac{4x}{5}$; and therefore $x = 90$, the number in each lot, or $2x = 180$, the whole number.

QUEST. 9. Two persons, A and B, engage at play. Before they begin, A has 80 guineas, and B has 60: but after a certain number of games won and lost between them, A rises with three times as many guineas as B: how many guineas did A win of B?

Denote by x the number of guineas won by A. Then they rise with $80 + x$ and $60 - x$ respectively. But by the question $80 + x = 3(60 - x)$: hence $x = 25$, the guineas won by A.

QUEST. 10. The sum of the three digits composing a certain number is 16; the sum of the left and middle digits is to the sum of the middle and right ones as 3 to $3\frac{2}{3}$; and if 198 be added to the number, the digits will be inverted in the expression of this sum.

Let x, y, z denote the digits; then $100x + 10y + z$ will express the number itself, and $100z + 10y + z$ will express the number having the same digits in an inverted order. Whence the three conditions are

$$x + y + z = 16,\ x + y : y + z :: 3 : 3\tfrac{2}{3}, \text{ and}$$

$$100x + 10y + z + 198 = 100z + 10y + x, \text{ or } z - x = 2.$$

Whence the solution is $x = 5$, $y = 4$, and $z = 7$; and the number itself is $5 . 100 + 4 . 10 + 7$, or 547.

QUEST 11. If N men of a certain degree of skill can do a piece of work in n days, N_{I} others of different skill in n_{I} days, N_{II} others in n_{II} days, and so on for m sets of men: in how many days would one of each set be able to do $\frac{1}{p}$ of the work, supposing they all worked together, without impeding each other's operations?

Each of the first set would do $\frac{1}{Nn}$ of the work per day;

.... second .... $\frac{1}{N_{I}n_{I}}$ ....

....

.... mth .... $\frac{1}{N_{m}\,n_{m}}$ ....

Suppose that one of each of these men working together could execute the pth part of the work in x days: then they would execute the whole in px days, or $\frac{1}{px}$ in one day. Hence equating the two expressions for their total work in one day, we have

$$\frac{1}{Nn} + \frac{1}{N_I n_I} + \frac{1}{N_{II} n_{II}} + \ldots \frac{1}{N_m n_m} = \frac{1}{px} : \text{ hence}$$

$$x = \frac{\frac{1}{p}}{\frac{1}{Nn} + \frac{1}{N_I n_I} + \frac{1}{N_{II} n_{II}} + \ldots \frac{1}{N_m n_m}}.$$

QUESTIONS FOR PRACTICE.

It is recommended that the student should also solve these questions generally, by taking literal symbols instead of the given numbers.

1. DETERMINE two numbers such, that their difference may be 4, and the difference of their squares 64. Ans. 6 and 10.

2. Find two numbers with these conditions, viz. that half the first with a third part of the second may make 9, and that a fourth part of the first with a fifth part of the second may make 5. Ans. 8 and 15.

3. Divide the number 2 into two such parts, that a third of the one part added to a fifth of the other may make $\frac{3}{5}$. Ans. 1·5 and ·5.

4. Find three numbers such, that the sum of the 1st and 2d shall be 7, the sum of the 1st and 3d 8, and the sum of the 2d and 3d 9. Ans. 3, 4, 5.

5. A father, dying, bequeathed his fortune, which was 2800*l*, to his son and daughter, in this manner; that for every half-crown the son might have, the daughter was to have a shilling: what then were their two shares?

Ans. the son 2000*l*, and the daughter 800*l*.

6. Three persons, A, B, C, make a joint contribution, which in the whole amounts to 400*l*: of which sum B contributes twice as much as A and 20*l* more; and C as much as A and B together: what sum did each contribute?

Ans. A 60*l*, B 140*l*, and C 200*l*.

7. A person paid a bill of 100*l* with half-guineas and crowns, using in all 202 pieces; how many pieces were there of each sort?

Ans. 180 half-guineas and 22 crowns.

8. Says A to B, if you give me 10 guineas of your money, I shall then have twice as much as you will have left; but says B to A, give me 10 of your guineas, and then I shall have 3 times as many as you: how many had each?

Ans. A 22, B 26.

9. A person goes to a tavern with a certain quantity of money in his pocket, where he spends 2 shillings; he then borrows as much money as he had left, and going to another tavern, he there spends 2 shillings also; then borrowing again as much money as was left, he went to a third tavern, where likewise he spent 2 shillings; and thus repeating the same at a fourth tavern, he then had nothing remaining: what sum had he at first, and what was he in debt?

Ans. at first 3*s* 9*d*, and had borrowed 4*s* 3*d*.

10. A man with his wife and child dine together at an inn. The landlord charged 1 shilling for the child; for the woman as much as for the child, and $\frac{1}{4}$ as much as for the man; and for the man as much as for the woman and child together: how much was that for each?

Ans. the woman 20*d*, and the man 32*d*.

11. A cask, which held 60 gallons, was filled with a mixture of brandy, wine, and cyder, in this manner, viz. the cyder was 6 gallons more than the brandy, and the wine was as much as the cyder and $\frac{1}{3}$ of the brandy: how much was there of each? Ans. brandy 15, cyder 21, wine 24.

12. A general, disposing his army into a square form, finds that he has 284 men more than a perfect square; but increasing the side by 1 man, he then wants 25 men to be a complete square: how many men had he under his command? Ans. 24000.

13. What number is that, to which if 3, 5, and 8 be severally added, the three sums shall be in geometrical progression? Ans. 1.

14. The stock of three traders amounted to 760*l*: the shares of the first and second exceeded that of the third by 240*l*; and the sum of the second and third exceeded the first by 360*l*: what was the share of each?

Ans. the 1st 200*l*, the 2d 300*l*, and the 3d 260*l*.

15. What two numbers are those, which, being in the ratio of 3 to 4, their product is equal to 12 times their sum? Ans. 21 and 28.

16. A certain company at a tavern, when they came to settle their reckoning, found that had there been 4 more in company, they might have paid a shilling each *less* than they did; but that if there had been 3 fewer in company, they must have paid a shilling each *more* than they did: what then was the number of persons in company, what did each pay, and what was the whole reckoning?

Ans. 24 persons, each paid 7*s*, and the whole reckoning was 8 guineas.

17. A jockey has two horses; and also two saddles, the one valued at 18*l*, the other at 3*l*. Now when he sets the better saddle on the 1st horse, and the worse on the 2d, it makes the 1st horse worth double the 2d; but when he places the better saddle on the 2d horse, and the worse on the 1st, it makes the 2d horse worth three times the 1st: what were the values of the two horses?

Ans. the 1st 6*l*, and the 2nd 9*l*.

18. What two numbers are as 2 to 3, to each of which if 6 be added, the sums will be as 4 to 5? Ans. 6 and 9.

19. What are those two numbers, of which the greater is to the less as their sum is to 20, and as their difference is to 10? Ans. 15 and 45.

20. What two numbers are those, whose difference, sum, and product, are to each other, as the three numbers 2, 3, 5? Ans. 2 and 10.

21. Find three numbers in arithmetical progression, of which the first is to the third as 5 to 9, and the sum of all three is 63. Ans. 15, 21, 27.

22. It is required to divide the number 24 into two such parts, that the quotient of the greater part divided by the less, may be to the quotient of the less part divided by the greater, as 4 to 1. Ans. 16 and 8.

23. A gentleman being asked the age of his two sons, answered, that if to the sum of their ages 18 be added, the result will be double the age of the elder; but if 6 be taken from the difference of their ages, the remainder will be equal to the age of the younger: what then were their ages? Ans. 30 and 12.

24. Find four numbers such, that the sum of the 1st, 2d, and 3d shall be 13; the sum of the 1st, 2d, and 4th, 15; the sum of the 1st, 3d, and 4th, 18; and lastly, the sum of the 2d, 3d, and 4th, 20. Ans. 2, 4, 7, 9.

25. Divide 48 into 4 such parts, that the first increased by 3, the second diminished by 3, the third multiplied by 3, and the fourth divided by 3, may be all equal to each other. Ans. 6, 12, 3, 27.

26. A cistern is to be filled with water from three different cocks: from the first it can be filled in 8 hours, from the second in 10, and from the third in 14: how soon would they all together fill it? Ans. in 3 h 22 min $24\frac{48}{83}$ sec.

27. Show at what periods the hands of a watch will be together during a complete revolution of the hour-hand.

Ans. at $\frac{12m}{11}$, where m is 1, 2, ... 11 successively.

28. A labourer engages to work for 3*s* 6*d* a day and his board, but to allow 9*d* for his board each day that he is unemployed. At the end of 24 days he has to receive 3*l* 2*s* 9*d*: how many days did he work? Ans. 19 days.

29. Three workmen are employed to dig a ditch of 191 yards in length. If A can dig 27 yards in 4 days, B 35 yards in 6 days, and C 40 yards in 12 days, in what time could they do it if they worked simultaneously? Ans. 12 days.

QUADRATIC EQUATIONS.

A QUADRATIC equation is that in which the unknown quantity is of the second degree, and is generally represented by $ax^2 + bx + c = 0$, where a, b, c may be any numbers positive or negative, integer, fractional or irrational.

When $b = 0$, it takes the form $ax^2 + c = 0$, and it is called a *pure quadratic*. It is treated as a simple equation, since in the solution no operation is required but the arithmetical one of extracting the square root of $-\frac{c}{a}$.

When all the terms are present, the equation is called an *adfected quadratic*.

There are two methods of solving such equations; one due to the Hindûs, the other to the early Italian algebraists. They are alike in principle, which is that of so modifying the first side as to render it a complete square; and by corresponding additions, subtractions, multiplications, or divisions, applied to the other side, to still retain the original truth of the equality. The operation is technically called *completing the square.*

1. *The Italian or common method.*

Transpose c, and divide every term by a: then we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Add the square of half the co-efficient of x, viz. of $\frac{b}{a}$, to both sides: then we have

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2},$$

and the first side is a complete square. Extract the roots, and resolve the resulting simple equation. This gives successively

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}, \text{ and } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This operation is often troublesome, on account of the reduction of the second side of the complete square, in actual numbers. When, however, $\frac{b}{a}$ is an even number, positive or negative, and $\frac{c}{a}$ also integer, it is the most convenient: but as this is seldom the case in the quadratics that arise in practice, we shall give a preferable one, viz.:

2. *The Hindû method.*

Let the equation be $ax^2 + bx = -c$. Multiply by $4a$, and add b^2 to the product on each side. Then we have

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac, \text{ and extracting}$$

$$2ax \;+ b = \pm \sqrt{b^2 - 4ac}, \text{ or } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which is precisely the same result as before obtained by the Italian method, though by a process which is *arithmetically* simpler *.

It has already been explained, that the square root of any quantity a^2 is either $+a$ or $-a$, and marked by writing the sign thus, $\pm a$, signifying that the root has both values $+a$ and $-a$. Hence the answers above given are to be understood as twofold in each case, viz.: $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$, and $\frac{-b-\sqrt{b^2-4ac}}{2a}$, either of which substituted for x in the given equation, will render all the terms on one side equivalent to those on the other, taken collectively.

Also, since the square root of a negative quantity cannot be actually extracted in real numbers, positive or negative, when $b^2 - 4ac$ is negative, the equation does not admit of resolution in numbers, or it is only symbolical. The roots, or values of x, are said in this case to be *imaginary;* and wherever such a result appears, it is the indication of contradictory conditions involved in the conditions of the problem which gave rise to the equation.

All equations, in which there are two terms including the unknown quantity, and which have the *index of the one just double that of the other*, are resolved like quadratics, by completing the square, as above.

Thus, $x^4 + ax^2 = b$, or $x^{2n} + ax^n = b$, or $x + ax^{\frac{1}{2}} = b$, or $(x^2 \pm ax)^2 \pm m (x^2 \pm ax) = b$, are analogous to quadratics, and the value of the unknown quantity may be determined accordingly.

It may also on some occasions be useful to remark, that any quadratic equation in which four times the product of the co-efficient of the first term by the third term is equal to the square of the coefficient of the second term, is already a complete square. For let $ax^2 + bx + c = 0$ be the equation: then if $4ac = b^2$, we shall have $c = \frac{b^2}{4a}$, and the equation becomes $ax^2 + b + \frac{b^2}{4a} = 0$, the root of which is $x\sqrt{a} + \frac{b}{2\sqrt{a}} = 0$.

The same of course is true if $ax^2 + bx + c = k$, where k designates any quantity or expression whatever.

EXAMPLES.

1. Given $x^2 + 4x = 60$, to find the values of x.

The *Italian method* applies to this example, and

$$x^2 + 4x + 4 = 64, \text{ or extracting, } x + 2 = \pm 8.$$

Whence $x = 6$, and $x = -10$ are the values of x.

2. Resolve the equation of $3x - 5x = 12$.

The *Hindû method* is applicable in this case.

$$\text{Then } 36x^2 - 60x + 25 = 144 + 25 = 169, \text{ and}$$

$$6x - 5 = \pm 13: \text{ hence } x = 3, \text{ and } x = -\frac{4}{3}.$$

* When the co-efficient of the second term is an even number, it will be sufficient to multiply all the terms by the co-efficient of the first, and add the square of half that of the second to both the products.

For let $a_ix^2 + 2b_ix = c_i$: then the completed equation is $a_i^2x^2 + 2a_ib_ix + b_i^2 = b_i^2 + a_ic_i$, the first side of which is the square of $a_ix + b_i$.

3. Given the equation $\frac{1}{2}x^2 - \frac{1}{3}x + 30\frac{1}{2} = 52\frac{2}{3}$.

Transposing and cancelling the denominators gives

$$3x^2 - 2x = 133.$$

Hence the special rule in the note on the Hindû method gives

$9x^2 - 6x + 1 = 400$, or $3x - 1 = \pm 20$. Whence

$x = 7$, and $x = -6\frac{1}{3}$.

4. Given $x^4 - 2ax^2 = b$, to find x. The *Italian method* can be used here.

Complete the square, then $x^4 - 2ax^2 + a^2 = a^2 + b$;

Extract the roots, then $x^2 - a = \pm \sqrt{a^2 + b}$;

Resolve simple equation $x^2 = a \pm \sqrt{a^2 + b}$;

Resolve pure quadratic $x = \pm \sqrt{a \pm \sqrt{a^2 + b}}$.

That is, x has the four values,

$+\sqrt{a + \sqrt{a^2+b}}, -\sqrt{a + \sqrt{a^2+b}}, +\sqrt{a - \sqrt{a^2+b}}$, and $-\sqrt{a - \sqrt{a^2+b}}$.

5. Given $\sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} = x$, to find x.

Transpose: then $\sqrt{1 - \frac{1}{x}} = x - \sqrt{x - \frac{1}{x}}$;

square and cancel: then $1 = x^2 - 2x\sqrt{x - \frac{1}{x}} + x$;

divide by x: then $\frac{1}{x} = x - 2\sqrt{x - \frac{1}{x}} + 1$;

transpose: then $\left(x - \frac{1}{x}\right) - 2\sqrt{x - \frac{1}{x}} + 1 = 0$;

extract: then $\sqrt{x - \frac{1}{x}} - 1 = 0$, or $\sqrt{x - \frac{1}{x}} = 1$;

square: then $x - \frac{1}{x} = 1$, or $x^2 - x = 1$;

complete the square by the Hindû method, and we get

$$4x^2 - 4x + 1 = 5, \text{ or } x = \frac{1}{2}\left\{1 \pm \sqrt{5}\right\}.$$

6. Given $\sqrt{x} - 4\left\{\sqrt{x} + 13\right\}^{\frac{1}{2}} + 7 = x - 2\sqrt{x} - 9$, to find x.

Add 10 to both sides: then we get both sides squares, viz.:

$\sqrt{x} + 13 - 4\sqrt{\sqrt{x} + 13} + 4 = x - 2\sqrt{x} + 1$, and extracting,

$\sqrt{\sqrt{x} + 13} - 2 = \pm\left\{\sqrt{x} - 1\right\}$.

To continue the solution, take the results + and − separately.

(1.) Take + : then $\sqrt{x} - 1 - \sqrt{\sqrt{x} + 13} = -2$, or adding 14 to both, $\sqrt{x} + 13 - \sqrt{\sqrt{x} + 13} = 12$, which is again of the quadratic form.

Complete by the Hindû method, and extract, which gives

$4\left\{\sqrt{x} + 13\right\} - 4\sqrt{\sqrt{x} + 13} + 1 = 49$, or $\sqrt{\sqrt{x} + 13} = \frac{1 \pm 7}{2} = 4$ or -3.

Squaring $\sqrt{\sqrt{x}+13}=4$, we get $\sqrt{x}+13=16$, or $\sqrt{x}=3$, and $x=9$.

Squaring $\sqrt{\sqrt{x}+13}=-3$, we get $\sqrt{x}+13=9$, or $\sqrt{x}=-4$, and $x=16$.

(2.) Take $-$: then $\sqrt{x}+13+\sqrt{\sqrt{x}+13}=16$; and completing the square (Hindû method) as before, extracting and reducing, we obtain ultimately two other values of x. Collecting the four together, we have $x=9$, $x=16$, and $x=\dfrac{57\pm7\sqrt{65}}{2}$.

This example exhibits a class of contrivances for completing the square of very frequent use; but no general rule can be laid down respecting it. The only general remark that can be made is, to endeavour to render the parts without the radical, the square of the radical itself; and then, if on completing the square of the side so transformed, the other side is also a square, the method will be effective.

7. Given $\sqrt[m]{(1+x)^2}-\sqrt[m]{(1-x)^2}=\sqrt[m]{(1-x^2)}$; to find x.

Dividing both members of this equa. by $\sqrt[m]{(1-x^2)}$, we have

$$\sqrt[m]{\frac{1+x}{1-x}}-\sqrt[m]{\frac{1-x}{1+x}}=1.$$

Mult. by $\sqrt[m]{\dfrac{1+x}{1-x}}$, and we have $\left(\dfrac{1+x}{1-x}\right)^{\frac{2}{m}}-1=\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{m}}$.

Theref. by transp. $\left(\dfrac{1+x}{1-x}\right)^{\frac{2}{m}}-\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{m}}=1$.

This is evidently in the form of a quadratic; by resolving which, we get,

$$\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}-\tfrac{1}{2}=\pm\tfrac{1}{2}\sqrt{5},\text{ or }\frac{1+x}{1-x}=\frac{(1\pm\sqrt{5})^m}{2^m}.$$

$$\text{and finally, } x=\frac{(1\pm\sqrt{5})^m-2^m}{(1\pm\sqrt{5})^m+2^m}.$$

EXAMPLES FOR PRACTICE.

1. Given $x^2-6x-7=33$; to find x. Ans. $x=10$ or -4.
2. Given $x^2-5x-10=14$; to find x. Ans $x=8$ or -3.
3. Given $5x^2+4x-90=114$; to find x. Ans. $x=6$ or $-6\frac{4}{5}$.
4. Given $\frac{1}{2}x^2-\frac{1}{4}x+2=9$; to find x. Ans. $x=4$ or $-3\frac{1}{2}$.
5. Given $3x^4-2x^2=40$; to find x. Ans. $x=2$ or -2.
6. Given $\frac{1}{3}x-\frac{1}{2}\sqrt{x}=1\frac{1}{2}$; to find x. Ans. $x=9$ or $2\frac{1}{4}$.
7. Given $\frac{1}{2}x^2+\frac{2}{3}x=\frac{3}{4}$; to find x. Ans. $x=-\frac{2}{3}\pm\frac{1}{6}\sqrt{70}$.
8. Given $x^6+4x^3=12$; to find x. Ans. $x=\sqrt[3]{2}$, and $x=-\sqrt[3]{6}$.
9. Given $x^2+4x=a^2+2$; to find x. Ans. $x=-2\pm\sqrt{a^2+6}$.
10. In $\dfrac{(x+5)^2-\sqrt{x}}{2}+5\sqrt{x}+x^{\frac{3}{2}}-57\frac{1}{2}=0$; $x=4, 9$, or $\dfrac{-29\pm\sqrt{-59}}{2}$.
11. Solve $\dfrac{1}{x-\sqrt{2-x^2}}-\dfrac{1}{x+\sqrt{2-x^2}}=1$. Ans. $x=\pm\sqrt{\dfrac{1\pm\sqrt{5}}{2}}$.
12. In $(x-a)^2+\dfrac{a}{2}(x-a)=\dfrac{a}{2x}-1$; $x=\dfrac{a}{2}$, and $x=\dfrac{a\pm\sqrt{a^2-4}}{2}$.
13. Solve $x^{\frac{6}{5}}+x^{\frac{3}{5}}=7\cdot56$; $3x^{\frac{16}{3}}-2\cdot5x^{\frac{8}{3}}+592=0$; and $\dfrac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2+x^2}}=b$

14. Resolve $2x\sqrt[3]{x} - 3x\sqrt[3]{\frac{1}{x}} = 20$, and $2x^2 + 3x - 5\sqrt{2x^2 + 3x + 9} = -3$.

15. Solve $\left\{(2x + 1)^2 + x\right\}^2 - x = (2x + 1)^2 + 90$, and $x = \frac{\sqrt{x^4 - a^4}}{a}$.

16. Resolve $(x + 1)(x^2 + 1)(x^3 + 1) = 30x^3$, and $x^4 - 2x^3 + x = a$.

n.
11. S. IV, 31

SIMULTANEOUS EQUATIONS OF THE SECOND AND HIGHER DEGREES.

The same conditions that had place in the case of simple equations also hold with respect to those of the higher degrees. The elimination of any number of them, and the actual determination of the values of any one of them, becomes, however, much more difficult. Even when the elimination of all but one of the quantities is effected, the equation in which it is involved is generally of a high degree, and the trouble of actually determining its numerical value becomes considerable: but except in very particular cases, when more than one of the simultaneous equations is above the first degree, the equation which results from the elimination takes a form which does not admit of any solution in terms of the literal quantities involved in the given ones, or at most by means of expressions of extreme complexity. In the case of the simultaneous equations involving two of the second degree, and the remaining ones simple, the solution is theoretically possible in the terms of the given symbols; but still the transformations to be made in the forms of the equations to effect it are too numerous to render it capable of practical application.

Into the general reasonings concerning the principles of elimination, the limits of this Course forbid our entering. Nevertheless, as cases of frequent occurrence, in almost every branch of mathematics, involve this problem under more or less confined conditions, it has been considered necessary to devote a page or two to the simpler uses of it, and to give a few exercises, as a praxis for the student, at the close of the chapter.

I. All the operations described in the section on Simultaneous Equations of the first degree are to be applied to others of a higher order, in any case that admits of it.

II. If all the terms of an equation contain the same number of unknown factors (in which case it is called a *homogeneous equation*), we may put one of the factors equal to the other, multiplied by a new unknown, assumed for the purpose. As for instance, in $3xy + 2y^2 = 4x^2$, we may put $y = vx$, which gives—

$$(3v + 2v^2)x^2 = 4x^2, \text{ or}$$
$$3v + 2v^2 = 4;$$

from which v may be obtained. The same substitution being made in another equation simultaneously given, but with the value of v, instead of v itself, gives an equation containing y only, and which may be resolved by the usual methods and known quantities. If there be three or more unknowns, so many independent but simultaneous substitutions must be assumed for them as there are of quantities besides the one selected above.

III. Sometimes we can effect the reduction by substituting for one of the unknowns the sum, and for the other the difference of two other quantities, of course unknown too. This method applies to the case of two unknowns, and

obviously is confined to it. Methods in some degree analogous to this have, however, been devised for the case of three or more unknowns.

IV. It often happens that by raising one equation to some power, several of its terms will be identical with those of some other power of another equation, and in this case the equations are simplified by subtracting one from the other. The same is true of multiples of one, and powers of another equation. Sometimes too it happens that adding some quantities to one side of an equation, to render it a complete square, cube, or higher power of a binomial, the other side, so increased, becomes also a square, cube, and so on. The roots then being taken, the equation is reduced to lower dimensions, and the ultimate elimination more easily effected.

Other rules and remarks will occur to the intelligent student as he proceeds; and the teacher will often, in the actual solution of individual problems, be able to enforce a rule, and to point out the circumstances under which it can be applied, which could scarcely be rendered intelligible in print without extreme prolixity.

Ex. 1. Given $x + y = a$, and $xy = b^2$, to find x and y.

By squaring (1) we have $x^2 + 2xy + y^2 = a^2$,

and multiplying (2) by 4, $\quad 4xy \quad = 4b^2$;

Subtracting, $x^2 - 2xy + y^2 = a^2 - 4b^2$,

and extracting, $x - y = \mp \sqrt{a^2 - 4b^2}$.

From this and (1) we get

$$x = \frac{a \pm \sqrt{a^2 - 4b^2}}{2}, \text{ and } y = \frac{a \mp \sqrt{a^2 - 4b^2}}{2}.$$

Ex. 2. Given $x + y = a$, and $x^3 + y^3 = b^3$, to find x and y.

Cube eq. (1): then $x^3 + 3x^2y + 3xy + y^3 = a^3$.

Subtract (2) from this: then $3xy(x + y) = a^3 - b^3$,

or by substituting (1) in this, $xy = \dfrac{a^3 - b^3}{3(x + y)} = \dfrac{a^3 - b^3}{3a}$.

Then by means of this and (1) proceed to find x and y as in the last example.

Or thus. Put $x = u + v$ and $y = u - v$: then we have $x + y = 2u = a$, or $u = \dfrac{a}{2}$. Insert the assumed values of u and v in $x^3 + y^3 = b^3$; then we get $2u^3 + 6uv^2 = b^3$, or putting in this the value found for u, we have

$$v^2 = \frac{4b^3 - a^3}{12a}, \text{ or extracting, } v = \pm \tfrac{1}{2}\sqrt{\frac{4b^3 - a^3}{3a}}.$$

Putting in the equations $x = u + v$ and $y = u - v$ these values, we have the result required.

Or, thus again. Put $y = ux$: then the equations become $x(1 + u) = a$, and $x^3(1 + u^3) = b^3$.

Divide the cube of the first by the second of these: then

$$\frac{1 + 2u + u^2}{1 - u + u^2} = \frac{a^3}{b^3}, \text{ or } (a^3 - b^3)u^2 - (a^3 + 2b^3)u + a^3 - b^3 = 0.$$

Hence u becomes known by the solution of the quadratic equation.

Also, inserting the values of u thus found in $x(1 + u) = a$, we get $x = \dfrac{a}{1 + u}$; and from this, again, $y = ux$ will be obtained.

Ex. 3. Given $x^2y + y^2x = 30$, and $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6}$, to find x and y.

Break (1) into factors, and cancel denominators in (2), then $xy(x + y) = 30$, and $6(x + y) = 5xy$.

Multiply the second of these by xy and the first by 6, and subtract: then $x^2y^2 = 36$, or $xy = \pm 6$.

As there is subsequent work to perform, it will be advisable to work with $xy = 6$ and $xy = -6$ separately *.

First, take $xy = 6$: then inserting this in (1) we get $x + y = 5$; and resolving this as in Ex. 1, we find $x - y = \pm 1$; and this again combined with (1) by addition and subtraction gives $x = 3$ or 2, and $y = 2$ or 3.

Secondly. Take $xy = -6$: then, proceeding as before, we have $x = 6$ or -1, and $y = -1$ or 6.

The latter pair of results not fulfilling the equation, do not come properly under the denomination of answers. They come into the work from the ambiguous root of $x^2y^2 = 36$; but it might have been inferred from this being $(+xy) \cdot (+xy) = 36$, that only $xy = +6$ was admissible.

Or thus. Put $x = u + v$ and $y = u - v$. Then substituting in the given equations and reducing, we find $u(u^2 - v^2) = 15$, and $12u = 5(u^2 - v^2)$.

Divide the first of these equations by the second: then $4u^2 = 25$, or $u = \frac{5}{2}$ (see remark on last solution).

Substitute this in either of the last equations: then we get $v = \pm \frac{1}{2}$. Whence we obtain the same results as before, $x = u + v = \frac{5 \pm 1}{2} = 3$ or 2, and $y = u - v = \frac{5 \mp 1}{2} = 2$ or 3.

Or, thus again. Put $y = ux$. Then the equations reduce to $uy^3(1 + u) = 30$, and $6(1 + u) = 5uy$. Equating the values of y^3 derived from these we obtain $36(1 + u)^4 = 625u^2$, or $6(1 + u)^2 = \pm 25u$: that is, $6u^2 - 13u + 6 = 0$, and $6u^2 + 37u + 6 = 0$.

The former gives $u = \frac{3}{2}$ and $\frac{2}{3}$, and the latter $u = -\frac{6}{1}$, and $-\frac{1}{6}$.

Take $u = +\frac{2}{3}$: then $y = \dfrac{6(1+u)}{5u} = 3$, and $x = uy = 2$.

.... $u = +\frac{3}{2}$: then $y =$ .... $= 2$, and $x = uy = 3$

.... $u = -6$: then $y =$ .... $= -30$, and $x = uy = 180$.

.... $u = -\frac{1}{6}$: then $y =$ ...... $= -6$, and $x = uy = 1$.

The two last results are, as in the other case, the consequence of the ambiguous sign in $6(1 + u)^2 = \pm 25$, and their inapplicability might have been inferred at the outset, as in the former solutions.

Ex. 4. Given $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 9$, $\frac{2}{x} + \frac{3}{y} = 13$, and $8x + 3y = 5$.

From (3), $\dfrac{3}{y} = \dfrac{9}{5 - 8x}$, and from (2), $\dfrac{3}{y} = \dfrac{13x - 2}{x}$; equating these values of $\frac{3}{y}$, and reducing, we have

$$104x^2 - 72x + 10 = 0; \text{ whence } x = \frac{1}{2} \text{ and } x = \frac{5}{26}.$$

---

* In questions of this kind, these separations should, for the sake of secure working, be always employed.

$$\text{Also } y = \frac{5 - 8x}{3} = \frac{1}{3} \text{ or } \frac{15}{13}.$$

Insert these values in (1) and reduce; then we obtain

$$\frac{1}{z} = 9 - \frac{1}{x} - \frac{1}{y} = 9 - 2 - 3 = 4, \text{ or } z = \frac{1}{4}, \text{ and}$$

$$= 9 - \frac{26}{5} - \frac{13}{15} = \frac{44}{15}, \text{ or } z = \frac{15}{44}.$$

Ex. 5. Given $\frac{4}{y^2} + \frac{4 + y}{y} = \frac{8 + 4y}{x} + \frac{12y^2}{x^2}$, and $4y^2 - xy = x$.

The first equation is convertible into one having both sides squares, viz. $x^2(2 + y)^2 - 4xy^2(2 + y) + 4y^4 = 16y^4$, and extracting $x(2 + y)\ 2y^2 = \pm 4y^2$; or taking them separately, and reducing, $6y^2 - xy = 2x$, and $2y^2 + xy = -2x$.

Combining each of these with the second given equation, we shall readily obtain $x = 2$ and y from the first combination; and $x = -\frac{50}{3}$, and $y = -\frac{5}{3}$ from the second.

Or thus. From the second given equation $\frac{1}{x} = \frac{1 + y}{4y^2}$.

Substitute this in the first, and reduce; then $3y^2 + 2y = 5$.

$$\text{Whence } y = \frac{-1 \pm 4}{3} = 1 \text{ or } -\frac{5}{3}.$$

From the second equation $x = \frac{4y^2}{1 + y} = 2$ or $-\frac{50}{3}$, the same as before.

EXAMPLES FOR PRACTICE.

1. Given $x^n + y^n = a^n$, and $xy = b^2$, to find x and y.

Ans. $x = \{\frac{1}{2}a^n \pm \frac{1}{2}\sqrt{a^{2n} - 4b^{2n}}\}^{\frac{1}{n}}$, and $y = \{\frac{1}{2}a^n \mp \sqrt{a^{2n} - 4b^{2n}}\}^{\frac{1}{n}}$.

2. Given $x + y = a$, and $x^4 + y^4 = d^4$, to find x and y.

Ans. $x = \frac{a}{2} \mp \sqrt{-\frac{3a^2}{4} \mp \sqrt{\frac{d^4 + a^4}{2}}}$, and $y = \frac{a}{2} \pm \sqrt{-\frac{3a^2}{4} \mp \sqrt{\frac{d^4 + a^4}{2}}}$.

3. In the following equations find the values of x and y, viz.

$$\frac{x + y + \sqrt{x^2 - y^2}}{x + y - \sqrt{x^2 - y^2}} = \frac{9}{8y}(x + y), \text{ and}$$

$$(x^2 + y)^2 + x - y = 2x(x^2 + y) + 506.$$

4. In $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{61}{\sqrt{xy}} + 1$, and $\sqrt[4]{x^3y} + \sqrt[4]{xy^3} = 78$, find x and y.

5. Given $\frac{1}{x} + \frac{1}{y} = \frac{1}{m}$, and $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{n^2}$, to find x and y.

6. Given $x^2 - y^2 = a^2$, and $(x + y + b)^2 + (x - y + b)^2 = c^2$, to find x, y.

7. Find the values of x, y, z in the three equations,

$$x(y + z) = a^2,\ y(z + x) = b^2, \text{ and } z(x + y) = c^2;$$

And likewise also from the three,

$$x + y + z = 10,\ x^2 + y^2 + z^2 = 38, \text{ and } y^2 + 1 = xz.$$

8. Find x, y, z n the three following equations,

$$x^2 + xy + y^2 = c^2,\ x^2 + xz + z^2 = b^2, \text{ and } y^2 + yz + z^2 = a^2.$$

9. Given $\frac{18x}{y} = \frac{8y}{x}$, and $3xy + 2x + y - 485 = 0$, to find x and y.

10. Given $\frac{ax}{y} = \frac{by}{x}$, and $cxy + dx + ey = h$, to find x and y.

11. Given $\frac{xyz}{x+y} = a$, $\frac{xyz}{y+z} = b$, and $\frac{xyz}{z+x} = c$, to find x, y, z;
Also, $xy = 10$, $z(4 - y) = 12$, and $(5 - x)(6 - z) = 21$.

12. Resolve the following pairs of simultaneous equations.

(1). $x^2 + y^2 + x + y = a$, and $x^2 - y^2 + x^2 - y = b$.

(2). $x + y = xy$, and $x + y + x^2 + y^2 = a$.

(3). $19 + \frac{14x}{y} = \frac{14y}{x}$, and $7x - 4y - \cdot 18 - \frac{4x}{y} = 0$.

(4.) $5y + \frac{1}{5}\{x^2 - 15y - 14\}^{\frac{1}{2}} = \frac{1}{3}x^2 - 36$,

and $\frac{x^2y^{-1}}{8} + \frac{2x}{3} = x\left\{\frac{xy^{-1}}{3} + \frac{1}{4}\right\}^{\frac{1}{2}} - \frac{y}{2}$

(5). $v + u^{\frac{1}{2}} - \sqrt{uv} = 2v^{\frac{1}{2}} + \frac{1}{4}(21 - u)$, and $u^{\frac{1}{2}} + v^{\frac{1}{2}} = 6$.

13. Given $x + y + z = a$, $x^2 + y^2 + z^2 = b^2$, and $y = \sqrt{xz}$.

14. Find u, x, y, z, from the equations $u + x + y + z = 15$, $u^2 + x^2 + y^2 + z^2 = 85$, $x^2 = uy$, and $y^2 = xz$.

15. $uv + xz = 444$, $ux + vz = 180$, $uz + vx = 156$, $uvxz = 5184$.

16. Given $x + y = a$, and $x^5 + y^5 = b^5$
Also $x - y = a_1$, and $x^5 - y^5 = b_1^5$ } to find x and y in both pairs.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

As none of these questions are difficult of solution after the equations are obtained, it has been considered unnecessary in general to do more than form the equations and put down the answers.

1. To find two numbers whose difference is 2, and product 80.

Let x and y denote the numbers: then the conditions are $x - y = 2$, and $xy = 80$.

Resolving, we have $x = 10$ and $y = 8$, or $x = -8$ and $y = -10$.

2. To divide the number 14 into two such parts that their product may be 48.

Let x and y be the numbers: then the conditions are $x + y = 14$ and $xy = 48$.

The answers are 6 and 8.

3. What two numbers are those whose sum, product, and difference of their squares are all equal.

Let x and y be the numbers: then $x + y = xy$, and $x + y = x^2 - y^2$, which are easy to resolve, and give $x = \frac{3 \pm \sqrt{5}}{2}$, and $y = \frac{1 \pm \sqrt{5}}{2}$.

4. There are four numbers in arithmetical progression the product of whose extremes is 22, and that of whose means is 40. What are those numbers?

Let x be the less extreme, and y the common difference: then the numbers are x, $x + y$, $x + 2y$, $x + 3y$. Whence the conditions are $x^2 + 3\,xy = 22$, and $x^2 + 3\,xy + 2y^2 = 40$.

The answers are 2, 5, 8, 11, or -11, -8, -5, -2.

5. To find four numbers in geometrical progression whose sum shall be 80, and the sum of whose squares shall be 3280.

Here, taking u, x, y, z for the numbers, we shall have, by geometrical progression, $uy = x^2$, $xz = y^2$; and by the given conditions $u + x + y + z = 80$, and $u^2 + x^2 + y^2 + z^2 = 3280$.

These equations are precisely cases of Ex. 14, p. 192, and may be resolved as those were.

But the problem admits of being expressed by only two equations. For let x be the first term, and u the ratio of the proportion; then x, ux, u^2x, and u^3x are the four numbers, and the conditions are expressed by

$$x\,(1 + u + u^2 + u^3) = 80, \text{ and } x^2\,(1 + u^2 + u^4 + u^6) = 3280.$$

By geometrical progression these are convertible into

$$\frac{x\,(1 - u^4)}{1 - u} = 80 \text{ and } \frac{x^2\,(1 - u^8)}{1 - u^2} = 3280.$$

Divide the second by the square of the first; then

$$\frac{1 - u^8}{1 - u^2} \cdot \frac{(1 - u)^2}{(1 - u^4)^2} = \frac{3280}{6400},$$

$$\text{or, } \frac{(1 + u^4)\,(1 - u)}{(1 - u^4)\,(1 + u)} = \frac{1 + u^4}{(1 + u)\,(1 + u + u^2 + u^3)} = \frac{41}{80}, \text{ or again}$$

By multiplication and transposition this becomes $39 - 82u - 82u^2 - 82u^3 + 39u^4 = 0$, and dividing by u^2 it becomes $39\,(u^2 + \frac{1}{u^2}) - 82\,(u + \frac{1}{u}) = 82$, or $39\,(u + \frac{1}{u})^2 - 82\,(u + \frac{1}{u}) = 160$.

Resolving by the Hindû method, considering $u + \frac{1}{u}$ as one quantity, we get

$$u + \frac{1}{u} = \frac{10}{3} \text{ or } -\frac{16}{13}.$$

From these we have four values of u, viz. 3, $\frac{1}{3}$, and $\frac{-8 \pm \sqrt{-105}}{13}$, the latter pair of which are imaginary, and therefore imply that all the four numbers are imaginary also. The two former values of u give the four numbers 2, 6, 18, 54, and 54, 18, 6, 2.

In very nearly the same way may the problem be solved when there are five numbers instead of four.

6. There is a number composed of four digits which are in arithmetical progression. The sum of the digits is 20. If 6174 be subtracted from the number, the remainder will have the same digits in an inverse order; and the product of the extreme digits is two-thirds of the product of the intermediate ones. What is the number?

Let u, x, y, z be the digits in order from right to left: then $1000u + 100x + 10y + z$ denotes the number, and $1000z + 100y + 10x + u$ denotes the number when the digits are inverted. But by the question we have $1000u + 100x$

$+ 10y + z - 6174 = 1000z + 100y + 10x + u$; or, $999u + 90x - 90y - 999z = 6174$; whence

$$111(u - z) + 10(x - y) = 686 \ldots\ldots (1)$$

The other conditions are readily formed, and are

$$u + z = x + y,\ u + x + y + z = 20, \text{ and } uz = \frac{2}{3}xy,$$

which admits of solution as before.

7. To find a number such that if 7 be subtracted from its square, and this root be added to twice the number, the sum shall be 5.

Let x be the number; then the conditions are expressed by the equation $2x + \sqrt{x^2 - 7} = 5$.

By transp. $\sqrt{x^2 - 7} = 5 - 2x$, and squaring $3x^2 - 20x + 32 = 0$, which, resolved by the Indian method, gives $x = 4$ and $x = 2\frac{2}{3}$.

By substituting these values in the equation, they are found not to fulfil the condition, though they do fulfil the condition $2x - \sqrt{x^2 - 7} = 5$. No numbers, indeed, can be found to fulfil the given condition; and it must be carefully borne in mind that *except the expressed condition be given free from radicals, a solution, even a symbolical one, cannot be depended on without subsequent verification.*

This circumstance has been a source of much perplexity to mathematicians, and was never cleared up till Mr. Horner addressed a letter to the present Editor of the Course, and which was published in the Philosophical Magazine for Jan. 1836. To this letter the reader is referred, as any intelligible account of the principles of his exposition would require more space than can be allotted to it here.

After the student has solved the following questions for the particular data, he should be required to solve them when, instead of the given numbers, literal symbols are substituted. He should also, as he proceeds through the numerical solutions, put down *all* the results, even though they be imaginary; he should be accustomed to seek the interpretation of those results, and especially to point out those which are true solutions in the form proposed, and which are solutions of some collateral problems that are involved in the same algebraical expression, as well as to assign those which involve contradictory data, and which give therefore merely symbolical results.

Sometimes questions are proposed, in which, though the equation has real answers, yet the question has not, from the answers, though real, being inconsistent with some condition either implied or expressed, which is not taken into account in the equation; as when a *fractional* number of men, or of terms of a progression, &c. results from the question; or when the number of things to be *added* turns out to be *subtractive*, or $-$; and so on.

In the literal solution, the student must point out the conditions that render a problem impossible; that is, which give rise to imaginary roots, or are incongruous with the ideas implied in the subject of the problem.

QUESTIONS FOR PRACTICE.

1. What number being added to its square will make 42? Ans. 6, or -7.

2. Find two numbers such, that the less may be to the greater as the greater is to 12, and that the sum of their squares may be 45. Ans. 3 and 6.

3. What two numbers are those, whose difference is 2, and the difference of their cubes 98? Ans. 3 and 5.

4. What two numbers are those, whose sum is 6, and the sum of their cubes 72? Ans. 2 and 4.

5. What two numbers are those, whose product is 20, and the difference of their cubes 61? Ans. 4 and 5.

6. Divide the number 11 into two such parts, that the product of their squares may be 784. Ans. 4 and 7*.

7. Divide the number 5 into two such parts, that the sum of their alternate quotients may be $4\frac{1}{4}$, that is, of the two quotients of each part divided by the other. Ans. 1 and 4.

8. Divide 12 into two such parts, that their product may be equal to 8 times their difference. Ans. 4 and 8.

9. Divide the number 10 into two such parts, that the square of 4 times the less part may be 112 more than the square of twice the greater. Ans. 4 and 6.

10. Find two numbers such, that the sum of their squares may be 89, and their sum multiplied by the greater may produce 104. Ans. 5 and 8.

11. What number is that, which being divided by the product of its two digits, the quotient is $5\frac{1}{3}$; but when 9 is subtracted from it, there remains a number having the same digits inverted? Ans. 32, or — 23.

12. Divide 20 into three parts, such that the continual product of all three may be 270, and that the difference of the first and second may be 2 less than the difference of the second and third. Ans. 5, 6, 9.

13. Find three numbers in arithmetical progression, such that the sum of their squares may be 56, and the sum arising by adding together 3 times the first, and twice the second, and 3 times the third, may be 32. Ans. 2, 4, 6.

14. Divide the number 13 into three such parts, that their squares shall have equal differences, and that the sum of those squares shall be 75. Ans. 1, 5, 7.

15. Find three numbers having equal differences, so that their sum shall be 12, and the sum of their fourth powers shall be 962. Ans. 3, 4, 5.

16. Find three numbers having equal differences, and such that the square of the least added to the product of the two greater shall make 28, but the square of the greatest added to the product of the two less shall make 44. Ans. 2, 4, 6.

17. Three merchants, A, B, C, on comparing their gains, find that among them all they have gained $1444l$; and that B's gain added to the square root of A's made $920l$; but if added to the square root of C's it made $912l$. What were their several gains? Ans. A 400, B 900, C 144.

18. Find three numbers in arithmetical progression, so that the sum of their squares shall be 93; also if the first be multiplied by 3, the second by 4, and the third by 5, the sum of the products may be 66. Ans. 2, 5, 8.

19. Find two numbers, such that their product added to their sum may make 47, and their sum taken from the sum of their squares may leave 62.

Ans. 5 and 7.

20. (1) The sum of two numbers is 2, and their product is also 2; what are they? (2) Also find two numbers whose sum is a and whose product is b^2.

Ans. (1) *impossible;* (2) $x = \dfrac{a \pm \sqrt{a^2 - 4b^2}}{2}$; $y = \dfrac{a \mp \sqrt{a^2 - 4b^2}}{2}$

21. The greatest term of an arithmetical series, the common difference, and

* In a great number of the following questions the answers will appear in two forms or in four forms, and it will often happen in even more numerous ones still. The present question has for its solutions $x = \dfrac{11 \pm 3}{2}$, $y = \dfrac{11 \mp 3}{2}$; and $x = \dfrac{11 \pm \sqrt{233}}{2}$, $y = \dfrac{11 \mp \sqrt{233}}{2}$. Though for the most part only the *positive* answers are set down to enable the student to verify his actual work, he should be required to give all the solutions that the equation admits of, in symbols at least, and reduced where they admit of reduction.

the sum of the series, in five several cases, as below, are given, to find the number of terms:—

	(1)	(2)	(3)	(4)	(5)
Greatest term	·25	8	·25	400	400
Common difference	·02	·02	·02	4	4
Sum	10·25	10·25	1·25	20200	20160

22. (1) There are two numbers whose sum, sum of their squares, and their product, are all equal; and (2) two others whose sum, product, and difference of their squares, are all equal. What are these pairs of numbers?

Ans. (1) *impossible;* (2) $\frac{1}{2}(3 \pm \sqrt{5})$ and $\frac{1}{2}(1 \pm \sqrt{5})$.

23. A gentleman bought a horse for a certain sum, and having re-sold it for 119*l.* found that he had gained as much per cent. by the transaction as the horse cost him; what was the prime cost of the horse?

Ans. 70*l.* or $-$ 170*l.* The latter is incongruous.

24. The arithmetical mean of two numbers exceeds the geometrical mean by 13, and the geometrical exceeds the harmonical mean by 12. What are those numbers?

25. A traveller sets out for a certain place, and travels one mile the first day, two miles the second, three the third, and so on: and five days afterwards another sets out and travels 12 miles a day. Show how far he must travel before he overtakes the first, and explain the other answer.

26. A wine merchant sold 7 dozen of sherry and 12 dozen of claret for 50*l.* and finds that he sold 3 dozen more of sherry for 10*l.* than of claret for 6*l.* What was the price of each? Explain the double answer.

27. A parcel contained 24 coins, valued 18*s*, part of them silver and the other copper. Each silver coin is worth as many pence as there are copper coins, and each copper coin is worth as many pence as there are silver coins. How many more were there of copper than of silver?

28. Find four numbers which exceed one another by unity, such that their continued product may be 120.

29. There is a number consisting of two digits, which, when divided by the sum of its digits, gives a quotient greater by 2 than the first digit; but if the digits be inverted, and the resulting number be divided by a number greater by unity than the sum of the digits, the quotient is greater by 2 than the preceding quotient. Find the congruous answer.

30. "Some bees were sitting on a tree; at one time the square root of half their number flew away. Again, eight-ninths of the whole flew away the second time; two bees remained. How many were there *?"

31. D sets out from F towards G, and travels 8 miles a day; after he had gone 27 miles, E sets out from G towards F, and goes every day $\frac{1}{20}$ of the whole journey; and, after he had travelled as many days as he goes miles in one day, he met D. What is the distance of the two places?

32. There is a number consisting of three digits, of which the first is to the second as the second to the third; the number itself is to the sum of its digits as 124 to 7; and if 594 be added to it, the digits will be inverted. Required the number.

---

* From Strachey's translation of the *Bija Ganita*, the work from which the rule for the solution of Quadratic Equations, given at p. 197, was taken. The scientific world is indebted for the publication of this very curious and interesting work to my lamented friend, the late Professor Leybourn, of the Royal Military College, Sandhurst.

33. Bacchus having caught Silenus asleep by the side of a full cask, seized the opportunity of drinking, which he continued for two-thirds of the time that Silenus would have taken to empty the whole cask. After that, Silenus awakes, and drinks what Bacchus had left. Had they both drunk together it would have been emptied two hours sooner, and Bacchus would have drunk only half of what he left for Silenus. Required the time in which each would have emptied the cask separately. Ans. Bacchus in 6, and Silenus in 3 hours.

34. A and B travelled on the same road, and at the same rate, from H to L. At the 50th milestone L, A overtook a drove of geese, which were proceeding at the rate of 3 miles in 2 hours, and 2 hours afterwards met a stage-waggon which was moving at the rate of 9 miles in 4 hours. B overtook the same drove of geese at the 45th mile-stone, and met the same stage-waggon exactly 40 minutes before he came to the 31st milestone. Where was B when A reached L?

Ans. at the 25th milestone.

THE SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS.

Although Horner's general method of approximating to the roots of numerical equations of all degrees supersedes the special methods adapted to particular classes, yet there are many occasions in the higher departments of mathematical inquiry in which it would be of great advantage to possess the symbolical values of the roots in terms of the literal coefficients of any given equation. Beyond those of the third and fourth degrees the labours of mathematicians have been altogether unsuccessful in assigning these symbolical values: and, indeed, there are strong reasons for believing that this want of success arises from circumstances that render the solution of the problem altogether impossible. Were it, however, otherwise, it is evident from a comparison of the rapid increase in the complexity of the expressions of the roots of equations of the first, second, third, and fourth degrees, that the roots of an equation of the fifth degree in terms of the coefficients would be so unwieldy as to preclude the possibility of substituting them in any other expression, and effecting a sufficient degree of reduction in the result to be of the slightest mathematical utility.

I. CARDAN'S SOLUTION OF THE CUBIC EQUATION.

1. *To prepare the equation*, it is necessary to transform it into another whose second term has the coefficient 0.

Let $ax^3 + bx^2 + cx + d = 0$ be the given equation. Assume two unknown quantities u and z, such that $u + z = x$. Substitute this value of x in the given equation; then it reduces to

$$az^3 + (3au + b)\,z^2 + (3au^2 + 2bu + c)\,z + au^3 + bu^2 + cu + d = 0.$$

Now that the coefficient of z^2 may be 0, we must have $3au + b = 0$, or $u = -\frac{b}{3a}$. Put this value for u, and reduce the expression; then we have finally for the transformed equation

$$27a^3z^3 - (9ab^2 - 27a^2c)\,z + 2b^3 - 9abc + 27a^2d = 0.$$

When $a = 1$, and $\frac{b}{3}$ an integer, the expression takes a much simpler form.

When the equation is of any higher degree, as the nth, the same process will give for the value of u which will remove the second term $u = -\frac{b}{na}$ *.

As an example of the cubic, let the equation $x^3 - 6x^2 + 10x - 8 = 0$ be transformed into one wanting the second term.

Here $a = 1$, $b = -6$, and hence $u = -\frac{-6}{3} = 2$, and $x = z + 2$; which upon substitution in the given equation gives $z^3 - 2z - 4 = 0$.

Again, for the biquadratic $x^4 - 6x^3 - 21x^2 + 146x - 120 = 0$, where $a = 1$, $n = 4$, $b = -6$; hence $u = -\frac{b}{na} = \frac{3}{2}$, and $x = z + \frac{3}{2}$. Substitute this, and there will result $z^4 - 34{\cdot}5z^2 + 56z + 36{\cdot}5625 = 0$.

EXAMPLES.

Remove the second terms from each of the following equations:

1. $x^4 - 4x^3 - 8x^2 + 32 = 0$, and from $y^4 + 4y^3 + 8y^2 - 32 = 0$
2. $z^3 + 3z^2 + 9z - 13 = 0$, and from $x^4 + \frac{3}{4}x^3 + 1{\cdot}5 = 0$
3. $y^5 - 2{\cdot}5\,y^4 + 16{\cdot}25\,y^3 - 18{\cdot}375y^2 - 18{\cdot}575 = 0$
4. $x^2 - \frac{9}{10}x + 3 = 0$; $x^2 + \frac{9}{100}x + 3\,({\cdot}1)^3 = 0$.

5. Transform $y^{-3} + 12y^{-2} - 64y^{-1} + 15y^1 = 0$, and $z^{\frac{3}{3}} - 10z^{\frac{2}{3}} + 15z^{\frac{1}{3}} - 18z^0 = 0$ †, into equations deficient of the second term.

2. *To resolve the transformed cubic equation.*

Let the transformed equation be $z^3 + 3ez - 2f = 0$, where e and f denote the coefficients of the transformed equation, divided by the coefficient of z^3.

Assume the two unknowns x and y to fulfil the equations $x + y = z$, and $xy = -e$. Substitute these in the given equation, we have

$$x^3 + y^3 = 2f; \text{ or squaring,}$$

$$x^6 + 2x^3y^3 + y^6 = 4f^2. \text{ Also } 4x^3y^3 = -4e^3.$$

Whence by subtraction, $x^6 - 2x^3y^3 + y^6 = 4(f^2 + e^3)$, and by extraction, $x^3 - y^3 = \pm 2\sqrt{f^2 + e^3}$.

From these values of $x^3 + y^3$, and $x^3 - y^3$, we have

$$x^3 = f \pm \sqrt{f^2 + e^3}, \text{ and } y^3 = f \mp \sqrt{f^2 + e^3}.$$

Hence extracting the cube roots, and putting the values of x and y in $x + y = z$, we have

$$z = \sqrt[3]{f \pm \sqrt{f^2 + e^3}} + \sqrt[3]{f \mp \sqrt{f^2 + e^3}},$$

which are both contained in the single expression

$$z = \sqrt[3]{f + \sqrt{f^2 + e^3}} + \sqrt[3]{f - \sqrt{f^2 + e^3}}.$$

And $z + u$ gives the value of the root of the original or untransformed equation, u being determined according to the process already explained.

---

* As a process for numerical work, it may be stated, that a much shorter one may be given than that of actual substitution; but as it will be detailed further on, (in Horner's method of solution,) it will be unnecessary to do more than refer to it here.

† In the first of these two equations, it will be necessary to reduce the equations to the form of integer indices before the application of the method; and in the second, to consider $z^{\frac{1}{3}}$ as the quantity according to which the arrangement is made. This last, however, may be put in a still different form, though not a more advantageous form.

When e is negative, and e^3 is greater than f^2, the radical $\sqrt{f^2 + e^3}$ becomes imaginary, and hence the numerical calculation of the root becomes impossible without some further contrivance. Many have been proposed; as the expansion of $\{f + k\}^{\frac{1}{3}} + \{f - k\}^{\frac{1}{3}}$ into series, by which the odd powers of k or $\sqrt{f^2 + e^3}$ mutually cancel, and leave only terms which are real; also by means of trigonometrical tables, and by special tables devoted to the purpose. All these methods are now so completely superseded by Horner's process as to need no remark here.

It will only be requisite to state, that the case now supposed is called *The Irreducible Case of Cardan's Cubic;* and that it is known from other considerations, that in this case all the three roots of the equation are real, whilst in that to which Cardan's formula applies, two of the roots are imaginary, and the real one is given by that formula.

As an example, let us take the equation $x^3 - 6x^2 + 10x - 8 = 0$, whose second term we had eliminated by our previous transformation, giving $z^3 - 2z - 4 = 0$. Here $e = -\frac{2}{3}$ and $f = 2$. Hence

$$z = \sqrt[3]{2 + \sqrt{4 - \tfrac{8}{27}}} + \sqrt[3]{2 - \sqrt{4 - \tfrac{8}{27}}} = \sqrt[3]{2 + \tfrac{10}{9}\sqrt{3}} + \sqrt[3]{2 - \tfrac{10}{9}\sqrt{3}}$$
$$= (1 + \tfrac{1}{3}\sqrt{3}) + (1 - \tfrac{1}{3}\sqrt{3}) = 2.$$

Hence $x = z + 2 = 4$, which is the real value of x in the given equation.

Other Examples for Practice.

2. Find the value of x in the equations $x^3 - 6x^2 + 18x = 22$, and $x^3 - 7x^2 + 14x = 20$. Ans. $x = 2{\cdot}32748$, and $x = 5$ respectively.

3. Find Cardan's roots of the equations $x^3 + 6x = 20$, and $x^3 - 12x^2 + 36x = 7$. Ans. 2 and 7 respectively.

4. Given $x^3 - 15x^2 + 71x = 297$; and $x^3 - 12x^2 + 57x - 94 = 0$, to find Cardan's roots. Ans. 11; and $4 + \sqrt[3]{3} - \sqrt[3]{9}$.

5. Given $y^9 + 18y^6 + 216y^3 = 3392$, to find Cardan's value of y. Ans. 2 *.

6. Find Cardan's root of $y^{\frac{3}{2}} + {\cdot}24y^{\frac{1}{2}} = {\cdot}245$. Ans. $y = {\cdot}25$.

7. Resolve $x^{3m} - 36x^m = 91$, and $x^{\frac{3}{n}} - 36x^{\frac{1}{n}} = 91$.

Ans. $x = 7^{\frac{1}{m}}$, and $x = 7^n$ respectively.

8. Solve those of the equations given at p. 199 for transformation, that come under the form adapted to Cardan's solution.

II. SIMPSON'S SOLUTION OF THE BIQUADRATIC EQUATION †.

Let the given equation be $x^4 + 2ax^3 + bx^2 + cx + d = 0 \ldots.$ (1), and assume

---

* In all cases where there are higher powers of the unknown, if they be in the ratio of 1, 2, 3, the problem is treated as a cubic. That here referred to is one in which y is not directly found by the formula, but y^3, which is $= 8$, and then from this again $y = 2$. The same is true of any inferior powers, where the indices are related in the same way, as in the next question.

† This method of solution was invented by Mr. Thomas Simpson, F.R.S. Professor of Mathematics in the Royal Military Academy from 1743 to 1761. It has often been erroneously ascribed to Dr. Waring, Lucasian Professor of Mathematics in the University of Cambridge, and even called by his name.

it equal to $(x^2 + ax + A)^2 - (Bx + C)^2 = 0 \ldots$ (2), where A, B, C are unknown. Expand (2) and equate the coefficients of the powers of x in the result with those of (1). This gives,

$$2A + a^2 - B^2 = b, \text{ or } B^2 = 2A + a^2 - b \ldots (3)$$
$$2aA - 2BC = c, \text{ or } 2BC = 2aA - c \ldots (4)$$
$$A^2 - C^2 = d, \text{ or } C^2 = A^2 - d \ldots (5)$$

Also multiplying four times (3) and (5) together, and equating the product to the square of (4) we have

$$8A^3 - 4bA^2 + 4(ac - 2d)A = 4a^2d - 4bd + c^2 \ldots (6)$$

Suppose A to be found from this cubic equation; then

$$\text{from (3) we have } B = \pm\sqrt{2A + a^2 - b} \ldots (7)$$
$$\ldots (4) \ldots C = \frac{2aA - c}{2B} = \pm\frac{2aA - c}{2\sqrt{2A + a^2 - b}} \ldots (8)$$

But from (2) we have $x^2 + ax + A = \pm(Bx + C)$, or

$$x^2 + (a \mp B)x + (A \mp C) = 0 \ldots (9)$$

Inserting (7), (8), in (9) it becomes

$$x^2 + (a \mp \sqrt{2A + a^2 - b})x + A \mp \frac{2aA - c}{2\sqrt{2A + a^2 - b}} = 0 \ldots (10)$$

in which the doubtful sign is to be taken the same in the second and third terms.

Note I. Whenever, by taking away the second term of a biquadratic, after the manner described at page 199, the fourth term also vanishes, the roots may immediately be obtained by the solution of a quadratic only.

NOTE II. A biquadratic may also be solved independently of cubics, in the following cases :—

1. When the difference between the coefficient of the third term, and the square of half that of the second term, is equal to the coefficient of the fourth term, divided by half that of the second. Then if p be the coefficient of the second term, the equation will be reduced to a quadratic by dividing it by $x^2 \pm \frac{1}{2}px$.

2. When the last term is negative, and equal to the square of the coefficient of the fourth term divided by 4 times that of the third term, *minus* the square of that of the second: then to complete the square, subtract the terms of the proposed biquadratic from $(x^2 \pm \frac{1}{2}px)^2$, and add the remainder to both its sides.

3. When the coefficient of the fourth term divided by that of the second term gives for a quotient the square root of the last term; then, to complete the square, add the square of half the coefficient of the second term to twice the square root of the last term, multiply the sum by x^2, from the product take the third term, and add the remainder to both sides of the biquadratic.

4. The fourth term will be made to go out by the usual operation for taking away the second term, when the difference between the cube of half the coefficient of the second term and half the product of the coefficients of the second and third terms, is equal to the coefficient of the fourth term.

EXAMPLES.

Ex. 1. Given $x^4 + 12x = 17$ to find x by Simpson's method.

Here the reducing cubic is $A^3 + 17A = 18$, or $A = 1$. Hence

$B = \pm \sqrt{2A + a^2 - b} = \pm \sqrt{2}$, and $C = \pm \frac{2aA - c}{2\sqrt{2A + a^2 - b}} = \pm \frac{-12}{2\sqrt{2}}$ $= \mp 3\sqrt{2}$. Consequently the two quadratics become

$$x^2 - x\sqrt{2} + 1 + 3\sqrt{2} = 0, \text{ and } x^2 + x\sqrt{2} + 1 - 3\sqrt{2} = 0.$$

The roots of these are $x = \frac{1}{2}\sqrt{2} \pm \sqrt{-\frac{1}{2} - 3\sqrt{2}}$, and

$$x = -\tfrac{1}{2}\sqrt{2} \pm \sqrt{-\tfrac{1}{2} + 3\sqrt{2}}, \text{ respectively.}$$

Ex. 2. Let $x^4 - 6x^3 - 58x^2 - 114x - 11 = 0$ be given to find x.

Here the reducing cubic is $A^3 + 29A^2 + 182A - 1256 = 0$, and from this $A = 4$. Whence $B = \pm 5\sqrt{3}$, and $C = \pm 3\sqrt{3}$.

Inserting these and reducing the quadratics, we get

$$x = \tfrac{3}{2} \pm \tfrac{5}{2}\sqrt{3} \pm \sqrt{17 \pm \tfrac{21}{2}\sqrt{3}}.$$

Ex. 3. Given $z^4 + 2az^3 - 37a^2z^2 - 38a^3z + a^4 = 0$, to find z.

Dividing all by a^4, and putting $\frac{z}{a} = x$, we have $x^4 + 2x^3 - 37x^2 - 38x + 1 = 0$.

The reducing cubic is $2A^3 + 37A^2 - 40A - 399 = 0$, and the roots (obtained by a method not yet explained, but they may be verified by substitution) are 3·5, -3, and -19. Either of these may be used in the quadratic equations, and the final result will in all cases be the same; and x can be obtained as before, and thence $z = ax$.

Ex. 4. Given $x^4 - 4ax^3 + 5a^2x^2 - 4a^3x + a^4 = 0$, to find x.

Ex. 5. Solve the three following equations by the methods explained in the notes, as well as by the general method.

(1). $x^4 - 25x^2 + 60x = 36$. Ans. $1, 2, 3, -6$.

(2). $x^4 + 2qx^3 + 3q^2x^2 + 2q^3x - r^4 = 0$.

Ans. $-\frac{1}{2}q \pm \sqrt{-\frac{3}{4}q^2 \pm \sqrt{q^4 + r^4}}$.

(3). $x^4 - 9x^3 + 15x^2 - 27x + 9 = 0$.

Ans. $\dfrac{9 \pm 3\sqrt{5} \pm \sqrt{78 \pm 54\sqrt{5}}}{4}$.

Ex. 6. The following equations are proposed for solution :—

(1). $x^4 - 8x^3 - 12x^2 + 84x = 63$. Ans. $2 \pm \sqrt{7} \pm \sqrt{11 \pm \sqrt{7}}$.

(2). $x^4 + 36x^3 - 400x^2 - 3168x + 7744 = 0$.

Ans. $-9 \pm \sqrt{137} \pm 3\sqrt{2}\sqrt{17 \mp \sqrt{137}}$.

(3). $x^4 + 24x^3 - 114x^2 - 24x + 1 = 0$. Ans. $\pm\sqrt{197} - 14$, and $2 \pm \sqrt{5}$.

(4). $x^4 - 12x = 5$. Ans. $1 \pm \sqrt{2}$, and $-1 \pm 2\sqrt{-1}$.

(5). $x^4 = 12x^3 + 47x^2 - 72x + 36 = 0$. Ans. $1, 2, 3, 6$.

THE SOLUTION OF EQUATIONS BY DOUBLE POSITION.

The method of double position has been explained in the arithmetic, under the form best adapted to its use there; which is, as the student will perceive by

solving the same questions algebraically, in those cases where the equation of the question is of the first degree. In all other cases it furnishes but gradual approximations to the true answer; and in these, according to the circumstances of the given equation, there will be very different degrees of rapidity in the approaches to the three values of the unknown. In algebraical equations, the rate of approximation is generally considered to be such as to give at each operation about as many figures more as we had already obtained. In other classes of equations *, however, the approximation is much more slow; and were it not the only method which we have the means to employ in such cases, its great slowness, and its being superseded in all respects in its application to algebraical equations, would call for its total omission from this work: but as the *modus operandi* is so easily perceived in the application of it to algebraic equations, and some practice in the use of it so essential in future inquiries, it will be retained in nearly the same form as heretofore †.

$$\text{Let } x^n + ax^{n-1} + bx^{n-2} + \dots\dots hx = k \dots\dots (1).$$

Suppose two numbers x_1 and x_2 are found by trial, or otherwise nearly equal to x, and substitute them in (1); giving

$$x_1^n + ax_1^{n-1} + bx_1^{n-2} + \dots hx_1 = k_1 \dots\dots (2)$$
$$x_2^n + ax_2^{n-1} + bx_2^{n-2} + \dots hx_2 = k_2 \dots\dots (3)$$

Subtract (3) from (1) and from (2), and divide the results. Then

$$\frac{(x^n - x_2^n) + a(x^{n-1} - x_2^{n-1}) + b(x^{n-2} - x_2^{n-2}) + \dots h(x - x_2)}{(x_1^n - x_2^n) + a(x_1^{n-1} - x_2^{n-1}) + b(x_1^{n-2} - x_2^{n-2}) + \dots h(x_1 - x_2)} = \frac{k - k_2}{k_1 - k_2}. \quad (4)$$

But each compound term of the numerator is divisible by $x - x_2$, and each one of the denominator by $x_1 - x_2$. Whence

$$\frac{(k_1 - k_2)(x - x_2)}{(k - k_2)(x_1 - x_2)} = \frac{x_1^{n-1} + x_1^{n-2}(x_2 + a) + x_1^{n-3}(x_2^2 + ax_2 + b) + \dots}{x^{n-1} + x^{n-2}(x_2 + a) + x^{n-3}(x_2^2 + ax_2 + b) + \dots\dots} \quad (5)$$

Now if the second side of this equation were unity, the solution of (5) for x would be accurate; but as it is only an approximation, x and x_1 being only *nearly* equal, the value of x determined on the hypothesis of $x = x_1$ will give a result differing from the truth by a small quantity dependent on this inequality.

If x, x_1, and x_2, agree to p places of figures, the numerator and denominator will generally agree to $np - 2$ places, and hence the quotient will not differ from unity till about $np - 2$ places. This, with equations of a low degree, and at the outset of the work, gives nearly the rate of approximation already spoken of: but it is evidently much more rapid in higher equations at the outset, and in all after a few steps in the approximation have been made. When, however, the given equations involve radicals, it is difficult to investigate without considerable detail the extent of the approximation.

Adopting then as an approximation that the right side of (5) is unity, we have

$$x - x_2 = \frac{k - k_2}{k_1 - k_2}(x_1 - x_2) \dots\dots\dots\dots (6)$$

---

* Such as, for instance, $10^x = 50$, in which $x = 1{\cdot}69897$; or $x^x = 100$, in which $x = 3{\cdot}597285$. This application of the method requires, however, a knowledge of the principles of logarithms and the use of tables, the foundation and structure of which have not yet been explained. On this account, the solution of such equations will be reserved for a Supplement to the use of the tables referred to.

† This method is due to John Bernoulli, see Butler's Mathematics, vol. ii. p. 155.

From this we have a closer approximation to x than either x_1 or x_2 were, viz.:

$$x = \frac{k - k_2}{k_1 - k_2}(x_1 - x_2) + x_2 \dots\dots\dots\dots\dots\dots (7)$$

Hence, putting this formula into words, we have the following

RULE.

1. FIND, by trial, two numbers, as near to the true root as you can, and substitute them separately in the equation instead of the given quantity; and find how much the terms collected together, according to their signs + or —, differ from the absolute known term of the equation, marking each error + if in excess, and — if in defect.

2. Multiply the difference of the two numbers, found or taken by trial, by either of the errors, and divide the product by the difference of the errors, having regard to the algebraical laws of the signs.

3. Add the quotient last found to the number belonging to that error, when its supposed number is too little, but subtract it when too great, and the result will give the true root *nearly*.

4. Take this root and the nearest of the two former, or any other that may be found nearer; and, by proceeding in like manner as above, a root will be obtained still nearer than before. And so on, to any degree of exactness required †.

Notes.

1. It is best to employ always two assumed numbers that shall differ from each other only by unity in the last figure on the right hand; because then the difference, or multiplier, is only 1. It is also best to use always the least error in the above operation.

2. It will be convenient also to begin with a single figure at first, trying several single figures till there be found the two nearest the truth, the one too little, and the other too great; and in working with them, find only one more figure. Then substitute this corrected result in the equation, for the unknown letter, and if the result prove too little, substitute also the number next greater for the second supposition; but contrariwise, if the former prove too great, then take the next less number for the second supposition; and in working with the second pair of errors, continue the quotient only so far as to have the corrected number to four places of figures. Then repeat the same process again with this last corrected number, and the next greater or less, as the case may require, carrying the third corrected number to eight figures; because each new operation commonly doubles the number of true figures. And thus proceed to any extent that may be wanted.

3. The actual labour of finding the errors which result from the suppositions will be greatly abridged by the application of Problem I. of the Chapter on the Solution of Equations, in those cases where all, or nearly all, the terms are present, and the equation ordered according to descending powers of the unknown

---

* The rule in this form was first given by Mr. Bonnycastle, late Professor of Mathematics in the Royal Military Academy, in the 8vo. edition of his Arithmetic (1810). It is a better form in cases of *approximation* than the one given in the article on Arithmetic, in the present vol. p. 93.

quantity. To that rule, for the sake of avoiding repetition, the student is at once referred. See p. 208.

EXAMPLES.

Ex. 1. FIND the value of x in the equation $x^3 + x^2 + x - 100 = 0$. Here $k = 100$, and it is easily discovered that the root lies between 4 and 5. Substitute these numbers for x_2 and x_1, and adopt the algorithm referred to in note 3. Then,

$$\begin{array}{rrrl} 1 + & 1 + & 1 - & 100\ (4 = x_2 \\ & 4 & 20 + & 84 = k_2 \\ \hline & 5 & 21 & -16 = k_2 - k \end{array} \qquad \begin{array}{rrrl} 1 + & 1 + & 1 - & 100\ (5 = x_1 \\ & 5 & 30 & 155 = k_1 \\ \hline & 6 & 31 & 55 = k_1 - k \end{array}$$

Hence $x_1 - x_2 = 1$, $k_1 - k_2 = 71$, $k - k_2 = 16$, and we have $x = x_2 + \dfrac{k - k_2}{k_1 - k_2}(x_1 - x_2) = 4 + \dfrac{16.1}{71} = 4{\cdot}2$ nearly.

Again, suppose $x_2 = 4{\cdot}2$ and $x_1 = 4{\cdot}3$. Then, proceeding as before, we have

$$\begin{array}{rrrl} 1 + & 1 + & 1 - & 100\ (4{\cdot}2 = x_2 \\ & 4{\cdot}2 & 20\,8 & 91{\cdot}36 \\ & & 1{\cdot}04 & 4{\cdot}568 \\ \hline & 5{\cdot}2 & 22{\cdot}84 & -4{\cdot}072 = k_2 - k \end{array} \qquad \begin{array}{rrrl} 1 + & 1 + & 1 - & 100\ (4{\cdot}3 = x_1 \\ & 4{\cdot}3 & 21{\cdot}2 & 95{\cdot}16 \\ & 5{\cdot}3 & 1{\cdot}59 & 7{\cdot}137 \\ \hline & & 23{\cdot}79 + & 2\,297 = k_1 - k \end{array}$$

Hence $x_1 - x_2 = {\cdot}1$, $k_1 - k_2 = 6\,369$, and $k - k_2 = 4{\cdot}072$. Hence, proceeding as before, we have $x = 4{\cdot}264$ nearly.

Working, thirdly, with $4{\cdot}264$ and $4{\cdot}265$ in the same manner, we have $x = 4{\cdot}2644299$ more nearly.

So long as the first three figures only appear in the approximations, and the equation is only cubic, the table of squares and cubes at the end of this volume will facilitate the work of finding the values of the substitutions.

Ex. 2. Find the value of x in $x^3 - 15x^2 + 63x = 50$.

Ans. 1·0280392317 . . .

Ex. 3. Let it be required to find the value of x in the equation

$$\sqrt{144x^2 - (x^2 + 20)^2} + \sqrt{196x^2 - (x^2 + 24)^2} = 114.$$

By a few trials it is soon found that the value of x is but little above 7. Suppose, therefore, first that x is $= 7$, and then $x = 8$.

$$\begin{array}{rcr} \text{First, when } x_2 = 7. & & \text{Second, when } x_1 = 8. \\ 47{\cdot}906 \ \ldots & \sqrt{144x^2 - (x^2 + 20)^2} & \ldots\ 46{\cdot}476 \\ 65{\cdot}384 \ \ldots & \sqrt{196x^2 - (x^2 + 24)^2} & \ldots\ 69{\cdot}283 \\ \hline k_2 = 113{\cdot}290 & \ldots\ldots & 115{\cdot}759 = k_1 \\ k = 114{\cdot}000 & \ldots\ldots & 114{\cdot}000 = k \\ \hline k_2 - k = -\,0{\cdot}710 & \ldots\ldots & +\,1{\cdot}759 = k_1 - k \\ k_1 - k_2 = +\,1{\cdot}759 & \ldots\ldots & \\ \hline \end{array}$$

$k_1 - k_2 = 2{\cdot}469$. Hence by the formula we find $x = 7\,2$, nearly.

Suppose again $x = 7\cdot2$, and then, since this turns out too great, take $7\cdot1$ and $7\cdot2$ for the trial numbers, as follows:

First, when $x_2 = 7\cdot2$, Second, when $x_1 = 7\cdot1$

$$47\cdot990 = \sqrt{144x^2 - (x^2+20)^2} = 47\cdot973$$
$$66\cdot402 = \sqrt{196x^2 - (x^2+24)^2} = 65\cdot904$$

$$k_2 = 114\cdot392 \qquad\qquad 113\cdot877 = k_1$$
$$k = 114\cdot000 \qquad\qquad 114\cdot000 = k$$

$$k_2 - k = \cdot392 \qquad\qquad -\cdot123 = k_1 - k$$

Hence $x = x_2 + \dfrac{k - k_2}{k_1 - k_2}(x_1 - x_2) = 7\cdot1 + \dfrac{\cdot1\,.\,\cdot123}{\cdot515} = 7\cdot124$ nearly.

Ex. 4. Resolve the equation $x^3+10x^2+5x=260$. Ans. $x=4\cdot11798574108$.

Ex. 5. To find the value of x in the equation $x^3-2x=50$. Ans. $3\cdot864854$.

Ex. 6. To find x in the equation $x^3+2x^2-23x=70$. Ans. $x=5\cdot13457$.

Ex. 7. To find x in the equation $x^3-17x^2+54x=350$. Ans. $x=14\cdot95407$.

Ex. 8. To find x in the equation $x^4-3x^2-75x=10000$. Ans. $x=10\cdot2609$.

Ex. 9. Find x in $2x^4-16x^3+40x^2-30x=-1$. Ans. $x=1\cdot284724$.

Ex. 10. Resolve $x^5+2x^4+3x^3+4x^2+5x=54321$. Ans. $x=8\cdot414455$.

Ex. 11. Given $2x^4-7x^3+11x^2-3x=11$, to find x. Ans. $x = 1\cdot8375506$.

Ex. 12. To find the value of x in the equation

$(3x^2 - 2\sqrt{x} + 1)^{\frac{3}{5}} - (x^2 - 4x\sqrt{x} + 3\sqrt{x})^{\frac{5}{9}} = 56$. Ans. $x = 18\cdot360877$.

THE NUMERICAL SOLUTION OF ALGEBRAIC EQUATIONS.

The solution of algebraic equations having general or literal coefficients has never been effected beyond the fourth degree; and methods to this extent have already been explained in this work, at pages 170, 185, 198, and 200. A considerable number of the properties of the roots of algebraic equation without limitation as to degree, when taken in connexion with the coefficients, have been investigated; but as these have been viewed more with a prospect of literal than mere numerical solution, they become in reference to our present object rather matters of curiosity than of utility. It is not proposed, then, to enter upon the general theory of equations in this work further than it conduces to the solution of those whose coefficients are numerically given.

As, whatever may be the number of equations simultaneously given and containing the same number of unknowns, these equations can be reduced to a single one containing only one unknown, the inquiry will in this place be restricted to this single equation, whether originally so given or obtained from the system of coexisting equations by elimination.

DEFINITIONS AND NOTATION.

1. An *algebraic equation* is any one which contains positive integer powers of the unknown quantity. The following is the general type, n being a positive

integer, and A, B, C, ... L, M, N, any numbers whatever, either positive or negative, or zero, but all free from the imaginary symbol.

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots + Lx^2 + Mx + N = 0.$$

Some classes of reasonings, however, are facilitated by having unity for the coefficient of x^n, and the equation is at once reduced to this form by division of all the terms by A.

2. For brevity of writing, this is often put in the contracted form

$$f(x) = 0, \text{ or } X = 0.$$

The symbols $f(x)$ and X are in this case called *functions of x*; meaning an expression into the composition of which x enters, or which depends upon the value of x.

3. *A root of an equation* is any number or expression which, on being substituted in the given equation, and all reductions being performed, fulfils the expressed condition of making both sides equal by the mutual cancel of all the terms on the first side.

4. If $r_1, r_2, r_3, \ldots$ be roots of an algebraic equation $f(x) = 0$, then the successive quotients of $f(x)$ by $x-r_1$, $x-r_2$, ... are called the *depressed equations*. It is seldom that these depressed equations require any special notation; but when they do, $\phi_1(x) = 0$, $\phi_2(x) = 0$, $\phi_3(x) = 0 \ldots$ are found convenient.

5. If $f(x)$ be divided by $x-x$, these quotients are called the *first*, *second*, *third*, ... *derivative functions* of x, or simply *derivatives*. They are respectively denoted by $f_1(x)$, $f_2(x)$, $f_2(x)$, ... $f_n(x)$ according to the number of successive divisions performed.

6. An equation is said to be *transformed* when it is changed into another whose roots have any assigned relation to those of the given one: as, for instance, when an equation is given, and another is formed from it whose roots shall be triple, one half, or any multiple or part of those of the given one; or again when the roots shall be all greater or less than those of the given equation by some given quantity; and so on. In the last-mentioned case, the new equation is called the *reduced equation*.

7. By a *permanence of signs*, or simply *a permanence*, is meant that two consecutive terms of a complete equation have the same sign prefixed; as + + or — —: and by a *variation*, that two consecutive terms have unlike signs prefixed, as + — or — +.

THEOREM 1. If $r, r_1, r_2, \ldots$ be roots of an equation, $f(x) = 0$, then $f(x)$ is exactly divisible by $x-r$, $x-r_1$, $x-r_2$, ... without remainder.

Let $f(x) = Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots + Mx + N = 0$ be the given equation, and let the first side be divided by $x-r$ by the *synthetic method*, p. 128.

$$\begin{array}{r|l} & A + B + C + D + \ldots\ldots + L + M + N \\ +r & \quad + Ar + B_1r + C_1r + \ldots\ldots + K_1r + L_1r + M_1r \\ \hline & A + B_1 + C_1 + D_1 + \ldots\ldots + L_1 + M_1 + N_1 \end{array}$$

where A, B_1, C_1, ... L_1, M_1, N_1 are the coefficients of the quotient, as far as the term x^{-1}; and we have to show that $N_1 = 0$, and hence that the quotient terminates at M_1.

By attending to the formation of the coefficients of the quotient, we see that

$$\begin{aligned} B_1 &= Ar + B, \\ C_1 &= B_1r + C = Ar^2 + Br + C, \\ D_1 &= C_1r + D = Ar^3 + Br^2 + Cr + D, \end{aligned}$$

. . . .

$$M_1 = L_1 r + M = Ar^{n-1} + Br^{n-2} + Cr^{n-3} + .. + Lr + M,$$
$$N_1 = M_1 r + N = Ar^n + Br^{n-1} + Cr^{n-2} + .. + Lr^2 + Mr + N.$$

But by hypothesis r is a root of the equation $f(x) = 0$; hence substituting it for x we have

$$Ar^n + Br^{n-1} + Cr^{n-2} + \dots + Lr^2 + Mr + N = 0,$$

and as this is exactly the value of N_1 found above we have $N_1 = 0$, and the division terminates.

In the same manner it may be shown that $f(x)$ is divisible by $x-r_1$, $x-r_2$,..

Cor. Since $f(x)$ is divisible separately by $x-r$, $x-r_1$, $x-r_2$,.. it is also divisible by their product.

C. 5. THEOREM II. The derivatives may be formed by inspection in the following manner:

Multiply each term by the index of the power of x in it, and diminish that index by unity, which will give the first derivative: operate upon the first derivative in the same manner to produce the second; upon the second to produce the third; and so on to the end.

For the division by $x-x$ gives the same coefficients in terms of x that the division by $x-r$ gives in terms of r; hence restoring x for r in the values of B_1, C_1,.. L_1, M_1, we have

$$\begin{aligned} Ax^{n-1} + Ax^{n-1} + Ax^{n-1} + \dots \text{ (to } n \text{ terms)} &= nAx^{n-1} \\ + Bx^{n-2} + Bx^{n-2} + \dots \text{ (to } n-1 \text{ terms)} &= (n-1)\,Bx^{n-2} \\ + Cx^{n-3} + \dots \text{ (to } n-2 \text{ terms)} &= (n-2)\,Bx^{n-3} \end{aligned}$$

$$\text{Hence, } \frac{f(x)}{x-x} = nAx^{n-1} + (n-1)\,Bx^{n-2} + \dots + 3Kx^2 + 2Lx + M = f_1(x)$$

The second derivative $f_2(x)$ will obviously be found from this in the same manner, since the same reasoning applies to all the derivatives in succession.

Hence the several derivatives are

$$\begin{aligned} f_1(x) &= nAx^{n-1} + (n-1)\,Bx^{n-2} + (n-2)\,Cx^{n-3} + \dots \\ f_2(x) &= n(n-1)\,Ax^{n-2} + (n-1)(n-2)\,Bx^{n-3} + (n-2)(n-3)\,Cx^{n-4} + \dots \\ f_3(x) &= n(n-1)(n-2)\,Ax^{n-3} + (n-1)(n-2)(n-3)\,Bx^{n-4} + \dots \\ &\dots\dots\dots\dots\dots\dots \\ f_{n-2}(x) &= n(n-1)\dots 3Ax^2 + (n-1)(n-2)\dots 2Bx + (n-2)(n-3)\dots 2.1 \\ f_{n-1}(x) &= n(n-1)\dots 2Ax + (n-1)(n-2)\dots 1.B \\ f_n(x) &= n(n-1)\dots 2.1.A \end{aligned}$$

PROBLEM I.

To calculate the value of an algebraical function when the value of x is given; that is, to find the value of $Ax^n + Bx^{n-1} + \dots + Mx + N$, when x is a given number, the coefficients being also given numerically.

Range the coefficients in a horizontal line, as for synthetic division, with their proper signs prefixed, taking care when any coefficients are absent to fill their places with ciphers. Multiply A by the given value of x, and add the result to B, making the sum B_1; multiply B_1 by a and add the result to C giving C_1, and so on till we arrive at N_1. Then N_1 is the value of the expression.

For this is precisely the operation performed in theorem 1; and by the reasoning of that theorem

$$N_1 = Aa^n + Ba^{n-1} + \dots + La^2 + Ma + N.$$

Thus if the expression or function were $4x^6 - 5x^4 + 6x^3 + 4x^2 - 10x + 15$,

and we were required to find its value when $x = 5$ and when $x = -2$, then the work would stand thus:

$$\begin{array}{l|l}
4 + 0 - 5 + 6 + 4 - 10 + 15 & 5 \\
\quad 20 + 100 + 475 + 2405 + 12045 + 60175 & \\
\hline
\quad 20 + 95 + 481 + 2409 + 12035 + 60190 = \text{value when } x = 5. &
\end{array}$$

$$\begin{array}{l|l}
4 + 0 - 5 + 6 + 4 - 10 + 15 & -2 \\
-8 + 16 - 22 + 32 - 72 + 164 & \\
\hline
-8 + 11 - 16 + 36 - 82 + 179 = \text{value when } x = -2. &
\end{array}$$

EXAMPLES FOR PRACTICE.

1. Find the values of the function $x^6 + 4x^3 - 6x + 10$, when x has severally the values 1, 10, 19, and -100.

2. Find the values of the function $x^5 - 2x^4 - 10x^3 + 30x^2 + 63x - 120$, when x has the values 1, 2, -1, -2, and -3.

3. Find the values of $x^4 - 25x^2 + 60x - 36$ when x is 3, 2, 1, and -6.

When $x = \pm \{\frac{1}{2}\sqrt{2} \pm \sqrt{\mp 3\sqrt{2} - \frac{1}{2}}$, what is $x^4 + 12x - 17$ equal to?

What is the value of $x^3 - 7x^2 + 14x - 20$, when x is 5, or $1 \pm \sqrt{-3}$?

PROBLEM II.

To transform an algebraic equation, into another having its roots less by a given quantity a than the roots of the given equation.

Divide the given function continually, employing the synthetic method, by $x-a$, always stopping at the term where $(x-a)^{-1}$ occurs: the several terminal quotients will be the coefficients of the reduced equation. The following is its general type.*

Let $Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots Lx^2 + Mx + N = 0$ be the given equation: then, dividing synthetically, we have the several operations as follow:

$$\begin{array}{c|l}
 & A + B + C + D + \ldots + K + L + M + N \\
a & \quad + Aa + B_1a + C_1a + \ldots + H_1a + K_1a + L_1a + M_1a \\
\hline
 & A + B_1 + C_1 + D_1 + \ldots + K_1 + L_1 + M_1 + N_1 \\
a & \quad + Aa + B_2a + C_2a + \ldots + H_2a + K_2a + L_2a \\
\hline
 & A + B_2 + C_2 + D_2 + \ldots + K_2 + L_2 + M_2 \\
a & \quad + Aa + B_3a + C_3a + \ldots + H_3a + K_3a \\
\hline
 & A + B_3 + C_3 + D_3 + \ldots + K_3 + L_3 \\
\vdots & \\
\vdots & \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
\vdots & \\
\hline
 & A + B_{n-2} + C_{n-2} + D_{n-2} \\
a & \qquad Aa + B_{n-1}a \\
\hline
 & A + B_{n-1} + C_{n-1} \\
a & \qquad Aa \\
\hline
 & A + B_n
\end{array}$$

and the transformed equation is

$$A(x-a)^n + B_n(x-a)^{n-1} + C_{n-1}(x-a)^{n-2} + \ldots\ldots L_3(x-a)^2 + M_2(x-a) + N_1 = 0.$$

* It will be more convenient to place the transforming number to the right of the coefficients, separated by the curve employed in division or extraction of roots, as in the numerical illustrations which follow.

For the results of the successive divisions performed as above, are

$$A x^{n-1} + B_1 x^{n-2} + C_1 x^{n-3} + \ldots + L_1 x + M_1 + \frac{N_1}{x-a} = 0.$$

$$A x^{n-2} + B_2 x^{n-3} + C_2 x^{n-4} + \ldots + L_2 + \frac{M_2}{x-a} + \frac{N_1}{(x-a)^2} = 0.$$

$$A x^{n-3} + B_3 x^{n-4} + C_3 x^{n-5} + \ldots + \frac{L_3}{x-a} + \frac{M_2}{(x-a)^2} + \frac{N_1}{(x-a)^3} = 0.$$

. . . .

$$A + \frac{B_n}{x-a} + \frac{C_{n-1}}{(x-a)^2} + \ldots + \frac{L_3}{(x-a)^{n-2}} + \frac{M_2}{(x-a)^{n-1}} + \frac{N_1}{(x-a)^n} = 0.$$

and multiplying at once by $(x-a)^n$, it becomes

$$A\,(x-a)^n + B_n\,(x-a)^{n-1} + C_{n-1}\,(x-a)^{n-2} + \ldots\, L_3\,(x-a)^2 + M_2\,(x-a) + N_1 = 0,$$

which establishes the truth of the rule.*

Cor. 1. If we *diminish* the roots of an equation by a quantity a_p which is greater than p of the positive roots $a, a_1, a_2, \ldots a_{p-1}$, then in the reduced equation, p of the positive roots corresponding to these will be negative, viz. $a - a_p$, $a_1 - a_p$, $\ldots\, a_{p-1} - a_p$, all of which are negative, since a_p is the greatest of all the quantities. In a similar manner, if the roots of an equation are *increased* by a quantity a_p which is greater than $a, a_1, \ldots a_{p-1}$, ($-a, -a_1, \ldots -a_{p-1}$, being p negative roots,) then the p roots of the reduced equation corresponding to these will become positive. Of course the transformed roots corresponding to those which were negative before the diminution, or positive before the increase of the roots, retain still the same character as their original corresponding roots.

Cor. 2. Had we been required to form an equation whose roots are greater than those of a given equation, the process would obviously have been the same, only dividing continually by $x + a$ instead of $x - a$; that is, by using the factor $-a$ instead of $+a$ used above in forming the several successive courses of coefficients.

As examples, let $2x^4 - 10x^3 + 20x^2 - 15x + 10 = 0$ be transformed into an equation whose roots are less by 4, and the equation $2y^4 + 22y^3 + 92y^2 + 177y + 142 = 0$ into one whose roots shall be greater by 4. The work will stand thus:—

$$\begin{array}{rrrrr|l}
2 & -\,10 & +\,20 & -\,15 & +\,10 & \;4 \\
 & 8 & -\,8 & +\,48 & +\,132 & \\ \hline
 & -\,2 & +\,12 & +\,33 & +\,142 & \\
 & 8 & +\,24 & +\,144 & & \\ \hline
 & 6 & +\,36 & +\,177 & & \\
 & 8 & +\,56 & & & \\ \hline
 & 14 & +\,92 & & & \\
 & 8 & & & & \\ \hline
 & 22 & & & &
\end{array}$$

Hence the transformed equation is
$2\,(x-4)^4 + 22\,(x-4)^3 + 92\,(x-4)^2 + 177\,(x-4) + 142 = 0.$

$$\begin{array}{rrrrr|l}
2 & +\,22 & +\,92 & +\,177 & +\,142 & -\,4 \\
 & -\,8 & -\,56 & -\,144 & -\,132 & \\ \hline
 & 14 & +\,36 & +\,33 & +\,10 & \\
 & -\,8 & -\,24 & -\,48 & & \\ \hline
 & 6 & +\,12 & -\,15 & & \\
 & -\,8 & +\,8 & & & \\ \hline
 & -\,2 & +\,20 & & & \\
 & -\,8 & & & & \\ \hline
 & -\,10 & & & &
\end{array}$$

Hence the transformed equation is
$2\,(y+4)^4 - 10\,(y+4)^3 + 20\,(y+4)^2 - 15\,(y+4) + 10 = 0.$

---

* Other methods requiring rather less work but involving principles rather less comprehensible by the student, (and given, too, without investigation,) may be seen in Leybourn's Repository, vol. v. pp. 42--44. The demonstrations of them will appear in my forthcoming publication of Mr. Horner's works on Equations.—EDITOR.

It will at once appear that the second given equation is but a transformation of the first, and *vice versâ*. The restoration of the original result proves the truth of the transformation and re-transformation.

When the transformation is to be made by a number comprising more than one figure, it may first be transformed by the first of them (regard being had to its place in the decimal scale), then this transformed equation again transformed by the next figure, then again this by the next figure, to any assigned extent, the same precaution respecting decimal place being observed in all. Thus, to reduce the roots of the equation $x^3 + 3x^2 + 3x - 140 = 0$ by $4{\cdot}23$, we shall have

$$\begin{array}{lllll}
1 + & 3 + & 3 - & 140 & \underline{\backslash 4} \\
 & 4 & 28 & 124 & \\ \hline
 & 7 + & 31 - & 16 & \\
 & 4 + & 44 & & \\ \hline
 & 11 + & 75 & & \\
 & 4 & & & \\ \hline
 & 15 & & &
\end{array}
\qquad
\begin{array}{lllll}
1 + & 15 & + 75 & - 16 & \underline{\backslash {\cdot}2} \\
 & {\cdot}2 & 3{\cdot}04 & 15{\cdot}608 & \\ \hline
 & 15{\cdot}2 & 78{\cdot}04 - & {\cdot}392 & \\
 & {\cdot}2 & 3{\cdot}08 & & \\ \hline
 & 15{\cdot}4 & 81{\cdot}12 & & \\
 & {\cdot}2 & & & \\ \hline
 & 15{\cdot}6 & & &
\end{array}$$

$$\begin{array}{lllll}
1 + & 15{\cdot}6 & + 81{\cdot}12 & - {\cdot}392 & \underline{\backslash {\cdot}03} \\
 & {\cdot}03 & {\cdot}4689 & 2{\cdot}447667 & \\ \hline
 & 15{\cdot}63 & 81{\cdot}5889 & 2{\cdot}055667 & \\
 & 03 & 4698 & & \\ \hline
 & 15{\cdot}66 & 82{\cdot}0587 & & \\
 & 03 & & & \\ \hline
 & 15{\cdot}69 & & &
\end{array}$$

And the transformed equation is

$$(x - 4{\cdot}23)^3 + 15{\cdot}69\,(x - 4{\cdot}23)^2 + 82{\cdot}0587\,(x - 4{\cdot}23) + 2{\cdot}055667 = 0\ *.$$

But in many cases, especially where the process does involve much intermixture of the signs + and −, the whole work may be more advantageously performed at once according to the following method:

$$\begin{array}{llll}
1 + 3 & + 3 & - 140 & (4{\cdot}23 \\
\quad 4{\cdot}23 & 28{\cdot}92 & 134{\cdot}3316 & \\
 & \ \ 1{\cdot}446 & \ \ 6{\cdot}71658 & \\
 & \quad 2169 & \ \ 1{\cdot}007487 & \\ \hline
1 + 7{\cdot}23 & 33{\cdot}5829 & \ \ 2{\cdot}055667 & \\
\quad 4{\cdot}23 & 45{\cdot}84 & & \\
 & \ \ 2{\cdot}292 & & \\
 & \quad {\cdot}3438 & & \\ \hline
1 + 11{\cdot}46 & 82{\cdot}0587 & & \\
\quad\ 4{\cdot}23 & & & \\ \hline
\quad 15{\cdot}69 & & &
\end{array}$$

* Though the decimal points are marked in this process, they will, after a little practice, be easily dispensed with by the pupil, as the regular arrangement and increase of the places to the right will always secure the figures falling rightly.

EXERCISES ON THE REDUCTION OF EQUATIONS.

1. Transform the equation $x^3 + 4x^2 + 2x - 2328 = 0$, into one whose roots shall be less by 10; and this into another whose roots shall be still less by 2. Then transform the result into one whose roots shall be greater by 12.

2. Reduce the roots of $x^3 + 8x - 34648584 = 0$* successively by 300, 20, and 6.

3. Reduce the roots of the equation $x^3 - 18609625 = 0$, successively by the numbers 200, 60, and 5; and then verify the process by increasing those of the result by 60, 200, and 5.

PROBLEM III.

To transform a given equation into another the roots of which shall be the same as those of the given one, but having all their signs reversed.

Change the signs of the alternate terms, beginning with either the first or second; then this will be the equation required.

For first, let the degree of the equation be even, as

$Ax^{2n} + Bx^{2n-1} + Cx^{2n-2} + \ldots. Kx^2 + Lx + M = 0,$

then writing $-x$ instead of x, we shall have

$A(-x)^{2n} + B(-x)^{2n-1} + C(-x)^{2n-2} + \ldots. + K(-x)^2 + L(-x) + M = 0$; or,

$Ax^{2n} - Bx^{2n-1} + Cx^{2n-2} - \ldots. + Kx^2 - Lx + M = 0.$

And, secondly, let it be of an odd degree, as

$Ax^{2n+1} + Bx^{2n} + Cx^{2n-1} + Dx^{2n-2} + \ldots. Kx^3 + Lx^2 + Mx + N = 0,$

in which, writing $-x$ for x we get

$A(-x)^{2n+1} + B(-x)^{2n} + C(-x)^{2n-1} + \ldots. + K(-x)^3 + L(-x)^2 + M(-x) + N = 0$; or,

$-Ax^{2n+1} + Bx^{2n} - Cx^{2n-1} + \ldots. - Kx^3 + Lx^2 - Mx + N = 0$; or,

$Ax^{2n+1} - Bx^{2n} + Cx^{2n-1} - \ldots. + Kx^3 - Lx^2 + Mx - N = 0.$

Thus if the roots of the equation $x^4 \pm 0x^3 - 25x^2 + 60x - 36 = 0$ be 3, 2, 1, -6; then those of $x^4 \mp 0x^3 - 25x^2 - 60x - 36 = 0$ are -3, -2, -1, and $+6$.

EXERCISES.

Change signs of the roots of the equations given in problems I. and II. the functions in problem I. being equated to 0.

PROBLEM IV.

To transform an equation into another whose roots shall be the reciprocals of the roots of the given one.

Reverse the order of the co-efficients: these will be the co-efficients of the new equation sought.

For, if in $Ax^n + Bx^{n-1} + \ldots.. + Lx^2 + Mx + N = 0$, we write $y = \frac{1}{x}$, or $x = \frac{1}{y}$, then substituting we get

---

* When any terms are deficient, their places must be kept and filled with 0; that being in such case the value of the coefficient of that term, as in the *Synthetic Division*.

$$\frac{A}{y^n} + \frac{B}{y^{n-1}} + \frac{C}{y^{n-2}} + \ldots + \frac{L}{y^2} + \frac{M}{y} + N = 0, \text{ or}$$
$$A + By + Cy^2 + \ldots\ldots + Ly^{n-2} + My^{n-1} + Ny^n = 0\,^*;$$

EXERCISES.

Find the equations whose roots are the reciprocals of those of the equations given in problems I. and II. equating the functions in the former to 0.

Scholium.

Any other proposed relations between the roots of a given equation and those of another to be determined, may be effected in an analogous manner, viz.: by first expressing the assigned relation between x and y, resolving for x in terms of y, and substituting for x its value in the given equation. After the simplification of the expression to the utmost degree, we shall obtain the equation sought.

For instance, to form an equation whose roots shall be m times those of the given equation, put $y = mx$, or $x = \frac{y}{m}$: then the equation

---

* When the coefficients of the direct and reciprocal equations are alike, that is, $N = A$, $M = B$, $L = C$, ... it is evident, that upon knowing the value of half the roots, the other half will be known from their being the reciprocals of these; or, in other words, if r_1, r_2, ... be roots, then also will $\frac{1}{r_1}$, $\frac{1}{r_2}$, ... be roots also. This circumstance, as it lessens the work of solving an equation of such form, is important to be remarked.

1. Let the equation be of an even degree, having the above relation; then it may be written

$$Ax^{2n} + Bx^{2n-1} + Cx^{2n-2} + \ldots\ldots Cx^2 + Bx + A = 0, \text{ or again,}$$
$$A\{x^{2n}+1\} + B\{x^{2n-1}+x\} + C\{x^{2n-2}+x^2\} + \ldots = 0; \text{ and dividing by } x^n,$$
$$A\left\{x^n + \frac{1}{x^n}\right\} + B\left\{x^{n-1} + \frac{1}{x^{n-1}}\right\} + C\left\{x^{n-2} + \frac{1}{x^{n-2}}\right\} + \ldots = 0 \ldots\ldots\ldots\ldots (1)$$
$$\text{Now } x^n + \frac{1}{x^n} = \left\{x + \frac{1}{x}\right\}^n - n\left\{x^{n-2} + \frac{1}{x^{n-2}}\right\} - \frac{n(n-1)}{1\cdot2}\left\{x^{n-4} + \frac{1}{x^{n-4}}\right\} - \ldots$$
$$x^{n-1} + \frac{1}{x^{n-1}} = \left\{x + \frac{1}{x}\right\}^{n-1} - (n-1)\left\{x^{n-3} + \frac{1}{x^{n-3}}\right\} - \frac{(n-1)(n-2)}{1\cdot2}\left\{x^{n-5} + \frac{1}{x^{n-5}}\right\} - \ldots$$

and so on.

By these successive reductions we can convert (1) into the form

$$Au^n + B'u^{n-1} + C'u^{n-2} + \ldots\ldots = 0, \left(\text{where } u = x + \frac{1}{x}\right) \ldots\ldots\ldots\ldots (2)$$

Then resolving (2) we find n values of u, and each of these substituted in $x + \frac{1}{x} = u$ gives two reciprocal values of x, and thereby furnish the $2n$ roots of the equation.

The solution of Ex. 5, p. 194, is an exemplification of this circumstance.

2. Let the equation be of an odd degree, as the $(2n+1)$th: then, since it is the same thing as

$$A\{x^{2n+1}+1\} + B\{x^{2n-1}+1\}x + C\{x^{2n-3}+1\}x^2 + \ldots\ldots = 0,$$

in which, as all the powers of x within the brackets are odd, every bracketted term is divisible by $x+1$. Hence $x + 1 = 0$ is one of the component factors of the equation; and the given equation being divided by $x + 1$, gives a depressed equation of the $2n$th degree; and hence also this is soluble by means of an equation of the nth degree, and a quadratic as in the last case. It would be just the same if the latter half of coefficients were written — instead of +, since it would only change the sign between each two bracketted terms, and every term would then be divisible by $x - 1 = 0$, and $x = 1$ would be the corresponding single root.

The problem suggested at p. 194 may be taken as an illustration. Equations in which this relation exists are called *reciprocal recurrents.*

$$Ax^n + Bx^{n-1} + \dots + Lx^2 + Mx + N = 0, \text{ becomes}$$
$$\frac{Ay^n}{m^n} + \frac{By^{n-1}}{m^{n-1}} + \frac{Cy^{n-2}}{m^{n-2}} + \dots \frac{Ly^2}{m^2} + \frac{My}{m} + N = 0, \text{ or}$$
$$Ay^n + mBy^{n-1} + m^2Cy^{n-2} + \dots m^{n-2} \, . \, Ly^2 + m^{n-1}My + m^n \, . \, N = 0.$$

Or again, to form an equation whose roots shall be $\frac{1}{m}$th of those of the given equation, we have $y = \frac{x}{m}$, or $x = my$, and the transformation leads to

$$Am^ny^n + Bm^{n-1}y^{n-1} + \dots \dots Lm^2y^2 + Mmy + N = 0.$$

These transformations are often useful in the solution of equations, and the arrangement of the work for effecting them is sufficiently obvious, without any examples.

PROBLEM V.

TO FORM THE EQUATION WHOSE ROOTS ARE ANY GIVEN NUMBERS.

1. Change the signs of all the given roots.

2. Put down 1, having annexed to it by its proper sign one of these changed roots; and multiply this binomial by another of the changed roots, beginning the products one place to the right, and add up the columns into one single horizontal row. These will be the co-efficients with their proper signs of the quadratic equation whose roots are those two roots already used.

3. Multiply this horizontal row by another of the changed roots, placing as before the first product in the second column. The several sums will be the co-efficients with their proper signs of a cubic equation, whose roots are the three roots already used.

4. Proceed thus through all the roots: then the equation will be completed. For this is evidently only working by detached co-efficients, and availing ourselves of the contrivance of allowing the multiplicand to stand as the product of the multiplicand by 1, and also of omitting the actual exhibition of the multiplier beneath the multiplicand. The second term of the multiplier may be put in the margin, as in the example annexed to the rule. This will appear quite clearly upon working out one example at length.

For this is the only application of the method of detached coefficients to the multiplication of $x - a = 0$, $x - b = 0$, together: and as $(x-a)\,(x-b)\dots\dots = 0$, is fulfilled by $x = a$, $x = b$, ..., therefore a, b, ... are roots of the equation, by the definition of a root.

Thus, suppose it were required to form the equation whose roots are -1, 1, 3, 4, and -5, the process would be as follows :—

$$\begin{array}{rrrrrrl}
1 & +1 & & & & & \lfloor -1 \\
 & -1 & -1 & & & & \\
\hline
1 & -0 & -1 & & & & \lfloor -3 \\
 & -3 & +0 & +3 & & & \\
\hline
1 & -3 & -1 & +3 & & & \lfloor -4 \\
 & -4 & +12 & +4 & -12 & & \\
\hline
1 & -7 & +11 & +7 & -12 & & \lfloor +5 \\
 & 5 & -35 & +55 & +35 & -60 & \\
\hline
1 & -2 & -24 & +62 & +23 & -60 &
\end{array}$$

Hence the equation is $x^5 - 2x^4 - 24x^3 + 62x^2 + 23x - 60 = 0$.

EXAMPLES.

1. Form the equation whose roots are 1, 5, — 4, — 3, — 1, and compare it with the example worked above by means of Problem III.

2. Form the equation whose roots are the reciprocals of each of these, viz. of $\frac{1}{1}$, $\frac{1}{5}$, $-\frac{1}{4}$, $-\frac{1}{3}$, and $-\frac{1}{1}$; and also of $-\frac{1}{1}$, $-\frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{1}$, by means of Problem IV.

3. Form the equation whose roots are severally the first nine digits, taken alternately + and —, viz. + 1, — 2, + 3, ... ; and then form that whose roots are the reciprocals of these.

4. The three roots of a cubic equation are in arithmetical progression, they are all integers, and their sum is equal to their continued product. What is that cubic equation?

5. If any number of pairs of roots be of the form $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$, $a_{\prime} + b_{\prime}\sqrt{-1}$, and $a_{\prime} - b_{\prime}\sqrt{-1}$, show that the co-efficients of the equation will be real; or, in other words, that the imaginary symbols will disappear from the result.

THEOREM III. If an equation given in terms of x be transformed into one in terms of $x - a$, then the several coefficients will be

$$\frac{f_n(a)}{1.2\ldots n},\quad \frac{f_{n-1}(a)}{1.2\,..\,(n-1)},\quad \frac{f_{n-2}(a)}{1.2\,..\,(n-2)},\ \ldots\ldots\ \frac{f_2(a)}{1.2},\quad \frac{f_1(a)}{1},\ f(a),$$

wherein a takes the place of x in the given equation and its successive n derivatives.

This is established at once by completing the transformation in terms of a, and resolving the numerical coefficients into factors in terms of n. The only difficulty in giving the successive steps of the work in this place, is the extent of page which it would require; but as this does not hold in the student's practice, he should exercise himself in the actual reduction of four or five of these coefficients *.

* This theorem admits of an elegant demonstration, by means of the binomial theorem, as follows:

Put $x = a + z$: then expanding the function, $\mathrm{A}x^n + \mathrm{B}x^{n-1} + \ldots$, so that like powers of z stand in the same vertical columns, we have

$$(a+z)^n = \mathrm{A}a^n + \frac{n\mathrm{A}a^{n-1}}{1}.z + \frac{n(n-1)\,\mathrm{A}a^{n-2}}{1.2}.z^2 + \frac{n(n-1)(n-2)\,\mathrm{A}a^{n-3}}{1.2.3}.z^3 + \ldots$$

$$(a+z)^{n-1} = \mathrm{B}a^{n-1} + \frac{(n-1)\,\mathrm{B}a^{n-2}}{1}.z + \frac{(n-1)(n-2)\,\mathrm{B}a^{n-3}}{1.2}.z^2 + \frac{(n-1)(n-2)(n-3)\,\mathrm{B}a^{n-4}}{1.2.3}.z^3 + \ldots$$

$$(a+z)^{n-2} = \mathrm{C}a^{n-2} + \frac{(n-2)\,\mathrm{C}a^{n-3}}{1}.z + \frac{(n-2)(n-3)\,\mathrm{C}a^{n-4}}{1.2}.z^2 + \frac{(n-2)(n-3)(n-4)\,\mathrm{C}a^{n-5}}{1.2.3}.z^3 + \ldots$$

and so on to the end.

In this we see at once that by adding vertically, and attending to the forms taken by the derivatives, the several coefficients of z are as stated in the text; and that we have

$$f(a+z) = f(a) + \frac{f_1(a)}{1}.z + \frac{f_2(a)}{1.2}.z^2 + \ldots\ldots + \frac{f_{n-1}(a)}{1.2\,..\,(n-1)}.z^{n-1} + \frac{f_n(a)}{1.2\ldots n}.z^n$$

If we restore the value of z, this becomes

$$f(x) = f(a) + \frac{f_1(a)}{1}(x-a) + \frac{f_2(a)}{1.2}(x-a)^2 + \ldots\ldots + \frac{f_n(a)}{1.2\ldots n}(x-a)^n;$$

or, again, reversing the order of the terms, it is

THEOREM IV. If the equation $f(x) = 0$ have p equal roots, then $f_1(x) = 0$ will have $p - 1$ of them, $f_2(x) = 0$ will have $p - 2$ of them, and so on; till $f_{p-1}(x) = 0$ will have $p - (p - 1) = 1$ of them, and $f_p(x) = 0$ will have $p - p = 0$ of them.

For, since $f(x) = 0$ contains p roots each equal to r, the function $f(x)$ is divisible p times successively by $x - r$ without remainder (theor. 1). Hence the p last coefficients, viz. N_1, M_2, L_3 ..... in the operation of Problem II. will be zero. But these coefficients are respectively

$$\frac{f_1(r)}{1},\ \frac{f_2(r)}{1.2},\ \frac{f_3(r)}{1.2.3},\ \ldots\ \frac{f_{p-1}(r)}{1.2\ldots(p-1)};$$

and as the denominators are all finite, it follows that the several numerators are equal to 0. That is, $f_1(r) = 0$, $f_2(r) = 0$, ... $f_p(r) = 0$; and therefore the value of r, which fulfils the equation $f(x) = 0$, fulfils also the $p - 1$ equations $f_1(x) = 0, f_2(x) = 0, \ldots . f_{p-1}(x) = 0$ *.

$$f(x) = \frac{f_n(a)}{1.2\ldots n}(x-a)^n + \frac{f_{n-1}(a)}{1.2\ldots(n-1)}(x-a)^{n-1} + \ldots \frac{f_2(a)}{1.2}(x-a)^2 + \frac{f_1(a)}{1}(x-a) + f(a),$$

which is the same result as above stated.

One important use of this problem in the older methods of solution of equations, was to enable us to remove any specified term from the equation by rendering its coefficient equal to zero. It only required us to solve an equation of an inferior degree, viz. of the first degree to remove the second term, of the second degree to remove the third term, and so on.

For evidently, if we find such a value of a in these several coefficients as would render them respectively zero, our object would be accomplished. That is, to solve the equation $f_p(a) = 0$, which is of the $(n-p)^{\text{th}}$ degree, for the unknown quantity a.

Thus, $f_{n-1}(a) = (nAa + B)(n-1)(n-2)\ldots\ldots 2.1 = 0$, or $nAa + B = 0$; or $a = -\frac{B}{nA}$.

$$f_{n-2}(a) = \{n(n-1)Aa^2 + (n-1)Ba + C\}(n-2)\ldots\ldots 2.1 = 0;\text{ or,}$$

$$n(n-1)Aa^2 + (n-1)Ba + C = 0.$$

Hence there are two values of a (either real or imaginary), which will remove the third term. In the same way by solving $f_{n-3}(a) = 0$, $f_{n-4}(a) = 0$, and so on, we may remove any term whatever. It may be remarked, that to remove the last term we must solve $f(a) = 0$, or the given equation.

This use is, however, superseded by improved methods of solution; but it is still applicable to many others of great importance, some of which will be made apparent in this work.

* The following method may perhaps be found more intelligible to some students than that in the text.

Since $f(x) = 0$ contains p roots equal to r, the function $f(x)$ is divisible by $x - r$ successively p times; and as r is a root of the depressed equations till the last, we may write x for it in each of them successively. But this transforms the several depressed equations into the several derivatives; and thus we have

$$\begin{array}{ll} Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots\ldots\ldots + Lx^2 + Mx + N & = 0 = f(x) \\ nAx^{n-1} + (n-1)Bx^{n-2} + (n-2)Cx^{n-3} + \ldots + 2Lx + M & = 0 = f_1(x) \\ n(n-1)Ax^{n-2} + (n-1)(n-2)Bx^{n-3} + \ldots\ldots\ldots + 2L & = 0 = f_2(x) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ n(n-1)\ldots\ldots(n-p+1)Ax^{n-p} + \ldots\ldots & = 0 = f_p(x) \end{array}$$

Now each depression by $x - r$ removes one of these roots, it will follow that

$f(x) = 0$ has p roots equal to r

$f_1(x) = 0$ has $(p - 1)$ roots equal to r

$f_2(x) = 0$ has $(p - 2)$ roots equal to r;

....

$f_{p-1}(x) = 0$ has $\{p - (p - 1)\}$ or one root equal to r.

$f_p(x) = 0$ has $(p - p)$ or 0 root equal to r.

Cor. Should there be other equal roots, as p_1 roots each equal to r_1, p_2 roots each equal to r_2, and so on: then the derivatives will contain $p_1 - 1, p_1 - 2, \ldots$ and $p_2 - 1, p_2 - 2, \ldots$ roots equal to r_1 and r_2 respectively.

PROBLEM VI.

To ascertain whether a given equation has equal roots, and if so to find them.

Form the equation $f_1(x) = 0$, and perform the operation of finding the greatest common measure of this and the given function. If this process leaves no final common measure in terms of x, there are no equal roots. If, on the contrary, there should be such a final divisor, it will always be of the form $(x - r)^p . (x - r_1)^{p_1} \ldots$; and there will be $p + 1$ roots equal to r, $p_1 + 1$ equal to r_1, and so on.

For, if the given equation $f(x) = 0$ contain p roots equal to r, p_1 roots equal to r_1, and so on: then $f_1(x) = 0$ will contain $p - 1$ roots equal to r, $p_1 - 1$ roots equal to r_1, and so on (Theor. 4). Whence $(x - r)^{p-1} . (x - r_1)^{p_1 - 1} \ldots$ will be a common measure of $f(x) = 0$ and $f_1(x) = 0$. Whence is deduced the above rule.

As an example, let $f(x) = x^6 + 3x^5 - 6x^4 - 6x^3 + 9x^2 + 3x - 4 = 0$ be the given equation. Find whether it has equal roots.

$$\text{Here } f(x) = x^6 + 3x^5 - 6x^4 - 6x^3 + 9x^2 + 3x - 4$$
$$f_1(x) = 6x^5 + 15x^4 - 24x^3 - 18x^2 + 18x + 3.$$

Of these two functions we find, as at p. 135, that the greatest common measure is $x^3 - x^2 - x + 1$; which resolved into factors is $(x - 1)^2 (x + 1)$, and hence $f(x)$ contains the factors $(x - 1)^3$ and $(x + 1)^2$, or three roots equal to $+ 1$, and two roots equal to $- 1$.

EXAMPLES FOR PRACTICE.

Find the equal roots of the following equations, if such exist:

1. $x^3 - 4x^2 + 5x - 2 = 0.$
2. $x^3 - 3a^2x - 2a^3 = 0.$
3. $x^7 + 5x^6 + 6x^5 - 6x^4 - 15x^3 - 3x^2 + 8x + 4 = 0.$
4. $x^7 - x^6 + 4x^5 + 4x^4 + 5x^3 - 5x^2 - 2x + 2 = 0.$

THEOREM V. If an equation whose coefficients are not imaginary have one root of the form $a + b\sqrt{-1}$, it will have another of the form $a - b\sqrt{-1}$.

For we have seen, (Theor. 3,) that if we put $a + z$ for x, we shall get

$$f(x) = f(a) + \frac{f_1(a)}{1} z + \frac{f_2(a)}{1 \cdot 2} z^2 + \ldots \frac{f_n(a)}{1 \cdot 2 \ldots n} . z^n = 0.$$

And this may be written, putting $b\sqrt{-1}$ for z, as follows:

$$(a) - \frac{f_2(a)}{1 \cdot 2} b^2 + \frac{f_4(a)}{1 \cdot 2 \cdot 3 \cdot 4} b^4 - \ldots \Big\} = - \Big\{ \frac{f_1(a)}{1} - \frac{f_3(a)}{1 \cdot 2 \cdot 3} b^2 + \frac{f_5(a)}{1 \cdot 2 \ldots 5} b^4 - . \Big\} b\sqrt{-1}$$

Now, as the first member can never be equal to the second, except the bracketed coefficients themselves fulfil the condition, and as by hypothesis $a + b\sqrt{-1}$ is a root of the equation, this equality must be fulfilled as a consequence of that hypothesis, we have simultaneously

$$f(a) - \frac{f_2(a)}{1 \cdot 2} b^2 + \frac{f_4(a)}{1 \cdot 2 \cdot 3 \cdot 4} b^4 - \ldots . = 0$$

$$\frac{f_1(a)}{1} - \frac{f_3(a)}{1 \cdot 2 \cdot 3} b^2 + \frac{f_5(a)}{1 \cdot 2 \ldots 5} \cdot b^4 - \ldots . = 0,$$

which is altogether independent of the *sign* of $b\sqrt{-1}$. Hence if one sign $+ b\sqrt{-1}$ fulfil the condition of the equation, the other sign $- b\sqrt{-1}$ will also fulfil it: that is, if $a + b\sqrt{-1}$ be a root of the equation, $a - b\sqrt{-1}$ is also a root.

Cor. 1. Roots of such forms, (generally termed *conjugate roots*,) if they enter into an equation at all, enter it in pairs, and their number is always even.

Cor. 2. If an equation be of an odd degree, there is at least one root free from the imaginary sign.

Scholium.

The same kind of reasoning will prove, that if one root be of the form $a + \sqrt{b}$, there will be another of the form $a - \sqrt{b}$. For the same reason will exist for the separate bracketted coefficients being zero in this case as in the other.

THEOREM VI. Change the signs of all the roots $r_1, r_2, r_3, \ldots . r_n$, of an equation of the form

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots . + Mx + N = 0,$$

and combine the roots so changed by way of multiplication, in twos, threes, fours, and so on: then the sum of these changed roots will be equal to A; the sum of their products in twos will be equal to B; the sum of their products in threes will be equal to C; and so on, till the coefficient of the nth term is the sum of all the products of the roots $(n-1)$ at a time, and their continued product will be equal to N *.

For, take two roots, r_1 and r_2: then $x - r_1 = 0$, and $x - r_2 = 0$, which multiplied together, give

$$x^2 - (r_1 + r_2)\,x + r_1 r_2 = 0,$$

in which the theorem holds good.

Next take three, and we get

$$x^3 - (r_1 + r_2 + r_3)\,x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)\,x - r_1 r_2 r_3 = 0,$$

in which, again, the statement is true.

Proceeding thus to any extent, and observing the formation of the coefficients, we see that the theorem is generally true.

Cor. 1. If any pair of roots had been conjugates, whether irrational or imaginary, we see that these several coefficients would have become rational or real, provided A, B, C, were so: for since $\{(a + b\sqrt{-1}) + (a - b\sqrt{-1})\}$ and $\{a + b\sqrt{-1}\}\,\{a - b\sqrt{-1}\}$ are both real, the imaginary parts which would have come into the values of A, B, C . . . disappear from the result.

---

* Some writers have considered it necessary to complete this proof, to show that, generally, if it be true for the pth coefficient, it will also be true for the $(p+1)$th. This is easily done by assuming the first, and proceeding by actual multiplication (literal) to the next; and the only reason for omitting it here is the space which the printing would occupy.

Cor. 2. It may hence be inferred, that all the roots of an algebraic equation may be represented either by real quantities, $r_1\ r_2 \ldots .$, or imaginary ones, $a \pm b\sqrt{-1}$, $a_, \pm b_, \sqrt{-1}, \ldots$

THEOREM VII. Every algebraic equation contains as many roots, either real or imaginary, as it has dimensions, and no more.

For by Theorem VI. Cor. 2, every root may be represented by r_1, r_2, ..., and $a \pm b\sqrt{-1}$, $a_{,} \pm b_{,} \sqrt{-1}$; hence so many factors of the first degree may be formed of it as there are units in the degree: and such equation admits of no other binomial factors but these, as then one of the binomial factors would be divisible by some other binomial factor, which is obviously absurd.

THEOREM VIII. No equation can have a greater number of positive roots than there are changes of sign from + to —, and from — to +, in the terms of its first member; nor can it have a greater number of negative roots than of permanences, or successive repetitions of the same sign.

To demonstrate this proposition, it will be necessary merely to show, that, if any polynomial, whatever be the signs of its terms, be multiplied by a factor $x - a$, corresponding to a *positive* root, the resulting polynomial will present at least one more *change of sign* than the original; and that if it be multiplied by $x + a$ corresponding to a negative root, the result will exhibit at least one more *permanence of sign* than the original.

Suppose the signs of the proposed polynomial to succeed each other in any given order, as, for instance,

$$+ - - + - + + + - - + ;$$

then the multiplication of the polynomial, by $x - a$, will give rise to two rows of terms, which, added vertically, furnish the product. The first row will, obviously, present the same lines of signs as the original; and the second, arising from the multiplication by the negative term $-a$, will present the same lines of signs as we should get by changing every one of the signs of the first row. In fact, the two rows of signs would be

$$\begin{array}{lcccccccccccc} & + & - & - & + & - & + & + & + & - & - & + & \\ & & - & + & + & - & + & - & - & - & + & + & - \\ \hline \text{and signs of prod.} & + & - & \pm & + & - & + & \pm & \pm & - & \pm & + & - \end{array}$$

We have written the ambiguous sign $\pm$ in the product when the addition of unlike signs in the partial products occurs, and it is very plain that these ambiguities will, in this and in every other arrangement, be just as numerous as permanences in the proposed; thus, in the present arrangement, the proposed furnishes four permanences, viz. — —, + +, + +, — —; and there are, accordingly, in the product four ambiguities, the other signs remaining the same as in the proposed, with the exception of the final sign, which is superadded, and which is always contrary to the final sign in the proposed. It is an easy matter, therefore, when the signs of the terms of any polynomial are given to write down immediately the signs in the product of that polynomial, by $x - a$, as far, at least, as these signs are determinable, without knowing the values of the quantities employed; for we shall merely have to change every permanency in the proposed into a sign of ambiguity, and to superadd the final sign changed. For instance, if the proposed arrangement were

$$+ - + + - - - + - + + + ,$$

the signs of the product would be

$$+ - + \pm - + \pm + - + \pm \pm - .$$

Again, if the signs of the proposed equation were in order

$$+ + + - + - + - - -,$$

the signs of the product would be in the order

$$+ \pm \pm - + - + - \pm \pm +.$$

As, therefore, in passing from the multiplicand to the product, it is the *permanences* only of the former can suffer any change, it is impossible that the *variations* can ever be diminished, however they may be increased; consequently, the most unfavourable supposition for our purpose is, that the permanences (omitting the superadded sign) remain the same in number; and, in this case, if the proposed terminate with a variation, the superadded sign in the product will introduce another variation; but if it terminate with a permanency, then the corresponding ambiguity in the result will, obviously, substitute for it what sign we will, form a variation, either with the preceding, or with the superadded sign. It follows, therefore, that no equation can have a greater number of positive roots than variations of signs.

To demonstrate the second part of the proposition, it will suffice to remark, that, if we change all the signs in an equation, we change the roots from positive to negative, and *vice versâ* (Theor. IV.) The equation thus changed would have its permanences replaced by variations, and its variations by permanences; and since by the foregoing the changed equation cannot have a greater number of positive roots than variations, the proposed cannot have a greater number of negative roots than permanences.

This proposition constitutes the rule of Harriot *, and serves to point out limits which the number of the positive and negative roots of an equation can never exceed. It does not, however, furnish us with the means of ascertaining how many real roots, of either kind, any proposed equation may involve; nor, indeed, does it enable us to affirm that even one positive or negative root actually exists in any equation; it merely shows, that *if* real roots exist, those which are positive, or those which are negative, cannot exceed a certain number; they may, however, fall greatly short of its number, and, indeed, all be imaginary. But the rule is not without its use, even in detecting imaginary roots, as it sometimes discovers discrepancies incompatible with the existence of real roots, in those equations which are incomplete, or have terms wanting. This will be

---

* By the foreign writers this rule is always attributed to *Descartes*, and most English writers follow their example. There is, however, undeniable evidence that the rule was obtained *indirectly* by Descartes from Harriot; and it may be mentioned in support of this view, that Harriot gives a reason for the rule, while Descartes gives none.

On the other hand, it has been alleged that the failure of its generality in consequence of the existence of imaginary roots was not perceived by Harriot, and that there is no evidence that he was even acquainted with the existence of imaginary roots. It must, however, be replied, that the *Ars praxis Analyticæ* was a posthumous work, edited by Warner, who does not appear to have fully understood Harriot's views, and who, therefore, thought he exercised a sound and kind discretion towards his friend in suppressing certain parts of the work: a suppression which we know did take place. We cannot, therefore, say more as to the views which Harriot entertained on this subject, till some of his papers, still in existence, are more completely examined than they have been. With respect, however, to his *knowledge* of imaginary roots, we have sufficient proof that he understood their forms and their meaning too. In the Supplement to the Works of Bradley, published by my estimable friend, the late Professor Rigaud, plate 5, will be seen a solution of the equation $1 - aa = -2a + 34$, and the solutions are separately put down; viz. $a = 1 + \sqrt{-32}$, and $a = 1 - \sqrt{-32}$. Even this, were this all, would remove the imputation of his *ignorance of the existence of imaginary roots.*—EDITOR.

made apparent in the proof, of De Gua's Criterion of imaginary roots, a little further on (see p. 224).

THEOREM IX. If r_1, r_2, r_3, ... r_{n-2p} be the real roots of the equation $f(x) = 0$ of the nth degree, in the descending order of their magnitudes, and quantities ρ, ρ_1, ρ_2, ... ρ_{n-2p} taken so that $\rho, r_1, \rho_1, r_2, \rho_2, \ldots r_{n-2p}, \rho_{n-2p}$ be also in descending order of magnitude: then we shall have

$$f(\rho) = +k, f(\rho_1) = -k_1, f(\rho_2) = +k_2,$$

and so on, the results being alternately + and −.

Let the p pairs of imaginary roots of $f(x)=0$ be $a_1 \pm b_1\sqrt{-1}$, $a_2 \pm b_2\sqrt{-1}$, ... $a_p \pm b_p \sqrt{-1}$: then the portion of $f(x)$ depending upon these is

$$F(x) = \{(a_1 - x)^2 + b_1^2\}\ \{(a_2 - x)^2 + b_2^2\} \ldots \{(a_p - x)^2 + b_p^2\};$$

in which, since every factor is essentially positive, their product will be +, whatever be the value given to x.

$(x - r_1)(x - r_2) \ldots (x - r_{2n-p}) \cdot F(x) = 0$ is the general form of $f(x) = 0$; and of this $F(x)$ being always +, it will not affect the signs depending on the values of x in the other factors.

Substituting then ρ, ρ_1, ... ρ_{n-2p} in the other part of the function, we have successively

$(\rho - r_1)(\rho - r_2)(\rho - r_3) \ldots (\rho - r_{n-2p}) = +k$, since all the factors are +.
$(\rho_1 - r_1)(\rho_1 - r_2)(\rho_1 - r_3) \ldots (\rho_1 - r_{n-2p}) = -k_1$. since only one factor is −.
$(\rho_2 - r_1)(\rho_2 - r_2)(\rho_2 - r_3) \ldots (\rho_2 - r_{n-2p}) = +k_2$, since only two factors are—.
and so on through all the substitutions.

Cor. If in the results of any two substitutions ρ' and ρ'' in $f(x)$ we find different signs, there are 1, 3, 5, or some odd number of roots in the interval ρ' and ρ''; and if the signs of those results be alike, then 0, 2, 4, or some even number of roots are situated in that interval.

THEOREM X. In any function $f(x)$ proceeding by decreasing powers of x, a value may be found for x which shall render the sign of the result the same with that of the first term.

Let $f(x) = x^n + Ax^{n-1} + \ldots + Lx + M$; and suppose the most unfavourable case, where all the coefficients after the first, are different from the first, and K that which is numerically greatest. Then we shall have

$K\{x^{n-1} + x^{n-2} + \ldots x + 1\}$ greater than $\{Ax^{n-1} + Bx^{n-2} + \ldots Lx + M\}$.

But $x^{n-1} + x^{n-2} + \ldots x + 1 = \dfrac{x^n - 1}{x - 1}$; and hence $K \cdot \dfrac{x^n - 1}{x - 1}$ is greater than all the terms of the function after the first. It will, therefore, be sufficient to show that such a value can be found for x as shall render x^n greater than $K \cdot \dfrac{x^n - 1}{x - 1}$, or $(x - 1)x^n$ greater than $K(x^n - 1)$. Now the value $K + 1$ given to x will render $(x - 1)x^n = K(K + 1)^n$, and $K(x^n - 1) = K\{(K + 1)^n - 1\}$: but $K(K + 1)^n$ is greater than $K\{(K + 1)^n - 1\}$, and hence such a value of x as was asserted possible has been found.

THEOREM XI. In any function, as $N + Mx + Lx^2 + \ldots + Bx^{n-1} + Ax^n$, values of x may be found which will render the result of the entire function of the same sign as the first term N.

Take, as before, the most unfavourable case, where all the terms after N have their signs different from that of N. Then if K be greater than either of these coefficients, we shall have

$Kx\{1 + x + x^2 + \ldots x^{n-1}\}$ greater than $x\{M + Lx + \ldots Bx^{n-2} + Ax^{n-1}\}$.

Now the greater of these is $\dfrac{Kx(1 - x^n)}{1 - x}$; and if x be less than unity, $\dfrac{Kx}{1 - x}$

is greater than $\frac{Kx(1-x^n)}{1-x}$; and hence still more is $\frac{Kx}{1-x}$ greater than $x\{M + Lx + \ldots\ldots Bx^{n-2} + Ax^{n-1}\}$.

If, therefore, we can find a value of x such that N is equal to, or greater than, $\frac{Kx}{1-x}$, we shall have effected the proof of the proposition. Now if $x = \frac{N}{N+K}$, we shall have $\frac{Kx}{1-x} = N$; and as this is *always* greater than all the terms following N in the given equation, the condition is entirely fulfilled.

THEOREM XII. The consecutive roots of $f_1(x) = 0$ lie each in succession between the consecutive roots of $f(x) = 0$.

Let $r_1, r_2, \ldots. r_n$ be the roots of $f(x) = Ax^n + Bx^{n-1} + \ldots = 0$,

and $\rho_1, \rho_2, \ldots. \rho_{n-1}$, be those of $f_1(x) = nAx^{n-1} + (n-1)Bx^{n-2} + \ldots = 0$.

Reduce the roots of $f(x) = 0$ by the indeterminate quantity r, (prob. II.) which will give the roots of the transformed equation respectively equal to $r_1 - r$, $r_2 - r, \ldots. r_n - r$. But this transformation gives $M_1 = f_1(r)$, and $N_1 = f(r)$.

Now the value of M_1 in this reduced equation is (theor. VI.)

$$\begin{aligned} M_1 = {} & (r-r_1)(r-r_2)(r-r_3)\ldots.(r-r_{n-1}) \\ & +(r-r_1)(r-r_3)(r-r_4)\ldots.(r-r_n) \\ & +(r-r_1)(r-r_2)(r-r_4)\ldots.(r-r_n) \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & +(r-r_2)(r-r_3)(r-r_4)\ldots.(r-r_n) \end{aligned}$$

Now in this expression there is but one group of factors from which any one of the given factors is absent, as, for instance, $r-r_1$. If then in M_1 or $f(r)$ we give the indeterminate quantity r the successive values $r_1, r_2, \ldots. r_n$, the several results will comprehend only one set of factors, viz. that in which $r_1, r_2, \ldots. r^n$ are thus rendered successively absent; and we will suppose them so ranged that $r_1, r_2, \ldots. r_n$ are in the order of descending values. This will give

$(r_1-r_2)(r_1-r_3)\ldots.(r_1-r_n) = +\kappa$, since all the factors are $+$.

$(r_2-r_1)(r_2-r_3)\ldots.(r_2-r_n) = -\kappa$, since one factor only is $-$.

$(r_3-r_1)(r_3-r_2)\ldots.(r_3-r_n) = +\kappa$, since two factors only are $-$.

And so on through the entire series of results.

But when a series of quantities $r_1, r_2, \ldots. r_n$ are substituted in an equation $f_1(x) = 0$, which give results alternately $+$ and $-$, there is in all cases one root of the equation $f_1(x) = 0$ comprehended between those numbers (theor. IX.) But $\rho_1, \rho_2, \ldots. \rho_{n-1}$ are the roots of $f_1(x) = 0$; and hence these values lie between $r_1, r_2, \ldots. r_n$ the roots of $f(x) = 0$. That is, the roots of both equations being ranged in the order of descending magnitude follow each other thus:

$$r_1, \rho_1, r_2, \rho_2, r_3, \ldots\ldots r_{n-1}, \rho_{n-1}, r_n.$$

Cor. In the same way it may be shown, that the roots of $f_2(a) = 0$ lie between those of $f_1(x) = 0$; those of $f_3(x) = 0$ between those of $f_2(x) = 0$, and so on.

Scholium.

These properties admit of various applications in the higher theory of algebraic equations, and are popularly known as *the limiting equations of Newton.* In actual numerical solution, however, they are now of little use; and they are only given here in justification of one or two processes employed. The equations are evidently the same which have been before treated under the name of the derivative equations.

There is one remarkable property of the limiting equation: viz. that when $f_1(\rho) = 0$, then $f(\rho)$ has a greater or less value than it has when $f_1(\rho)$ is either a positive or negative quantity a little different from 0; or, in more technical language, (though belonging to a more advanced subject of study,) $f(\rho)$ is a maximum or a minimum.

THEOREM XIII. If one of the roots r_1 of an equation $Ax^n + Bx^{n-1} + \ldots\ldots Lx^2 + Mx + N = 0$, be very small in comparison with all the others, we shall have, *nearly*, $r_1 = -\dfrac{N}{M}$.

For, let $r_2, r_3, \ldots\ldots r_n$ be the other roots; then (theor. VI.) we have

$M = \pm \{r_1(r_2 r_3 \ldots\ldots r_{n-1} + r_3 r_4 \ldots\ldots r_n + r_4 r_5 \ldots\ldots r_n r_2 + \ldots\ldots) + r_2 r_3 r_4 \ldots\ldots r_n\}$

$N = \mp r_1 r_2 r_3 \ldots\ldots r_n.$

Now since r_1 is very small in comparison with the other roots, the vinculated term which contains it as a factor in M is small in comparison with the term $r_2 r_3 \ldots\ldots r_n$, which does not contain it. Neglecting, therefore, this term, we have

$$\frac{N}{M} = \frac{\mp r_1 r_2 r_3 \ldots\ldots r_n}{\pm r_2 r_3 \ldots\ldots r_n} = -r_1, \text{ or } r_1 = -\frac{N}{M}.$$

Scholium.

This theorem enables us, after we have obtained a first distinct approximation, to obtain a closer one by mere division; and thence to still further reduce the roots of the equation, and especially that to which in any case we may be approximating.

PROBLEM VII.

To find the limits between which are situated the roots of any given equation, $Ax^n + Bx^{n-1} + \ldots + Lx^2 + Mx + N = 0$.

1. Find a reducing number k which will render the signs of all the coefficients positive: for then the roots of the transformed equation will be negative, (Theor. VIII.) and hence k is greater than the greatest root of the equation, (Cor. 1. Prob. III.) or it is the superior limit of the positive roots.

2. Find similarly a number which will render all the coefficients of the transformed reciprocal equation positive; the reciprocal of this number will be less than the least positive root, or will be an inferior limit of the positive roots.

3. Change the alternate signs, and find the superior and inferior limits of the positive roots of this transformed equation: these limits, taken with negative signs, will be the limits between which all the negative roots lie.

4. To find how many roots lie within any *given* limits, a and b, of which a is the greater, reduce the roots of the given equation by the less of those numbers b; then, again, reduce the roots of this transformed equation by $a - b$. The difference of the number of variations of sign in these two transformed equations indicates the number of positive roots in the interval.

In a similar manner, after changing the signs of the alternate terms, and of the two negative limits, we may find the numbers of variations in each reduced equation; the difference of which will be the number of negative roots in that interval.

The first part of this rule becomes evident from Theor. VIII. and Cor. 1. Prob. III.: and the latter, from combining it with Prob. III. itself.

Scholium 1.

As a practical course of procedure, it will be advisable to reduce by 1, 10, 100,

. . . rather than by intermediate numbers to these, till the utmost limit is obtained; and then to work with such intermediate numbers, as may be thought, from the state of the coefficients, most likely to make a small number of changes in the state of the coefficients as to order and number of signs.

Proceeding with these till we have obtained two limiting consecutive numbers for one or more roots, the object of this problem will be attained.

Scholium 2.

When by narrowing the intervals of the substituted numbers, we find more than one variation continually disappearing in each of the substitutions made; these roots may be equal, or they may have minute differences, or any even number of them may be imaginary.

If there be equal roots, the process of Problem VI. will find them.

The only question, then, is to find whether an equation known to have only unequal roots, has any number of them imaginary, and how many; the remaining ones, of course, being real, and having differences less than that of the limiting substituted numbers between which they are indicated. Several methods of solving this problem have been proposed; but we shall here give only three of these *criterions,* those of De Gua, Budan, and Sturm; though those of Lagrange and Fourier well deserve to be studied by every one whose time and inclination lead him to pursue the subject further. See Lagrange, *Resolution des Equations Numériques,* p. 6, and Fourier, *Analyse des Equations Determinées,* p. 87.

THEOREM XIV. *De Gua's Criterion of Imaginary Roots* *.

This criterion is generally stated incorrectly. It should also be expressed more in detail than is usually done. Before stating it, however, it will be desirable to enter upon the examination of the principles from which it flows.

1. It is very clear from the reasoning in theor. VIII. that the rule of Harriot is true for all cases in which the roots of an equation are real, or constituted merely by the signs $+$ or $-$ prefixed to a real number, either integer, fractional, or irrational.

2. It follows, then, that all cases in which this assumption being made leads to contradictory results, indicate, according to the number of those contradictions, so many of the roots not being constituted as above expressed; that is, so many of the roots are imaginary.

3. When we have a cipher-coefficient, such as $0x^m$, it is either $+0x^m$ or $-0x^m$; the values of the expression in which it occurs being precisely the same in both cases.

4. The greatest number of negative roots in an equation which contains cipher-coefficients between any two actual coefficients, will be when all the ciphers are written with the same signs as one of the terms which form the extremes between which the ciphers are situated.

5. The least number of negative roots under the same circumstances will be

---

* This property of cipher coefficients was first given by the Abbé de Gua, in the Memoirs of the French Academy, for 1743. All writers, after the original author, have, however, committed an oversight in estimating the number of conditions implied in this theorem, which they uniformly assert to be $\frac{n(n+1)}{2}$ for n cipher coefficients. The conditions may, indeed, appear under different simultaneous forms: but their number cannot be more than as stated in the text above.

when the ciphers are taken with alternate signs, the first cipher being taken with the contrary sign to its adjacent actual coefficient.

6. The difference between the greatest and least number of negative roots indicated by taking the ciphers as in (4), (5), is the number of imaginary roots indicated by the sequences of the cipher-coefficients in the equation.

7. If cipher-coefficients occur in any of the transformed equations, the same rules apply, since no kind of transformation by real numbers can eliminate the imaginary part of the root.

It is, however, to be carefully kept in view, that no proof is offered of the imaginary roots, indicated by the transformed equation, being different from, or the same with, those indicated by the given equation.

The statement, then, of De Gua's rule will be as follows :—

I. If between terms having *like* signs, $2n$ or $2n - 1$ cipher-coefficients intervene, there will be $2n$ imaginary roots indicated thereby.

II. If between terms having *different* signs $2n + 1$ or $2n$ cipher-coefficients intervene, there will be $2n$ imaginary roots indicated thereby.

(1) Let there be $2n$ ciphers between like signs; then writing them as expressed in (7) we have

$+ h + 0 + 0 + 0 + \ldots\ldots + 0 + k$, giving $2n + 1$ negative roots.
$+ h - 0 + 0 - 0 + \ldots\ldots + 0 + k$, giving 1 negative root.

Whence since only one root is negative in the latter case, and $2n + 1$ roots in the former, there is a contradiction in the sign of $2n$ roots; which are, therefore, imaginary.

Had the signs of the extreme terms, h, k, been both $-$ instead of $+$, the series would have taken the form

$- h - 0 - 0 - 0 - \ldots\ldots - 0 - k$, giving $2n + 1$ negative roots.
$- h + 0 - 0 + 0 - \ldots\ldots - 0 - k$, giving 1 negative root.

And the same contradiction with respect to $2n$ of these roots would have resulted.

(2) Next, let there be $2n - 1$ ciphers between like signs: then

$+ h + 0 + 0 + 0 + \ldots\ldots + 0 + 0 + k$, giving $2n$ negative roots.
$+ h - 0 + 0 - 0 + \ldots\ldots + 0 - 0 + k$, giving 0 negative root.

Hence there are, as before, $2n$ imaginary roots.

(3) Let there be $2n$ ciphers between unlike signs: then

$+ h + 0 + 0 + 0 + \ldots\ldots + 0 + 0 - k$, giving $2n$ negative roots.
$+ h - 0 + 0 - 0 + \ldots\ldots - 0 + 0 - k$, giving 0 negative root.

Hence, as before, there are $2n$ imaginary roots.

(4) Let there be $2n + 1$ ciphers between unlike signs: then

$+ h + 0 + 0 + 0 + \ldots\ldots + 0 + 0 + 0 - k$, giving $2n + 1$ negative roots.
$+ h - 0 + 0 - 0 + \ldots\ldots - 0 + 0 - 0 - k$, giving 1 negative root.

Hence, again, there are $2n$ imaginary.

The theorem of De Gua is, therefore, established universally, and the statement given in its most general form *.

---

* It has been well remarked by Mr. Horner, (Math. Repos. vol. v. p. 27,) that though the direct application of this "criterion can only occur incidentally, yet its application is capable of an extension beyond what is at once apparent. To cite an example; when the mth coefficient changes its sign in passing from one set to another, while those which immediately precede and follow it are and continue to be identical, the existence of zero between like signs somewhere in

For example, $x^3 + x + 1 = 0$ having the coefficients $1 \pm 0 + 1 + 1$, has two imaginary roots; and the equation $x^3 \pm 5 = 0$ having the coefficients $1 \pm 0 \pm 0 \pm 5$, has also two imaginary roots.

EXAMPLES FOR PRACTICE.

1. How many imaginary roots are there in the equations $x^4 - 6x^2 + x = 0$, $x^8 + 1 = 0$, and $x^4 - 6x^2 + 2x + 12 = 0$.

2. The equation $x^5 - 5x^4 + 20x^3 + x = 100$ has one pair of imaginary roots indicated by its present state, and reducing the roots by a certain number to be found, will show another pair. What is that reducing number, and which are the places at which the imaginary roots are indicated?

3. Has the equation $x^4 - 4x^3 + 8x^2 - 16x + 20 = 0$ any imaginary roots?

THEOREM XV. *Budan's Criterion *, as arranged by Horner.*

The theorem may be stated thus:—

If in transforming an equation by any number r, there be n variations *lost*, and if in transforming the reciprocal equation by $\frac{1}{r}$ there be m variations *left;* then there will be at least $n - m$ imaginary roots in the interval $0, r$.

(1) There can be no root of an equation infinitely great except the absolute term be also infinite. The roots of equations, such as generally occur, are, therefore, finite.

(2) The reduction of the roots of an equation by $\frac{1}{0}$ or infinity, must therefore render all the roots negative; and hence give only permanencies of sign in the transformed equation.

(3) Imaginary roots enter an equation in pairs, as has been shown in Theor. V.

---

the interval may be suspected. Should all the previous signs in these sets be alike, the probability is increased." He might, indeed, have spoken even more decidedly; as by other means we can show that it amounts to all but absolute certainty.

In successive transformations this remark will often be of considerable utility to be borne in mind.

* This criterion was first given by Budan in his *Nouvelle Méthode pour la Résolution des Equations Numériques*, p. 36; but the form in which it now appears is due to Mr. Horner, and in any other it is next to useless. It would almost appear from a note to Lagrange's *Traité de Résolution des Eq. Num.* (p. 169,) that he did not seize its import: at all events, he formed an inadequate notion of it, and raised an objection to it which is altogether invalid.

The following observations upon using it will be useful to the student:—

1. It will always in practice be most convenient to take the transforming interval r equal to 1; as then the reciprocal $\frac{1}{r}$ is also equal to 1. When the interval is a prime number different from 2 or 5, it often becomes troublesome to reduce the fractional remainders; and it is not often safe to neglect them, as the whole force of the criterion may be destroyed by a very small quantity. Besides, though imaginary roots can never be indicated by this criterion except they exist, they may exist without being indicated by a specified interval. The smaller the interval, therefore, the greater the probability of their detection. Hence, except in rare cases, it is better to take the interval 1.

2. When no indication is supplied by the interval 1, it will be most convenient to use

(4) There are as many positive roots in the interval 0, r, of the direct equation, as there are between $\frac{1}{r}$ and $\frac{1}{0}$ of the reciprocal equation. For the roots of the reciprocal equation are the reciprocals of the roots of the direct equation; and hence must lie between the reciprocals of those limits of the direct equation.

(5) If then the number of variations n, lost in the direct equation by passing over the interval r, be greater than the number left m in the reciprocal equation, after transformation by $\frac{1}{r}$, there will be a contradiction with respect to the character of a number of them, equal to the difference $n - m$. These roots, therefore, are imaginary.

To take one example, find whether the equation $x^5 - 8x^4 + 11x^3 - 39x^2 + 31x = 101$ has imaginary roots.

Reduce by 1; then the work will stand thus:

$$\begin{array}{l} 1 - 8 + 11 - 39 + 31 - 101 \;(1 \\ \quad - 7 + \;\; 4 - 35 - \;\; 4 - 105 \\ \quad - 6 - \;\; 2 \;\ldots\ldots \end{array}$$ where four variations are already lost.

·5, which narrows the limit still more, as we have the intervals 0, ·5 and ·5, 1; and the reciprocal of ·5 being 2, we have an easy working number for the reciprocal equation, if both roots be still left doubtful. If there be one variation, and one only, lost, the roots are separated, one being between 0 and ·5, and the other between ·5 and 1. Should it still be uncertain, it will be convenient to reduce either the given equation or that in ·5 by ·2, since the reciprocal 5 is an easy number for the reciprocal transformation.

3. It will sometimes conduce to clearness of conception to multiply the roots by 10, 100, 1000, &c., in accordance with what is said at p. 213 of this work. This, however, is only the case in early practice of the method, since the process itself is not virtually altered by it. This step simply adds one cipher to the second coefficient, two to the third, three to the fourth, &c.

4. It will also be desirable in these successive transformations to keep De Gua's Criterion in view, both in its strict form, and with the modification suggested at the foot of page 225.

5. It will generally happen when we have taken too wide an interval that the ambiguity may be very simply removed as follows:—

Reduce the reduced reciprocal equation by unity at a time till the variations are lost in pairs, or the roots separated belonging to the doubtful interval. Apply the criterion to this reciprocal equation; and if the roots be indicated as imaginary, they are so; and then the roots of the original equation, which are functions of these, are also imaginary.

To take an example, let $x^4 + x^3 + 4x^2 - 4x + 1 = 0$ be given.

Direct transformation.	*Reciprocal transformation.*
$1 + 1 + 4 - 4 + 1 \;(1$	$1 - 4 + 4 + 1 + 1 \;(1$
$+ 2 + 6 + 2 + 3$	$- 3 + 1 + 2 + 3$
$3 + 9 + 11$	$- 2 - 1 + 1$
$4 + 13$	$- 1 - 2$
5	± 0

In the direct transformation two variations are lost, and in the second two variations are left; hence no conclusion whether they are or are not imaginary can be drawn from this result. Take the reduced reciprocal equation, therefore, and we shall have

$1 + 0 - 2 + 1 + 3 \;(1$	$3 + 1 - 2 + 0 + 1 \;(1$
$1 - 1 + 0 + 3$	$4 + 2 + 2 + 3$
$2 + 1 + 1$	$7 + 9 + 11$
$3 + 4$	$10 + 19$
4	13

In these two transformations we have now two imaginary roots by Budan's criterion; and hence the two corresponding positive roots of the given equation are imaginary.

Reduce the reciprocal by 1 : then

$$\begin{array}{rrrrrrl} -101 & +31 & -39 & +11 & -8 & +1 & (1 \\ & -70 & -109 & -98 & -106 & -105, & \end{array}$$

where there are no variations left. Hence $n - m = 4 - 0 = 4$ imaginary roots.

EXAMPLES FOR PRACTICE.

1. $x^3 - 2x = 5$.
2. $x^3 - 7x + 7 = 0$.

See other examples at p. 232.

THEOREM XVI. *Sturm's Criterion.*

Let $X = 0$ be an equation of the nth degree, and let $X_1 = 0$ be its first derivative; and let the given equation be free from equal roots. Perform the operation employed in finding the greatest common measure, always, however, changing the signs of the several divisor-remainders : and denote the series of functions which result, together with the given ones, by $X, X_1, X_2, \ldots X_m \ldots\ldots X_n$. Then if a and b be any two numbers substituted in these several functions, the difference of the numbers of variations of sign in the results of these substitutions expresses the number of real roots lying between a and b.

1. Denote the successive quotients by $Q_1, Q_2, \ldots\ldots Q_{n-1}$: then

$$\begin{aligned} X &= Q_1 X_1 - X_2 \\ X_1 &= Q_2 X_2 - X_3 \\ X_2 &= Q_3 X_3 - X_4 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ X_{n-2} &= Q_{n-1} X_{n-1} - X_n. \end{aligned}$$

Now as the degree of the function is diminished a unit at each stage, the final remainder is clear of x : and as the equation $X = 0$ contains no equal roots, X_n cannot be zero, or in other words, X_n is a number.

2. No two consecutive functions can become zero for the same value of x. For if it be possible, let them be X_{m-1} and X_m. Then since

$$X_{m-1} = Q_m X_m - X_{m+1},$$

and $X_{m-1} = 0$, and $X_m = 0$ at the same time, we also have $X_{m+1} = 0$. But $X_m = Q_{m+1} X_{m+1} - X_{m+2}$, and hence, for the same reason, $X_{m+2} = 0$; and similarly all which follow X_m become zero for the same value of x, amongst which is X_n. Now it has already been shown in (1) that X_n cannot become zero : and hence also X_m cannot become zero at the same time with X_{m-1}. No two consecutive functions can, therefore, vanish with the same value of x.

3. If any value of x cause one of the functions, as X_m, to vanish, it gives to the two adjacent ones equal values with contrary signs. This is evident from the connecting equation $X_{m-1} = Q_m X_m - X_{m+1}$, which in this case becomes $X_{m-1} = -X_{m+1}$.

4. If such a value be given to x as shall make one of the intermediary functions $X_1, X_2 \ldots\ldots X_{n-1}$, vanish, without making $X = 0$, there will be a change in the *order* of the signs produced at this stage of the variable values of x, but no change in the *number* of variations.

For let X_m be that in which any special value of x makes the result zero. Then, considering the three consecutive functions, X_{m-1}, X_m and X_{m+1}, we have seen that in such case X_{m-1} and X_{m+1} have contrary signs (3); and the series of signs will therefore be

$$+ X_{m-1}, \pm 0, - X_{m+1} \text{ or } - X_{m-1}, \pm 0, + X_{m+1}.$$

That is, writing only the signs, we have the following combinations:

In the first case $+ + -$, or $+ - -$; .... (1)

In the second $- + +$, or $- - +$ ...... (2)

Now denoting by a_m a root of $X_m = 0$, it is established, (Theor. IX) that if numbers greater and less than a_m be substituted in X_m, the results will have contrary signs; and hence, in passing through $X_m = 0$, the signs will undergo the changes indicated in (1) or (2): but so far as the *number* of variations is concerned, these are all precisely alike, and differ only in the *order of their succession*. No variation, therefore, is lost or gained in the passage of X_m from a value greater than a_m to one less, or the converse.

Moreover, that which holds true for X_m holds true for any other function X_{m+t}; and hence it holds true universally. No variation, therefore, can be lost or gained amongst the intermediate functions, though their order of succession should be changed in any way whatever.

5. Every time a value of x coincides with a real root r, of the equation $X = 0$, one variation, and one only, is gained in a descending series of values, or lost in an ascending series.

Since (4) no variation can be lost or gained amongst the intermediate series of functions $X_1, X_2, \ldots. X_n$, it follows that the only case in which a change in the number of variations arises, is by its occurring between X and X_1.

Now, by hypothesis $f(r) = 0$; and hence, taking the quantities $r - h$ and $r + h$ to represent the substituted numbers less and greater respectively than r, in the equations $X = 0$, and $X_1 = 0$, we get

$$f(r-h) = \qquad -f_1(r)\frac{h}{1} + f_2(r)\frac{h^2}{1.2} - f_3(r)\frac{h^3}{1.2.3} + \ldots.$$

$$f_1(r-h) = f_1(r) - f_2(r)\frac{h}{1} + f_3(r)\frac{h^2}{1.2} - f_4(r)\frac{h^3}{1.2.3} + \ldots.$$

$$f(r+h) = \qquad +f_1(r)\frac{h}{1} + f_2(r)\frac{h^2}{1.2} + f_3(r)\frac{h^3}{1.2.3} + \ldots.$$

$$f_1(r+h) = f_1(r) + f_2(r)\frac{h}{1} + f_3(r)\frac{h^2}{1.2} + f_4(r)\frac{h^3}{1.2.3} + \ldots.$$

But in all these functions we may find a value of h so small, that the value of the whole series shall be the same with that of its first term. If, then, we take an ascending series of values for the substitutions, the signs before and after the passage of x through r will be $- +$ and $+ +$; that is, a variation will be converted into a permanence, or one variation is lost. It has also been shown, that no variation can be lost amongst the intermediate functions; and hence only one variation can be lost in passing through a root of $X = 0$, from less to greater values of x.

Had we taken the descending series of values for x, the reverse would have taken place; viz. $+ +$ converted into $- +$, or one variation only be gained in the passage through r *.

---

* During the passage in the values of x in the several functions from one root of $X = 0$ to another, we have seen that any difference of order of succession may occur, but no difference in the total number of variations: and it may conduce to the clearness of our conception *how* a variation may be introduced or lost in passing through the next lower or higher root of $X = 0$.

We have seen (theor. IX.) that if $r_1, r_2, \ldots\ldots r_n$, ranged in the order of magnitude, be the roots of $X = 0$, and $\rho_1, \rho_2, \ldots\ldots \rho_{n-1}$, be those of $X_1 = 0$, ranged similarly, then these several values ranged in the order of magnitude will be

$$r_1, \rho_1, r_2, \rho_2, r_3, \rho_3, \ldots\ldots\ldots\ldots r_{n-1}, \rho_{n-1}, r_n.$$

It hence follows, that in passing from r_1 to r_2, we pass through ρ_1 a root of $X_1 = 0$; and

6. Let now a and b be any two numbers substituted in the series of functions, there will be one variation, and one only, gained or lost, according as we use ascending or descending values of x in passing from one value to the other, every time we pass through a real root of $X = 0$. There will hence be as many variations gained or lost as there are real roots between a and b, and no more than these can be gained or lost. There will be as many more variations in the series of signs of the functions arising from one substitution, than there are in the series of signs arising from the other, as there are real roots, neither more nor less. The difference in the number of variations in the two series of functions under these two substitutions, expresses, therefore, exactly the number of real roots in that interval. Sturm's criterion is, hence, fully established.

Cor. 1. Applying this to find at once the *entire number of imaginary roots* of an equation, we have only to take a positive limit greater than the greatest positive root of an equation, and a negative limit numerically greater than the greatest negative root of the same equation: then the difference in the number of variations of the two series of results of the substitution of these limits in X, X_1, X_2, . . ., X_n, will be the number of real roots of the equation, the remaining ones being imaginary.

But as $+\frac{1}{0}$ and $-\frac{1}{0}$, though limits too wide for approximation, are sufficiently near for the present purpose: and as in this, like the more restricted limits above spoken of, the signs of the entire functions will be the same with those of their first terms, we may at once obtain the number of real roots, which is the difference of the number of variations in the true series.

For the particular character of the roots in any given interval, it will, however, be necessary to form the values of the functions for the two numbers which form the limits.

Scholium.

It is very obvious, that except the changed signs, Sturm's functions are precisely those which occur in seeking for equal roots; and hence, if such occur, they will be made apparent, and the given equation depressed by these roots may be resumed as an original equation, which, from its being of lower dimensions, will create less difficulty in finding the functions *.

Let us take as an example the following equation:

Given $x^5 + 4x^4 - 2x^3 + 10x^2 - 2x - 962 = 0$.

---

hence the signs of X_1, when quantities less and greater than ρ_1 are substituted for x, must be changed, till we arrive at ρ_2. But before arriving at ρ_2 we pass through r_2, and hence in this interval $X = 0$. From the value, therefore, ever so little above r_1, to the value ever so little below ρ_1, X and X_1 retain their signs unchanged; but at this point, ρ_1, the sign of X_1, changes to the opposite, and continues to retain it till we arrive at ρ_2, when it again changes. The series, therefore, will be, supposing r_1 the greatest root of $X = 0$,

greater than r_1	between r_1 and ρ_1	between ρ_1 and r_2	between r_2 and ρ_2	between ρ_2 and r_3	between r_3 and ρ_3	And so on, as the two last are a repetition of the first two signs, and the same order will continue to the end.
X +	−	−	+	+	−	
X_1 +	+	−	−	+	+	

* Sturm investigates a modification of his method, which dispenses with the resumption of the process with respect to the depressed equation. Nothing in point of labour is, however, saved by it.

$$\begin{aligned}
\text{Here } X &= x^5 + 4x^4 - 2x^3 + 10x^2 - 2x - 962; \\
X_1 &= 5x^4 + 16x^3 - 6x^2 + 20x - 2; \\
X_2 &= 14x^3 - 29x^2 + 20x + 4007; \\
X_3 &= -325x^2 + 11358x + 59159; \\
X_4 &= -10052526x - 47309473; \\
X_5 &= +
\end{aligned}$$

$X \; X_1 \; X_2 \; X_3 \; X_4 \; X_5$

When $x = -\frac{1}{0}$, we have the signs $- + - - + +$, or 3 variations;

$x = +\frac{1}{0}$, we have the signs $+ + + - - +$, or 2 variations.

Hence, the difference of the number of variations of sign being $3 - 2 = 1$, the equation has but one real root.

However, it will be necessary to narrow the limits between which the substitutions are made in order to effect the entire solution.

If $x = 0$ we have the signs $- - + + - +$, or three variations; and hence there is no negative *real* root: but by Harriot's law of signs, there are two of the roots negative, there being two permanencies of signs in the given equation. These two negative roots are therefore imaginary.

Again, take 3 and 4 for the values of x in the functions: then, when $x = 3$ the signs are $- + + + - +$, and there are no variations lost in the series of signs; hence there is no real root between 0 and 3.

When $x = 4$, the signs are $+ + + + - +$, and there is one variation lost; hence there is one real root between 3 and 4.

Moreover, the signs undergoing no change for any greater value of x, there is no real root greater than 4; or the only real root lies between 3 and 4. We shall resume this example presently for the purpose of completion of the entire process of solution *.

EXAMPLES FOR PRACTICE.

1. Given $x^3 + 11x^2 - 102x + 181 = 0$.
2. Given $x^3 - 2x = 5$, and $x^3 - 7x + 7 = 0$, to find whether the roots are real or not in each.
3. In $x^4 - 2x^3 - 7x^2 + 10x + 10 = 0$, all the roots are real.
4. In $2x^5 + 2x^4 - 13x^3 - 3x^2 - 9x = 19$, there is one real root, two equal roots, and two imaginary roots.

---

* It may be useful to the student to compare the determination of the character of the roots by Budan's criterion with that in the text by Sturm's method.

(1) Reduce the roots by 1; then we have $+ + + + + -$, or 2 lost.

(2) Reduce the reciprocal by 1; then we have $- - - - - -$, or 0 left.

Hence there are two imaginary roots in the interval 0, 1.

(3) Reduce the roots of (1) by 2: then the signs are $+ + + + + -$, or 0 lost.

(4) Reduce the roots of (3) by 1: then we get $+ + + + + +$, or 1 lost.

Hence there is one real root between 3 and 4.

To find the negative roots, change the alternate signs: then

(5) Reduce by 3; and we have the signs $+ + + + - +$, or 0 lost.

(6) Reduce (5) by 1; and we get $+ + + + + +$, or 2 lost.

(7) Reduce the reciprocal of (5) by 1; then $+ + + + + +$, or 0 left.

There are hence two negative roots between 3 and 4, and they are imaginary.

5. Given $X = x^3 + px + q = 0$, to find the conditions which will render all the roots real.

6. For the same purpose, given $x^4 + px^2 + qx + r = 0$.

MISCELLANEOUS EXAMPLES ON CRITERIA, AND INITIAL APPROXIMATION.

1. $x^6 - 12x^5 + 60x^4 + 123x^2 + 4567x - 89012 = 0$.
2. $x^6 - 8x^5 + 32x^4 - 74x^3 + 104x^2 - 80x + 25 = 0$.
3. $x^7 - 9x^6 + 40x^5 - 106x^4 + 178x^3 - 184x^2 + 105x = 25$.
4. $89012x^6 - 4567x^5 - 123x^3 - 60x^2 + 12x = 1$.
5. $2x^5 + 2x^4 + 3x^3 - 2x^2 - 3x - 1 = 0$.
6. $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$.
7. $x^3 - 15x^2 + 63x - 50 = 0$.
8. $x^6 - 7x^5 + 22x^4 - 53x^3 + 117x^2 - 160x + 150 = 0$.
9. $x^6 - 1 = 0$, $x^6 + x^3 + 1 = 0$, and $x^{10} + 3x^5 - 2x = 0$.

PROBLEM VIII.

Horner's method of Continuous Approximation.

Let the given equation be $Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots\ Lx^2 + Mx + N = 0$; and let a_1 be a distinct approximation * to one of its roots; it is required to evolve the remaining figures in succession.

1. Reduce the roots of the equation by a_1, and denote the reduced equation by

$$A_1x_1^n + B_1x_1^{n-1} + C_1x_1^{n-2} + \ldots\ L_1x_1^2 + M_1x_1 + N_1 = 0.$$

2. Find a new root-figure from $-\dfrac{N_1}{M_1} = a_2$, that is, from $-\dfrac{f(a_1)}{f_1(a_1)} = a_2$; and reduce the last equation by this, giving

$$A_2x_2^n + B_2x_2^{n-1} + C_2x_2^{n-2} + \ldots\ L_2x_2^2 + M_2\,x_2 + N_2 = 0.$$

If, however, upon transforming, the sign of N_2 should prove to be different from that of N_1, whilst that of M_2 is the same with that of M_1, the value a_2 is too great, and the next smaller number of the same decimal denomination should be tried, till one is found which fulfils the requisite condition.

3. From $-\dfrac{N_2}{M_2} = a_3$ find a new root-figure, and transform as before: and proceed thus till all the figures are found, if the root terminate, or as many as may be necessary if the root be interminable.

4. When the root is negative, change the signs of all the roots (Prob. IV.)

---

* By *a distinct approximation* is meant a value a_1 which is nearer to r_1, a root of $f(x) = 0$, than the corresponding root ρ_1, of $f_1(x) = 0$ is to r_1. In this case $f(a)$ and $f_1(a)$ have contrary signs; and for the most part (almost always after transformation by the first decimal of the root) the quotient $-f(a) \div f_1(a)$ gives the next figure of the root accurately, and the more especially if none of the intermediate coefficients are comparatively large numbers. This is obvious from Theor. XIII. as then one of the roots of the reduced equation is relatively very small; and as we proceed to diminish the roots still further, by succeeding decimals of the root to which we are approximating, it becomes accurate to several decimal places.

and find the positive roots above. These written negatively, will be the negative roots of the equation.

5. In actually working out the transformations, it will be convenient to mark the resulting coefficients of the transformed equation by oblique lines, as in the example on p. 234, instead of recommencing the work by writing them anew in a horizontal line.

6. After obtaining two or three decimals of the root, the work may be very much contracted, analogously to that employed in contracted multiplication and division of decimals, in the following manner:

(*a*). Let $A_p x_p^n + B_p x_p^{n-1} + \ldots + L_p x_p^2 + M_p x_p + N_p = 0$ be the reduced equation after which the contractions are to commence: then draw a vertical line on the right of the figures in N_p; a vertical line cutting off one figure from M_p; a vertical line cutting off two figures from L_p; and so on, till $n - 1$ figures are cut off from B_p, and n figures from A_p.

(*b*). Find with this contraction the next figure of the root a_{p+1}; and reduce by this, taking the figures to the *left* of the vertical lines, with one of those on their right as the multiplicand in each case, (taking care, however, to estimate the effect for the purpose of "carrying" of the rejected ones, as near as possible,) and put the results down in the corresponding places of the next column; viz. beginning with that on the right of the vertical line. The additions to be performed as when there was no contraction of the work. This will give $A_{p+1} x^n_{p+1} + B_{p+1} x^{n-1}_{p+1} + \ldots M_{p+1} x_{p+1} + N_{p+1} = 0$.

(*c*). In the next transformation, cut off one, two, three, ... $(n - 1)$ figures from the contracted coefficients, M_{p+1}, $L_{p+1} + K_{p+1}$, ... B_{p+1}; and proceed as before.

In these processes, as the greatest number of figures is cut off from those columns which originally contained the fewest, these will diminish very rapidly; and after a few transformations, an equation of a high degree is reduced in point of simplicity to one of a low degree; and generally the last half of the entire figures of the root are obtained by contracted division only.

Moreover, if p figures have been found, and n be the degree of the equation, the number of figures of the root which may be trusted to as quite accurate, will be $np - 1$. Thus in an equation of the 5th degree, if three decimals have been found, the contraction will give $3 \,.\, 5 - 1 = 14$ places true in all.

The theory of this contraction is very simple, but it does not admit of being concisely laid down in words: but a little consideration will enable the student to perceive that the *effect* of the parts cut off falls entirely to the *right*, in all the columns of the correction column, or of that which follows the column of figures to the right of the vertical lines.

In illustration of the entire process, let the equation $x^5 + 4x^4 - 2x^3 + 10x^2 - 2x - 962 = 0$ be proposed, in which a distinct approximation to one of the roots is 3. Then, performing the reduction by 3, we have

$$x_1^5 + 19x_1^4 + 136x_1^3 + 478x_1^2 + 841x_1 - 365 = 0.$$

To find a_2, we have $a_2 = -\dfrac{-365}{841} = \cdot 4$; but upon reducing the equation by $\cdot 4$, we find $N_2 = + 40{\cdot}68064$, or the sign of the final term is changed. Whence try $a_2 = \cdot 3$, and the condition is fulfilled. In all the subsequent stages $-\dfrac{-N_p}{M_p} = a_p$ is found upon performing the reductions, to fulfil the required condition. The following is the process.

1 + 4	−2	+10	−2	−962(3·354848699
3	21	57	201	597
7	19	67	199	−365.....
3	30	147	642	29914833
10	49	214	841....	−6585167
3	39	264	1561611	5987807 5
13	88	478...	9971611	−597359 5
3	48	42537	1694514	492583 6
16	136..	520537	1166612 5	−104775 9
3	579	44301	30949 0	98761 6
19.	14179	564838	1197561 5	−6014 3
3	588	46092	31354 0	4940 1
193	14767	6109 30	122891 5 5	−1074 2
3	597	80 5	254 4	988 0
196	15364	6189 8	123145 9	−86 2
3	606	81 0	254 8	74 1
199	15 970	6270 8	12340 0 7	−12 1
3	1	81 5	5 1	11 0
202	16 1	63 52 3	12345 2	1 1
3	1	1	5 1	
.205	16 2	63 6	12 3 5 0 3	
	1	1		
	16 3	63 7		
	1	1		
	.16 4	63 8		

This result is rather too small, owing to the contracted corrections of the coefficients, but especially of the fourth, being kept uniformly above the truth: but they have been retained to show the manner of conducting the operation, instead of throwing out the zeros which would have taken the place of the units now to the left of the line.

GENERAL RECAPITULATION AND REMARKS.

1. Count the number of variations and the number of permanencies of sign in the given equation; there will be as many positive roots as variations of sign, and as many negative roots as permanencies of sign. (Theor. VIII.)

2. If there be an odd number of positive roots, one at least of them will be real; and if an odd number of negative roots, one at least of these will be real.

3. As an initial experiment, reduce the roots of the equation, both as it is given, and with its alternate signs changed by the factors of the absolute term; since, if there be any integer roots, they are factors of that term, and in such case, the first horizontal line of operations will render $N_1 = 0$. If any such be

found, then employ in like manner the factors of M_2 of the reduced equation; and so on, as long as the division terminates. The integer roots will all be thus easily found. See Example 4, p. 236.

4. No equation having all its coefficients integers, and A different from unity, can have a fractional root: all such equations must, therefore, have their real roots either integers or interminable decimals.

5. If one variation be lost in passing from a transformation from an integer a to its consecutive integer $a + 1$, then the number represented by a is the principal part, or "first figure" of the root.

6. If two, four, or any even number of variations be lost in the transition, there are two, four, or some corresponding even number of roots, in the interval, of which one, two, or some corresponding number of pairs may be imaginary, and the remainder real.

7. In this case consider which criteria are most likely to be applicable to the determination of the number of real roots.

If De Gua's apply, it will be the most simple; or if any inference can be drawn from it under the aspect presented in the note on p. 225, let it be done.

In case of still doubting the character of the roots, apply Budan's criterion first of all, as directed in the statement, p. 226, 1 being the reciprocal of 1. If there be still uncertain roots, (or m less than n,) proceed as in the foot-notes, using either the reduced reciprocal equation, or a narrower interval in the original direct one.

Should there still be any doubt, which can never be the case except there be equal roots, or roots having very minute differences, have recourse to Sturm's Criterion. This in its progress, by giving some one of the modified remainders $X_p = 0$, will furnish the component equation containing the equal roots; and if there be not equal roots, the functions so derived will furnish a complete criterion for every part of the series of values between r, the greatest, and r_n, the least, of the roots *.

8. When there are two real roots in a small interval, it will always be more convenient to seek one or two figures of the corresponding roots of its reciprocal equation in the outset, as suppose a and b: then the leading parts of the roots of the given equation will be found from $\frac{1}{a}$ and $\frac{1}{b}$, and the approximation continued as usual from these.

9. When the root of an equation has been *accurately* determined, use the *depressed equation* for finding the other roots: but when a root has been ap-

---

* This may be in some degree an apparent inversion of the natural order of proceeding to obtain a complete solution of the equation. The most obvious course would be: (1), to form Sturm's functions $X, X_1, X_2, \ldots X_n$, which in their progress would detect the equal roots (2), to apply the limiting integers a and $a + 1$ to these functions in all such cases as presented a doubt: (3), to employ narrower intervals for finding the distinct approximation: and (4), to develop the roots whose initial values had been found by Horner's method. However, the very great labour attendant on finding Sturm's functions, renders it desirable to evade their use if it can possibly be dispensed with; and this can almost always be done by taking narrower limits for the transformation, as we thereby, for the most part, separate the *pairs of roots* which occur at small distances in the numerical scale from each other; and there is never any difficulty in determining by Budan's criterion, as modified in the notes, whether these be real or not. The order of working, therefore, pointed out in the text, contributes greatly to expedition, whilst it is much less likely to be productive of numerical error than the complicated operations and unwieldy numbers that are essential to Sturm's operation.

proximately determined, return to the original one (or an accurately depressed equation, if such has been found, by means of an accurate root) to find the other roots.

10. Equations whose coefficients are rational, cannot have irrational or imaginary *equal* roots, without their conjugates; or, in other words, if there be equal roots of the form $a + b\sqrt{\pm 1}$, there will also be as many of the form $a - b\sqrt{\pm 1}$.

EXAMPLES FOR PRACTICE.

1. Solve the equation $x^5 - 13089034x^2 + 26178063x - 13089030 = 0$.
Ans. 1, 1, 235, and the roots of $x^2 + 237x + 55698 = 0$.

2. Find the roots of $x^6 + 4x^5 - 8x^4 - 25x^3 + 35x^2 + 21x - 28 = 0$.
Ans. -4, -1, 1, 1·356896, 1·692021, and $-3·048917$.

3. Find the roots of $x^2 - 1·01 = 0$, and of $x^2 - 10x = 100$, by the general method.
Ans. $\pm 1·00498756$ in the former, and 16·18034, and $-6·18034$ in the latter.

4. Solve $x^8 - 6x^7 - 12x^6 + 134x^5 - 289x^4 + 480x^3 - 660x^2 - 608x + 960 = 0$.
Ans. 1, 3, 4, 4, -1, -5, and $\pm 2\sqrt{-1}$.

5. Solve the equation $x^5 + 7x^4 + 20x^3 + 155x^2 - 10000 = 0$.
Ans. 4·54419552, and four imaginary roots.

6. Determine completely the characters of the roots, and the values of the real ones in the following equations:
(1) $x^4 - 12x^2 + 12x - 3 = 0$; (all real.)
(2) $x^4 - 19x^3 + 132x^2 - 302x + 200 = 0$; (two imaginary.)
(3) $x^3 - 17x^2 + 54x - 350 = 0$; (two imaginary.)

7. Find what roots are imaginary, and find the real ones, in
(1) $x^5 - 3x^4 - 24x^3 + 95x^2 - 46x = 101$,
(2) $x^4 - 4x^3 - 3x + 23 = 0$,
(3) $x^3 + 2x^2 - 3x + 2 = 0$,
(4) $x^4 - x^3 + 4x^2 + x = 4$,
(5) $x^7 - 2x^5 - 3x^3 + 4x^2 - 5x + 6 = 0$,
(6) $x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0$,
(7) $x^5 - 10x^3 + 6x + 1 = 0$.

8. To find the values of x and y, there are given the two following:
$$4x^2 + 5xy - 10y^2 - 4x - 10y + 250 = 0$$
$$-10x^2 + 100xy - 8y^2 + 100x + 100y + 2356 = 0.$$

9. Given $xy^2 + x^2y + 1·75 = 0$, and $x^2 + y^2 = 4·25$, to find x and y.

10. Given $x^2 + yz = 16$, $y^2 + zx = 17$, and $z^2 + xy = 18$, to find x, y, z.

INDETERMINATE COEFFICIENTS.

In all the inquiries which have preceded the present, in this course, we have been required to find the special values of one or more unknown quantities, so as to satisfy the given equations, or the conditions which they expressed. There is, however, a distinct class of inquiries, in which we are required to change the *form* of any given compound expression into a series of single terms. If the indicated operation be one which we know how to perform, this change may in general be effected by an actual performance of those operations: but it will

generally happen that the labour attendant upon it will be intolerably great. It may happen, too, that we may be in possession of no rule for actually performing the indicated operation, and in this case the problem could not be solved at all. As, for instance, to extract the fifth root of $a + x$, or more generally the nth root of $a + x$, to which no rule of extraction given in the earlier part of this work is applicable. A general method of expanding such expressions in a series of single terms, will be given a little farther on, by means of indeterminate coefficients, as well as solutions of one or two other problems which cannot be dispensed with in our future investigations.

In this method, the development is *assumed* to be of a particular form, so far as indices are concerned, and equations by which the corresponding coefficients are obtained, are deduced from considerations presently to be explained. If these equations give *real* values of the coefficients, the development is effected by the solution of them: though, independently of an investigation specially directed to the decision of the point, we cannot be justified in affirming that the development so obtained is the *only one* that can possibly be made.

The principle upon which the doctrine of indeterminate coefficients turns is this:

That in all developments that are made according to successive powers of any quantity (whether integer or fractional), in whatever *form* the coefficients may appear, the one which belongs to any specified power of that quantity in one form must be equal in *value* to that which belongs to the same power in the other development. Thus, if the function fx could be written in two different ways, or so that the coefficients took different forms in the two developments, the following equation must be fulfilled independently of the specific value of x, viz.:

$$A + Bx + Cx^2 + \dots = A_1 + B_1x + C_1x^2 + \dots$$

For, by transposition, we have

$$(A - A_1) + (B - B_1)x + (C - C_1)x^2 + \dots = 0.$$

Now, if we have not simultaneously $A - A_1 = 0$, $B - B_1 = 0$, $C - C_1 = 0$, ..., we should have an equation in x of some degree, n, and x would in this case have n values only, and not admit of all values indiscriminately.

But since in development we have only to change the *forms* of the expressions, the quantities themselves must be capable of all possible values, and hence cannot be restricted to the special ones which constitute the roots of any given equations. It follows, therefore, that unrestricted values of x in the preceding equation are essential to the idea of a development, and hence that the coefficients of the several powers of x in the simplified equation are separately equal to zero.

We may also make another remark which will hereafter be found useful. If the following equation be true for $n + 1$ values of x, it is true for all values of x, viz.:

$$A + Bx + Cx^2 + \dots Nx^m = A_1 + B_1x + C_1x^2 + \dots N_1x^m.$$

For if it be true for $m + 1$ values of x, it is true for a value which is not a root of the equation of the nth degree; and hence it can only be true when $A - A_1 = 0$, $B - B_1 = 0$, ... $N - N_1 = 0$. But in this case the equation is true whatever x may be, or for all values of x.

It will at once appear, that to render this method effective, the development in symbolic coefficients must be of essentially different forms; and that in all cases where this cannot be effected, it will indicate that the powers of x, whose coefficients they are, do not enter into the true developement; or, in other

words, that the *assumed* law of the indices of x is not possible. The same is true if any of the values of the coefficients become imaginary.

A few examples will be necessary to illustrate the nature of the method.

Ex. 1. Extract the square root of $1 + x^2$.

Assume the root to be in positive powers of x; or $\sqrt{1 + x^2} = A + Bx + Cx^2 + \ldots\ldots$ Then squaring both sides,

$$\begin{aligned} 1 + x^2 = A^2 &+ ABx + ACx^2 + ADx^3 + AEx^4 + \ldots \\ &+ ABx + B^2x^2 + BCx^3 + BDx^4 + \ldots \\ &\qquad + ACx^2 + BCx^3 + C^2x^4 + \ldots \\ &\qquad\qquad + ADx^3 + BDx^4 + \ldots \\ &\qquad\qquad\qquad + AEx^4 + \ldots \end{aligned}$$

The coefficients of the two sides of the equation being of different forms, and those of the first side given, we may proceed to equate those of the like powers of x. We have, therefore,

In x^0, $A^2 = 1$, or $A = \pm 1$.

In x^1, $2AB = 0$, or $B = 0$.

In x^2, $2AC + B^2 = 1$, or $C = \dfrac{1 - B^2}{2A} = \dfrac{1}{\pm 2} = \pm \dfrac{1}{2}$.

In x^3, $2AD + 2BC = 0$, or $D = -\dfrac{BC}{A} = 0$.

In x^4, $2AE + 2BD + C^2 = 0$, or $E = -\dfrac{2BD + C^2}{2A} = -\dfrac{(\pm\frac{1}{2})^2}{2(\pm 1)} = \mp \dfrac{1}{8}$,

and so on. Whence the expansion is

$$\sqrt{1 + x^2} = \pm \left\{ 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \ldots\ldots \right\}$$

as may be easily verified by actual extraction, according to the method pointed out at p. 148.

Moreover, had we foreseen that only even powers entered into the expansion, we might have obtained twice as many terms of it as above with the same quantity of work, by assuming

$$\sqrt{1 + x^2} = A_{,} + B_{,}x^2 + C_{,}x^4 + \ldots$$

Ex. 2. Expand $\dfrac{(1 - x + x^2)(1 + x - x^2)}{1 + x + x^2}$ into a series.

Assume it equal to $A + Bx + Cx^2 + \ldots$; then multiply both sides by $1 + x + x^2$, and multiply the two factors of the numerator together. This gives

$$\begin{aligned} 1 - 0x - x^2 + 2x^3 - x^4 = A &+ Bx + Cx^2 + Dx^3 + Ex^4 + \ldots \\ &+ Ax + Bx^2 + Cx^3 + Dx^4 + \ldots \\ &\qquad + Ax^2 + Bx^3 + Cx^4 + \ldots \end{aligned}$$

Equating the coefficients of the like powers of x, we have

$$A = 1,\ A + B = 0,\ A + B + C = -1,\ B + C + D = 2,$$
$$C + D + E = -1,\ D + E + F = 0,\ E + F + G = 0,$$

and so on. Resolving these equations, we have

$$A = 1,\ B = -1,\ C = -1,\ D = 4,\ E = -4,\ F = 0,$$

and so on. Whence the development is

$$1 - x - x^2 + 4x^3 - 4x^4 + 0x^5 + 4x^6 - \ldots$$

as may be easily verified by *Synthetic Division.*

These questions, and others of similar kinds, admit, however, of easier solution by other methods, and they are only instanced here for showing the nature of the operations to be performed, whilst their results admit of verification by

those easier methods. The following is one of a class to which this method is the only one that can be applied without great labour; and it is one of perpetual occurrence in integration.

Ex. 3. Given $\dfrac{x^{n-1} + ax^{n-2} + \ldots\ldots kx + l}{x^n + a_i x^{n-1} + \ldots\ldots k_i x + l_i}$, where the denominator is resolvable into factors of the first, second, third, &c. degrees: to transform it into a series of partial fractions, each of which shall be one of those factors in succession.

As the method will be quite as well seen by a numerical example, it will be sufficient to so transform $\dfrac{1}{x^3 - x^2 - 2x}$.

Here the denominator is $x(x + 1)(x - 2)$ we may assume its partial fractions to be

$$\frac{1}{x^3 - x^2 - 2x} = \frac{A}{x + 1} + \frac{B}{x} + \frac{C}{x - 2},$$

or reducing to a common denominator,

$$= \frac{(A + B + C)\,x^2 - (2A + B - C)\,x - 2B}{x(x + 1)(x - 2)}.$$

Whence, equating the homologous coefficients of the numerator, we get

$$\left.\begin{aligned} A + B + C &= 0 \\ 2A + B - C &= 0 \\ -2B &= 1 \end{aligned}\right\} \text{ or, } A = \frac{1}{3},\ B = -\frac{1}{2},\ C = \frac{1}{6},$$

and hence $\dfrac{1}{x^3 - x^2 - 2x} = \dfrac{1}{3(x + 1)} - \dfrac{1}{2x} + \dfrac{1}{6(x - 2)}$,

which may be easily verified by reducing to a common denominator.

It is necessary to remark, that the power of x in the assumed numerators must in all cases be one degree lower than in the denominator.

EXAMPLES FOR EXERCISE.

4. Expand $\dfrac{1}{(1 + x)^2}$ or $\dfrac{1}{1 + 2x + x^2}$ in a series according to positive, and then to negative, powers of x.

5. Expand $\dfrac{1 - x}{1 + x}$, and $-\dfrac{1 - x^{-1}}{1 + x^{-1}}$.

6. Expand $\dfrac{2ab}{a + b}$, $\dfrac{a}{1 - a}$, $\dfrac{a^2}{a^2 - b^2}$, and $\dfrac{a^2}{a - x}$.

7. Extract the square root of $4 - 6x + 5x^2 - 9x^3$.

8. Expand $\sqrt{\dfrac{1 - x}{1 + x}}$, $\sqrt{1 + 2x + 3x^2}$, $\sqrt{2 - 3x + 5x^2}$ and $\sqrt{1 + x + x^2 + x^3}$.

9. Resolve into partial fractions the following expressions:

$$\frac{x^5 - 9x^4 + x^2 - 3x + 5}{x^6 + 4x^5 - 8x^4 - 25x^3 + 35x^2 + 21x - 28}$$

and

$$\frac{3x^4 - 9x^3 + 6x^2 + 4x - 20}{x^8 - 6x^7 - 12x^6 + 134x^5 - 289x^4 + 480x^3 - 660x^2 - 608x + 960}$$

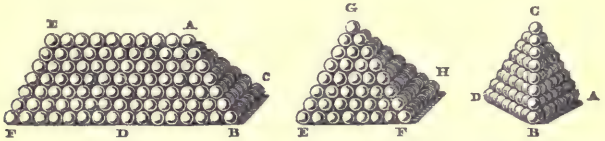
the denominators being the same as those in Examples 2 and 4, p. 236.

The preceding simple applications of the method will suffice to show its character; and we pass on to some of its applications in the investigation of general theorems for summation and expansion.

PILING OF BALLS.

The usual forms of piles of balls are the triangular, the square, and the rectangular, and they take their names from the figure of their lowest courses. In all cases the successive courses have one ball less in each of their sides than the one upon which they respectively rest; the highest course being in the triangular and square piles a single ball, and in the rectangular, a line of balls.

A pile is said to be incomplete or broken when it has either not been finished, or when some of the balls have been removed from a finished pile. The following figures represent the three kinds, as named below them respectively. The number of courses, therefore, is the same as the number of balls in the shortest side of the lowest course.



Rectangular pile. Square pile. Trian. pile.

1. *To find the number of balls in a triangular pile of n courses.*

Here we have $C_n = \frac{n^2}{2} + \frac{n}{2}$

$$C_{n-1} = \frac{n^2}{2} - \frac{n}{2}$$

$$C_{n-2} = \frac{n^2}{2} - \frac{3n}{2} + 1.$$

$$\dots\dots\dots\dots\dots\dots$$

Hence, as there are n courses expressible in the same manner, we may infer that the highest power of n that enters the expression for the sum ~~to~~ will be n^3. Let, then, $S_n = C_n + C_{n-1} + C_{n-2} + \dots C_2 + C_1 = An^3 + Bn^2 + Cn + D$.

Then, substituting in this the first four values of n, we shall have

$S_1 = A + B + C + D = 1$, for the first course;
$S_2 = 8A + 4B + 2C + D = 4$, for the first two courses;
$S_3 = 27A + 9B + 3C + D = 10$, for the first three courses.
$S_4 = 64A + 16B + 4C + D = 20$, for the first four courses.

Hence, resolving these equations, as at p. 180, we have

$$A = \frac{1}{6},\ B = \frac{1}{2},\ C = \frac{1}{3},\ \text{and } D = 0.$$

Whence $S_n = \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3} = \frac{n(n+1)(n+2)}{6}$,

which gives the sum of the n courses.

2. *To find the number of balls in a square pile of n courses.*

In this we have

$$C_n = n^2$$
$$C_{n-1} = n^2 - 2n + 1$$
$$C_{n-2} = n^2 - 4n + 4$$
$$\dots\dots\dots\dots\dots\dots$$

for the n courses: and for the same reason as before the form of the function will be $S_n = An^3 + Bn^2 + Cn + D$. Substitute for n the successive values 1, 2, 3, 4, as in the last case: then

$$\begin{aligned}
S_1 &= A + B + C + D = 1, \text{ for the first course;}\\
S_2 &= 8A + 4B + 2C + D = 5, \text{ for the first two courses;}\\
S_3 &= 27A + 9B + 3C + D = 14, \text{ for the first three courses;}\\
S_4 &= 64A + 16B + 4C + D = 30, \text{ for the first four courses.}
\end{aligned}$$

From these equations we have $A = \frac{1}{3}$, $B = \frac{1}{2}$, $C = \frac{1}{6}$, $D = 0$; and therefore

$S_n = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+1)}{6}$, the number in a square pile.

3. *The rectangular pile.* Let n be the number of balls in the shorter side, and $n + m$ the number in the longer side of the lowest rectangular course.

Then it will be obvious, by reference to the structure of the pile, that it is composed of a square pile of n courses, with m oblique triangular courses added successively to one of the oblique faces. Hence the entire pile is

$$\begin{aligned}
S_n &= \text{square pile} + m \text{ triangular courses.}\\
&= \frac{n(n+1)(2n+1)}{6} + m \cdot \frac{n(n+1)}{2}\\
&= \frac{n(n+1)(3m+2n+1)}{6},
\end{aligned}$$

which gives the balls in the rectangular pile.

It may be well to recollect that the ridge of the rectangular pile has $m + 1$ balls.

4. *The incomplete pile.* This will require the whole pile to be computed, and then the partial pile removed, both by the appropriate formula for the kind of pile. The difference is the number in the incomplete pile. Formulæ, indeed, may be given for all the cases in terms either of the number of courses taken off or the courses left, and the original number of courses: but these would be much more complex, and the implied numerical operations more laborious, than the unreduced formula and the work which it involves. Such formulæ, therefore, would be without utility, and from having little mathematical elegance, would be destitute of sufficient interest to justify their introduction here.

Ex. 1. Find the number of balls in a triangular and in a square pile, each composed of twelve courses.

Triangular pile.

$$S_{12} = \frac{12 \cdot 13 \cdot 14}{6} = 364.$$

Square pile.

$$S_{12} = \frac{12 \cdot 13 \cdot 25}{6} = 650.$$

Ex. 2. In a rectangular pile are 18 courses, and the number in the ridge is 45. How many balls are there in the entire pile?

Here $m + 1 = 45$, or $m = 44$, and $n = 18$. Hence by the formula,

$$S_{18} = \frac{18 \cdot 19 \cdot \{3 \cdot 44 + 2 \cdot 18 + 1\}}{6} = \frac{18 \cdot 19 \cdot 169}{6} = 9633.$$

Ex. 3. Of the preceding rectangular pile there are to be taken away 1031 balls; how many complete courses must be removed?

Let x be the number: then

$$\frac{x(x+1)(3 \cdot 44 + 2x + 1)}{6} = 1031, \text{ or by reduction,}$$

$$2x^3 + 135x^2 + 133x = 6186.$$

The real root of this is between 6 and 7; and hence six complete courses, together with 96 balls more, must be removed; for

$$\begin{array}{rrrl}
2 + & 135 + & 133 - & 6186 \mid 6\\
& 12 + & 882 & 6090\\
\hline
& 147 & 1015 & -96
\end{array}$$

EXAMPLES FOR EXERCISE.

1. How many more balls are there in a square pile of 15 courses than in a triangular one? Ans. 560.

2. A square pile which has as many courses as a triangular pile, contains half as many more balls. How many balls were there in both?

Ans. 4 courses in each, and 50 balls in all.

3. The upper and lower courses of an incomplete square pile have 15 and 25 balls in each side. Find the number in the original pile, and the number left.

Other examples may be easily formed to suit the ability of the pupil.

THE BINOMIAL THEOREM.

This theorem affirms that every expression of the form $(a + x)^n$ can be developed in a series of positive integer powers of either a or x, and assigns the coefficients of those powers of a or x in the development. It does not, however, affirm that this is the only form of development possible. The expression is either

$$(a + x)^n = a^n + \frac{n}{1} \cdot a^{n-1} x + \frac{n(n-1)}{1 \cdot 2} a^{n-2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3} x^3 + \ldots$$

or

$$(x + a)^n = x^n + \frac{n}{1} x^{n-1} a + \frac{n(n-1)}{1 \cdot 2} x^{n-2} a^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3} a^3 + \ldots$$

according as we consider a or x the leading term of the given expression.

1. *To find the first term of the development.*

$$\text{Assume } (a + x)^n = A_n + B_n x + C_n x^2 + D_n x^3 + \ldots\ldots \quad (a)$$

Then, using similar notation where $2n$ is written for n, we have

$$(a + x)^{2n} = A_{2n} + B_{2n} x + C_{2n} x^2 + D_{2n} x^3 + \ldots\ldots \quad (b)$$

But $(a + x)^{2n} = \{(a + x)^n\}^2$, in which, substituting from (a) and (b), we get

$$A_n^2 + 2A_n B_n x + \ldots\ldots = A_{2n} + B_{2n} x + \ldots\ldots$$

whence, equating the coefficients of x^0, we get $A_n^2 = A_{2n}$: which is fulfilled by $A_n = a^n$, and the first term is found to be a^n universally.

2. *To find the second term of the development.*

It will simplify the process to write the expression $(a + x)^n = a^n \left\{1 + \frac{x}{a}\right\}^n$ $= a^n(1 + v)^n$; and it will evidently be sufficient to expand the compound factor, and multiply every term of the expansion by a^n to obtain the complete development.

First, let n be a positive integer: then we may assume as before,

$$(1 + v)^n = 1 + A_n v + B_n v^2 + C_n v^3 + \ldots\ldots\ldots\ldots \quad (c).$$

For since $a^n \{1 + A_n v + \ldots\ldots\}$ gives $a^n + a^n A_n v + \ldots\ldots$, the first term of the bracketted series being 1, fulfils the condition respecting the development deduced in the former part of the investigation.

Divide, synthetically, equation (c) continually by $1 + v$: then

$$(1+v)^{n-1} = 1 + (A_n - 1)\,v + B_{n-1}\,v^2 + C_{n-1}\,v^3 + \ldots$$
$$(1+v)^{n-2} = 1 + (A_n - 2)\,v + B_{n-2}\,v^2 + C_{n-2}\,v^3 + \ldots$$
$$(1+v)^{n-3} = 1 + (A_n - 3)\,v + B_{n-3}\,v^2 + C_{n-3}\,v^3 + \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(1+v)^{n-m} = 1 + (A_n - m)\,v + B_{n-m}\,v^2 + C_{n-m}\,v^3 + \ldots.$$

and if we take $m = n - 1$, we have

$$1 + v = 1 + \{A_n - (n-1)\}\,v + B_1\,v^2 + C_1\,v^3 + \ldots.$$

Equating the coefficients of the homologous powers of v, we have for those of the first powers, or v^1,

$$1 = A_n - (n-1), \text{ or } A_n = n.$$

When, therefore, n is a positive integer, $A_n = n$, and the coefficient of the second term is found.

Secondly, let n be a negative integer; or the expression to be developed be $(1+v)^{-n}$. This is the same thing as

$$\frac{1}{(1+v)^n} = \frac{1}{1 + nv + B_n\,v^2 + \ldots.} = 1 - nv + B_{-n}\,v^2 + C_{-n}\,v^3 + \ldots.$$

by actual division. Whence in this case we have $(1+v)^{-n} = 1 - nv + \ldots\ldots$, and the coefficient of the second term is found.

Thirdly, let n be a fraction denoted by $\pm\frac{p}{q}$. Then, as before, assume that

$$(1+v)^{\pm\frac{p}{q}} = 1 + A_n\,v + B_n\,v^2 + C_n\,v^3 + \ldots.$$

Raise both sides to the qth power: then we have

$$1 \pm pv + B_p\,v^2 + \ldots = 1 + qv\,(A_n + B_n\,v + \ldots) + B_q\,v^2\,(A_n + B_n\,v + \ldots)^2 + \ldots.$$

and equating the homologous coefficients of v, we have for that of the first power,

$$\pm p = qA_n, \text{ or } A_n = \pm\frac{p}{q}.$$

Whence in this case also the coefficient of the second term is found: and in all cases it is equal to the index of the power.

3. *To find the third and subsequent terms of the development.*

Put $v = y + z$: then we have, denoting the coefficient of $(y+z)^r$ by A_r throughout,

$$(1+y+z)^n = 1 + n(y+z) + A_2\,(y+z)^2 + A_3\,(y+z)^3 + \ldots A_{r+1}\,(y+z)^{r+1} + \ldots$$

But we have also

$$(1+y+z)^n = (1+y)^n \left\{1 + \frac{z}{1+y}\right\}^n$$
$$= (1+y)\left\{1 + \frac{nz}{1+y} + \frac{A_2\,z^2}{(1+y)^2} + \ldots \frac{A_r\,z^r}{(1+y)^r} + \ldots\right\}.$$

Now by indeterminate coefficients, the coefficients of each separate power of z in these two developments are equal. Equate them for z^1: then

$$n + 2A_2\,y + 3A_3\,y^2 + \ldots (r+1)\,A_{r+1}\,y^r + \ldots = (1+y)^n \cdot \frac{n}{1+y};$$

or multiplying out, and changing sides,

$$n(1+y)^n = (1+y)\,\{n + 2A_2\,y + 3A_3\,y^2 + \ldots (r+1)\,A_{r+1}\,y^r + \ldots.\};$$

or again reducing, the two sides of the equation become respectively

$$n\{1 + ny + A_2\,y^2 + \ldots.\,A_r\,y^r + \ldots\} \text{ and}$$
$$n + (2A_2 + n)\,y + (3A_3 + 2A_2)\,y^2 + (4A_4 + 3A_3)\,y^3 + \ldots.\{(r+1)\,A_{r+1} + rA_r\}\,y^r + \ldots$$

Again, equating the homologous coefficients in these two expansions, we get, generally,

$$nA_r = rA_r + (r+1)\,A_{r+1}, \text{ or } (r+1)\,A_{r+1} = (n-r)\,A_r.$$

Substituting in this the successive values of r, we have

$$\begin{aligned} rA_r &= \left\{n-(r-1)\right\} A_{r-1} \\ (r-1)\,A_{r-1} &= \left\{n-(r-2)\right\} A_{r-2} \\ (r-2)\,A_{r-2} &= \left\{n-(r-3)\right\} A_{r-3} \\ &\cdots\cdots\cdots\cdots \\ 3A_3 &= (n-2)\,A_2 \\ 2A_2 &= (n-1)\,A_1 \\ 1A_1 &= nA_0, \end{aligned}$$

where, by the preceding investigation of the second term, we have $A_0 = 1$.

Multiply these columns vertically, and there results

$$1\,.\,2\,.\,3\,\ldots\ldots\, rA_r = n(n-1)\,(n-2)\,\ldots\ldots \left\{n-(r-1)\right\}, \text{ or}$$

$$A_r = \frac{n(n-1)\,(n-2)\,\ldots\ldots\left\{n-(r-1)\right\}}{1\,.\,2\,.\,3\,\ldots\ldots\, r},$$

which is the general form for the coefficient of the r^{th} power of v; and giving to r the successive values, 1, 2, 3, ... we have the corresponding coefficients, $\frac{n}{1}$, $\frac{n(n-1)}{1\,.\,2}$, $\frac{n(n-1)\,(n-2)}{1\,.\,2\,.\,3}$, ... the forms assigned in the enunciation of the theorem. Inserting these, restoring the value of v, viz. $\frac{x}{a}$, and multiplying all the terms by a^n, we have the formula as there given. Thus we obtain

$$(a+x)^n = a^n + \frac{n}{1}\,a^{n-1}x + \frac{n(n-1)}{1\,.\,2}\,a^{n-2}\,x^2 + \frac{n(n-1)\,n-2}{1\,.\,2\,.\,3}\,a^{n-3}\,x^3 + \ldots$$

for the expansion sought.

Similarly, since $a - x = a + (-x)$, we have in the expansion all the *odd powers* of x negative; that is, $\left\{a+(-x)\right\}^n = (a-x)^n = a^n - \frac{n}{1}\,a^{n-1}\,x + \frac{n\,(n-1)}{1\,.\,2}\,a^{n-2}\,x^2 - \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,a^{n-3}\,x^3 + \ldots$

4. When n is a positive integer, this series terminates with the $(n+1)^{\text{th}}$ term, and the coefficients of the terms reckoned from either extremity of the series are equal. In all other cases the series will never terminate.

For the $(n+2)^{\text{th}}$ term contains the factor 0, and this factor entering into all the succeeding terms, they also become 0. Whence the series terminates with the $(n+1)^{\text{th}}$ term.

Again, the r^{th} term from the end is the $(n-r+2)^{\text{th}}$ from the beginning; and therefore its coefficient is

$$P = \frac{n(n-1)\,(n-2)\,\ldots\ldots\, r}{1\,.\,2\,.\,3\,\ldots\,(n-r+1)};$$

and the coefficient of the r^{th} term from the beginning is

$$Q = \frac{n(n-1)\,(n-2)\,\ldots\ldots\,(n-r+2)}{1\,.\,2\,.\,3\,\ldots\ldots\,(r-1)}.$$

It, therefore, only remains to show that $P = Q$. Now we have very obviously

$$\frac{P}{Q} = \frac{n(n-1)\,(n-2)\,\ldots\ldots\, r}{1\,.\,2\,.\,3\,\ldots\,(n-r+1)} \cdot \frac{1\,.\,2\,.\,3\,\ldots\ldots\,(r-1)}{n(n-1)\;(n-2)\,\ldots\,(n-r+2)}$$

$$= \frac{n(n-1)\,(n-2)\,\ldots\ldots\, r(r-1)\,(r-2)\,\ldots\ldots\, 3\,.\,2\,.\,1}{1\,.\,2\,.\,3\,\ldots\ldots\,(n-r+1)\,(n-r+2)\,(n-r+3)\,\ldots\ldots\,(n-1)\,n}$$

in which the numerator and denominator are manifestly equal. Hence $P = Q$, as above affirmed.

5. It still remains to point out the arithmetical forms which are most convenient in the practical application of this theorem.

Suppose, as an instance, we had to develop $(a+x)^{\pm 10}$, the calculations will be as follow:

Write the terms without coefficients $a^{10} \quad a^9x \quad a^8x^2 \quad a^7x^3 \ldots$ with spaces between to receive the coefficients and their signs. Then when,

$$\begin{array}{rrrl}
n = 10 & & n = -10 & \\
n-1 = \ \ 9 & & n-1 = -11 & \\
2\underline{|90} & & 2\underline{|110} & \\
45 & \ldots\ldots\ldots\ldots & 55 & \ldots\ldots = \dfrac{n(n-1)}{1.2} \\
n-2 = \ \ 8 & & n-2 = -12 & \\
3\underline{|360} & & 3\underline{|-660} & \\
120 & \ldots\ldots\ldots\ldots & -220 & \ldots\ldots = \dfrac{n(n-1)\,(n-2)}{1.2.3} \\
n-3 = 7 & & n-3 = -13 & \\
4\underline{|840} & & 4\underline{|2860} & \\
210 & \ldots\ldots\ldots\ldots & 715 & \ldots\ldots = \dfrac{n(n-1)\,(n-2)\,(n-3)}{1.2.3.4},
\end{array}$$

and so on to the required extent: the literal parts of this throughout being merely explanatory, and need not be put down in actual working. The process is simply multiplying by the decreasing series, and dividing by the increasing one alternately. Each successive quotient is the successive coefficient of the series, which inserted in its place, gives the expansion sought.

This vertical alineation is not, however, convenient when n is a fraction, a horizontal one being much preferable. Thus, to expand $(1+v)^{-\frac{1}{2}}$, we may continually work as follows:

1st coeff.	2nd coeff.	3rd coeff.	4th coeff.	5th coeff.
$\dfrac{1^*.\,-\frac{1}{2}}{1} =$	$\dfrac{-\frac{1^*}{2}\,.\,-\frac{3}{2}}{2} =$	$\dfrac{\frac{3^*}{8}\,.\,-\frac{5}{2}}{3} =$	$\dfrac{-\frac{5^*}{16}\,.\,-\frac{7}{2}}{4} =$	$\dfrac{35^*}{128}\,.$

where the asterisks are placed at the several successive coefficients. The apparent continuity of equality may be, were it necessary, cut off, by drawing vertical lines after each sign of equality that is to be destroyed by the next operation, as in the example above. This cutting off is better than crossing out: but neither of them is absolutely necessary.

EXAMPLES IN POSITIVE INTEGER POWERS.

Ex. 1. Raise $a-x$ to the 10th power.
Ex. 2. Find the sixth power of $a-x$.
Ex. 3. Find the fourth power of $a-x$.
Ex. 4. Involve $a-x$ to the ninth, and $a+b-c$ to the fourth power.
Ex. 5. Find the cube of $\sqrt{a}-\sqrt{b}$.
Ex. 6. Find the fifth term of $(3y-2x)^9$.
Ex. 7. Raise $-a-b$ to the fourth power.
Ex. 8. Find the fifth power of $-a-b$.

EXAMPLES OF NEGATIVE AND FRACTIONAL POWERS.

Ex. 1. Extract the square root of $a^2 + b^2$ in a series; or evolve $(a^2 + b^2)^{\frac{1}{2}}$.

$$\text{Ans. } a + \frac{b^2}{2a} - \frac{b^4}{8a^3} + \frac{b^6}{16a^5} - \frac{5b^8}{128a^7} + \ldots$$

Ex. 2. Show that $\frac{1}{(a-x)^2} = \frac{1}{a^2} + \frac{2x}{a^3} + \frac{3x^2}{a^4} + \ldots.$

Ex. 3 *. Also that $\frac{a^2}{a-x} = a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \ldots.$

Ex. 4. And that $\sqrt{\frac{1}{a^2+x^2}}$ or $\frac{1}{(a^2+x^2)^{\frac{1}{2}}} = \frac{1}{a} - \frac{x^2}{2a^3} + \frac{3x^4}{8a^5} - \frac{5x^6}{16a^7} + \ldots$

Ex. 5. The expansion of $\frac{a^2}{(a-b)^2}$ is $1 + \frac{2b}{a} + \frac{3b^2}{a^2} + \frac{4b^3}{a^3} + \frac{5b^4}{a^4} + \ldots$

Ex. 6. And that of $\sqrt{a^2 - x^2}$ is $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \ldots$

Ex. 7. Show that $(a^3 - b^3)^{\frac{1}{3}} = a - \frac{b^3}{3a^2} - \frac{b^6}{9a^5} - \frac{5b^9}{81a^8} - \ldots.$

Ex. 8. The expansion of $\sqrt[5]{a^5 + x^5}$ is $a + \frac{x^5}{5a^4} - \frac{2x^{10}}{25a^9} + \frac{6x^{15}}{125a^{14}} - \ldots$

Ex. 9. And that of $\frac{a-b}{a+b}$ is $1 - \frac{2b}{a} + \frac{2b^2}{a^2} - \frac{2b^3}{a^3} + \ldots..$

Ex. 10. The cube root of $\frac{a^3}{a^3+b^3}$ is $1 - \frac{b^3}{3a^3} + \frac{2b^6}{9a^6} - \frac{14b^9}{81a^9} + \ldots$

Ex. 11. Expand $\sqrt[3]{a + \sqrt{b-c}} + \sqrt[3]{a - \sqrt{b-c}}$.

Ans. $2a^{\frac{1}{3}}\left\{1 - \frac{d^2}{9a^2} - \frac{10d^4}{243a^4} - \frac{154d^6}{6561a^6} - \frac{935d^8}{59049a^8} - \ldots\right\}$, where $\sqrt{b-c} = d$.

Ex 12. Assign the first eleven coefficients of $(1+z)^{1+z}$.

Ex. 13. Expand $\frac{1-x^{-1}}{1+x^{-1}}$, and $\frac{a - x\sqrt{-1}}{a + x\sqrt{-1}}$ into series; and find the sum and difference of $\frac{1}{\sqrt{a}+\sqrt{b}}$ and $\frac{1}{\sqrt{a}-\sqrt{b}}$ in their expanded states.

Ex. 14. It is required to find the square and the cube of the expression $\sqrt[3]{-x-y\sqrt{-1}} - \sqrt[3]{-x+y\sqrt{-1}}$.

Scholium.

The binomial development may be employed also in the extraction of roots of numbers, and sometimes with considerable advantage. It is especially the case when the number whose root is to be extracted does not differ greatly from the same power of some whole number, as in such cases the convergency is so rapid as to give eight or ten figures of the root true, by means of three or four terms of the development.

---

* Examples 3, 5, 9, may be verified by Synthetic Division; and in all cases where it can be applied, and the result merely is required, this method is rather simpler than the binomial theorem.

Let us, for instance, seek the 11th root of 2044. Here $2^{11} = 2048$, and hence we have

$$2044 = 2048 - 4 = 2048\left(1 - \frac{4}{2048}\right) = 2^{11}\left(1 - \frac{1}{8^3}\right)$$

the 11th root of which is $2 \times \left(1 - \frac{1}{8^3}\right)^{\frac{1}{11}}$.

The values of $\frac{n}{1}$, $\frac{n-1}{2}$, ... are $\frac{1}{11}$, $-\frac{10}{22}$, $-\frac{21}{33}$, $-\frac{32}{44}$, ... ad infinitum.

Hence $\left(1 - \frac{1}{8^3}\right)^{\frac{1}{11}} = 1 - \frac{1}{11}\cdot\frac{1}{8^3} - \frac{1}{11}\cdot\frac{10}{22}\cdot\frac{1}{8^6} - \frac{1}{11}\cdot\frac{10}{22}\cdot\frac{21}{33}\cdot\frac{1}{8^9} - \ldots$

By calculating only the terms here written down, we obtain $\sqrt[11]{2044} = 1{\cdot}999644570706$, true, probably, within one unit in the last figure.

When, however, the quantity v is nearly $= 1$, or indeed above $\cdot5$, the convergency becomes very slow; as it is obvious that the successive factors of the several coefficients continually increase and approach towards unity as their common limit, in all the roots. The convergency depends then on the smallness of v, which causes its powers to diminish rapidly in value.

This difficulty, however, may be completely evaded by taking two figures instead of one for the first approximation. It will be well to take 4, 5, 6, or 7, for the second figure, according as the first taken with it shall form a number decomposable into factors never greater than 12, and such as shall be supposed most likely to approximate closely to the true root. Thus, for instance, the square root of 6 gives, whilst we take only one figure, $2\,(1 + \frac{1}{2})^{\frac{1}{2}}$, or $3\;(1-\frac{1}{3})^{\frac{1}{2}}$, which would converge slowly: but if we take two figures, as $2{\cdot}5$, we have $2{\cdot}5\;(1 - \frac{1}{25})^{\frac{1}{2}}$, which converges very rapidly. And, in all cases, the binomial theorem enables us to secure this rapid convergency *.

Ex. Let the student extract the roots which follow by this method. $\sqrt[3]{7}$; $\sqrt[3]{9}$; $\sqrt{17}$; $\sqrt[5]{246}$; and calculate to six decimals the values of the following binomial surds: $3 + \sqrt[10]{\cdot9}$; $\frac{3}{4} - \sqrt[7]{\cdot006564}$; and $\sqrt{-1} \times \sqrt[3]{16}$.

THE EXPONENTIAL THEOREM.

The expression a^x takes the form of a simple term: but it is of great importance to develop it in a series proceeding according to powers of x, as in the last case. That is, to find the coefficients of the series in

$$a^x = A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots. \qquad (1)$$

Assuming this particular form, we have also

$$a^y = A_0 + A_1y + A_2y^2 + A_3y^3 + \ldots. \qquad (2)$$

But $a^xa^y = a^{x+y}$; and

$$a^{x+y} = A_0 + A_1(x + y) + A_2(x + y)^2 + A_3(x + y)^3 + \ldots \qquad (3)$$

Multiply (1) (2) together, and equate it to (3): then equating the coefficients of y^1 in the result, we have

$$A_1\left\{A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots\right\} = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \ldots$$

Again equating the homologous coefficients of x in this, we have the results

* This is the method most commonly employed by foreign mathematicians for approximating to the roots of numbers, when more figures are required than can be obtained by the logarithmic tables. *Bourdon*, *Algèbre*, p. 290.

$$A_1A_0 = A_1, \text{ or } A_0 = 1$$

$$2A_2 = A_1^2, \text{ or } A_2 = \frac{A_1^2}{1 \,.\, 2}$$

$$3A_3 = A_1A_2, \text{ or } A_3 = \frac{A_1A_2}{3} = \frac{A_1^3}{1 \,.\, 2 \,.\, 3}$$

$$4A_4 = A_1A_3, \text{ or } A_4 = \frac{A_1A_3}{4} = \frac{A_1^4}{1 \,.\, 2 \,.\, 3 \,.\, 4}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$mA_m = A_1A_{m-1}, \text{ or } A_m = \frac{A_1A_{m-1}}{m} = \frac{A_1^m}{1 \,.\, 2 \,\ldots\, m}.$$

Whence, omitting the subscribed accent from A_1, the development is

$$a^x = 1 + \frac{Ax}{1} + \frac{A^2x^2}{1 \,.\, 2} + \frac{A^3x^3}{1 \,.\, 2 \,.\, 3} + \frac{A^4x^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \ldots\ldots$$

It still remains to determine the value of A in terms of a, which may be thus effected.

Put $a^x = \left\{1 + (a-1)\right\}^x$; then expanding the binomial, we have

$$a^x = 1 + \frac{x}{1}(a-1) + \frac{x(x-1)}{1 \,.\, 2}(a-1)^2 + \frac{x(x-1)\,(x-2)}{1 \,.\, 2 \,.\, 3}(a-1)^3 + \ldots.$$

Now the coefficients of x^1 in the several terms are as follows:

$+ \quad (a-1)$ in the second term,
$- \frac{1}{2}(a-1)^2$ ...... third ....,
$+ \frac{1}{3}(a-1)^3$ ...... fourth ...,
$- \frac{1}{4}(a-1)^4$ ...... fifth ....,
....

Whence $A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \ldots.$

For the purposes of *calculation*, this expression is generally useless, on account of its want of convergency. As an analytical expression, however, it is an essential element in the deduction of the formula for logarithms; and the necessity of its calculation here is avoided by taking a different subsequent course.

LOGARITHMS.

I. DEFINITIONS AND ELEMENTARY PROPERTIES.

In the equation $a^x = N$, a is called the base of the system, N the number, and x the logarithm of N to the base a. This is generally denoted by the equation,

$$x = \log_a N, \text{ or } x = l_a N,$$

where the base of the system is written as a subscribed letter to the contractions "log" or "l" of the word logarithm. Logarithms are said to be of different systems, according to the value of the base a.

As logarithms are, by the definition, only indices of the powers of the base a, it will be obvious that the fundamental operations will be the same as those of indices already explained. It will, nevertheless, be advantageous to collect into one place, and with appropriate phraseology, the simple propositions relative to these indices which we shall have occasion to employ. They are, in fact, the rules for the use of logarithms; and the only difficulty in the inquiry is the actual calculation of the logarithms themselves.

1. The sum of the logarithms of two numbers is equal to the logarithm of their product.

For let $a^x = N$, and $a^y = N_1$: then $a^x a^y = a^{x+y} = N N_1$, or $x + y = \log_a N N_1$.

2. The difference of the logarithms of two numbers is equal to the logarithm of their quotient.

Let $a^x = N$, and $a^y = N_1$. Then $\frac{a^x}{a^y} = a^{x-y} = \frac{N}{N_1}$; or $x - y = \log_a \frac{N}{N_1}$.

3. The logarithm of the nth power of any number is equal to n times the logarithm of that number.

Let $a^x = N$; then $a^{nx} = N^n$, or $nx = \log_a N^n$.

4. The logarithm of the nth root of any number is the nth part of the logarithm of that number.

Let $a^x = N$; then $a^{\frac{x}{n}} = N^{\frac{1}{n}}$; or $\frac{x}{n} = \log_a N^{\frac{1}{n}}$.

5. If a series of numbers be taken in geometrical progression, their logarithms are in arithmetical progression.

For any number may be represented by a^m. Let a^m be the first term of the geometrical series, and a^n the ratio: then the series are

For the numbers a^m, a^{m+n}, a^{m+2n}, a^{m+3n}, ...
and for the logs. m, $m + n$, $m + 2n$, $m + 3n$ ....

and it is obvious that these logarithms are in arithmetical progression, whatever the base of the system may be.

6. The logarithm of the base in every system is 1.

For $a^1 = a$, or $\log_a a = 1$.

7. The logarithm of 1 in every system is 0.

For $a^0 = 1$, or $\log_a 1 = 0$.

8. If a table of logarithms be calculated to any one system, those for another given system can be obtained from these by the use of a constant multiplier for all the logarithms of the first table.

For let $a^M = a_1$: then $a_1{}^x = a^{Mx} = N$.

Whence, taking log of N in both systems, we have $\log_{a_1} N = x$, and $\log_a N = Mx$, where M depends upon the bases a_1 and a, and is constant for all values of x, so long as the systems remain the same.

It will therefore follow, that if we can more easily compute logarithms to one base a_1, than to any other a, we may avail ourselves of it, and convert them to another system by means of the proper multiplier M.

9. As a general mode of finding M_a, we have, from the last equation,

$$M_a = \frac{\log_a N}{\log_{a_1} N}.$$

Whence, if we can compute the logs. of any one number N in the two systems, we can obtain the requisite multiplier for all the other transformations.

The number M is called the *modulus* of the system of logarithms; and refering to the base a, it is written M_a, signifying *the modulus to the base a*.

10. If a, b be the bases of two systems, and N, N_1 any numbers whatever: then $\frac{\log_a N}{\log_a N_1} = \frac{\log_b N}{\log_b N_1}$.

For let e be the base whose modulus is unity: then we have

$$\log_a N = M_a \log_e N,\ \log_a N_1 = M_a \log_e N_1,\ \log_b N = M_b \log_e N,$$
$$\text{and } \log_b N_1 = M_b \log_e N_1.$$

Hence also $\frac{\log_a N}{\log_a N_1} = \frac{M_a \log_e N}{M_a \log_e N_1} = \frac{\log_e N}{\log_e N_1} = \frac{M_b \log_e N}{M_b \log_e N_1} = \frac{\log_b N}{\log_b N_1}$.

II. LOGARITHMIC SERIES.

In the equation $a^x = N$, to find an expression for the value of x in terms of a and N.

Raise both sides to the zth power; then we have $a^{xz} = N^z$. Develop both sides by the exponential theorem: then we obtain

$$1 + \frac{Axz}{1} + \frac{A^2x^2z^2}{1\,.\,2} + \ldots . = 1 + \frac{A_1z}{1} + \frac{A_1^2z^2}{1\,.\,2} + \ldots .$$

in which

$A=(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\ldots$; and $A_1=(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\ldots$

Equating the homologous coefficients of the indeterminate quantity z, we have from *any one* of the resulting equations, as that of z^1, for instance,

$$Ax = A_1, \text{ or } x = \frac{A_1}{A} = \frac{(N-1) - \frac{1}{2}(N-1)^2 + \frac{1}{3}(N-1)^3 - \ldots .}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \ldots}$$

which is an expression for x, the logarithm of N to the base a.

It is more usual to write n instead of $N - 1$, and M_a instead of $\dfrac{1}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \ldots}$ and this reduces the expression to

$$x = \log_a (1 + n) = M_a \left\{n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \ldots . \right\}$$

This series is not in a form well adapted for calculation, except when n is a small fraction. The following process will transform it into another adapted to any number whatever.

Substitute $-n$ for n, and write the two equations,

$$\log_a (1 + n) = M_a\left\{+ n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \ldots . \right\}$$
$$\log_a (1 - n) = M_a\left\{- n - \tfrac{1}{2}n^2 - \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 - \ldots . \right\}$$

Hence by subtraction,

$$\log_a (1 + n) - \log_a (1 - n) = \log_a \frac{1+n}{1-n} = 2M_a\left\{n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 + \ldots . \right\}$$

Let now $n = \dfrac{1}{2p+1}$: then $\dfrac{1+n}{1-n} = \dfrac{p+1}{p}$, and we have

$$\log_a \frac{p+1}{p} = \log_a (p+1) - \log_a p = 2M_a\left\{\frac{1}{2p+1} + \frac{1}{3(2p+1)^3} + \frac{1}{5(2p+1)^5} + \ldots\right\}$$

or finally, $\log_a(p+1) = \log_a p + 2M_a\left\{\dfrac{1}{2p+1} + \dfrac{1}{3(2p+1)^3} + \dfrac{1}{5(2p+1)^5} + \ldots\right\}$

Hence, whenever we can calculate $\log_a p$, we can, by means of this series, calculate $\log_a(p+1)$; and the series converges the more rapidly as p becomes greater.*

---

* Several improvements of this formula, at least in respect of practical application, have been proposed by different writers; but as the tables have already been computed and verified, they are, in this point of view, of little importance. Nevertheless, it may not be out of place to merely indicate one or two of them, referring for more ample details to the elegant little treatise of Professor Young, on the "Computation of Logarithms," second edition. 1835.

1. Put $n = \dfrac{2}{p^3-3p}$: then $\dfrac{1+n}{1-n} = \dfrac{(p-1)^2(p+2)}{(p+1)^2(p-2)}$, and we have $\log(p+2) = \log(p-2)$ $+2\log(p+1) - 2\log(p-1) + 2M\left\{\dfrac{2}{p^3-3p} + \dfrac{1}{3}\left(\dfrac{2}{p^3-3p}\right)^3 + \dfrac{1}{5}\left(\dfrac{2}{p^3-3p}\right)^5 + \ldots\ldots\right\}$

which is *Borda's theorem*, and essentially the same as *Leslie's*.

2. Put

III. ON THE COMPUTATION OF LOGARITHMS.

1. *To find the value* e *of* a *which will render* $M_{,} = 1 = \frac{1}{A}$, *or* $A = 1$.

In the exponential e^x where $A = 1$, we have $\frac{1}{A} = 1 = M_{,}$, and $A = 1$. Whence, whatever x may be, we shall have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \ldots.$$

$$\text{Whence } e^1 = 1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \ldots.$$

Computing this series to thirteen terms, we have $e = 2{\cdot}718281828 \ldots\ldots$

Logarithms calculated for the base e, or modulus 1, are called napierean, from their inventor Lord Napier. They are also often called hyperbolic logarithms, from an analogy which exists between them and the spaces contained by the rectangular hyperbola and its asymptotes. Under the latter name they are given in Hutton's Tables.

2. *To calculate the hyperbolic logarithms.*

The general series for the logarithm of $p + 1$ is, in this case,

$$\text{Log. } (p+1) = \text{log. } p + 2\left\{\frac{1}{2p+1} + \frac{1}{3(2p+1)^3} + \frac{1}{5(2p+1)^5} + \ldots\right\}$$

Now, (theor. 7, p. 249) log. $1 = 0$, and hence

$$\text{Log. } 2 = \quad 0 + 2\left\{\frac{1}{3} + \frac{1}{3\,.\,3^3} + \frac{1}{5\,.\,3^5} + \ldots\right\} = {\cdot}6931472$$

$$\text{Log. } 3 = {\cdot}6931472 + 2\left\{\frac{1}{5} + \frac{1}{3\,.\,5^3} + \frac{1}{5\,.\,5^5} + \ldots\right\} = 1{\cdot}0986123$$

$$\text{Log. } 4 = 1{\cdot}0986123 + 2\left\{\frac{1}{7} + \frac{1}{3\,.\,7^3} + \frac{1}{5\,.\,7^5} + \ldots\right\} = 1{\cdot}3862944$$

....

$$\text{Log. } 10 = 2{\cdot}1972246 + 2\left\{\frac{1}{19} + \frac{1}{3\,.\,19^3} + \frac{1}{5\,.\,19^5} + \ldots\right\} = 2{\cdot}3025851$$

And in the same way, the series may be continued to any extent required.

But since $(a^x)^n = a^{nx}$, $(a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$, $a^x\, a^y = a^{x+y}$, and $\frac{a^x}{a^y} = a^{x-y}$

the logarithms of numbers which are either products or quotients, powers or

---

2. Put $\frac{m}{n}$ for p in the equation of the text; and then if we make $m = x^4 - 25x^2$ and $m + n = x^4 - 25x^2 + 144$, we get $\log(x+5) = \log(x+3) + \log(x-3) + \log(x+4) + \log(x-4) - \log(x-5) - 2\log x - 2\,M\left\{\frac{72}{x^4-25x^2+72} + \frac{1}{3}\left(\frac{72}{x^4-25x^2+72}\right)^3 + \ldots\ldots\right\}$ which is the formula of *Haros*.

3. Put $\frac{p}{q} = \frac{1+n}{1-n}$; then we get $n = \frac{p-q}{p+q}$. Put also $p = x^6 - 98x^4 + 2401x^2$, and $q = x^6 - 98x^4 + 2401x^2 - 14400$; from which is obtained the *formula of Lavernède*, $2\log x + 2\log(x+7) + 2\log(x-7) - \log(x+8) - \log(x-8) - \log(x+5) - \log(x-5) - \log(x+3) - \log(x-3) = 2M\left\{\frac{7200}{x^6-98x^4+2401x^2-7200} + \frac{1}{3}\left(\frac{7200}{x^6-98x^4+2401x^2-7200}\right)^3 + \ldots\ldots\right\}$

roots, of numbers whose logarithms are already computed, may be obtained by much simpler means. Thus since

$e^{\cdot 6931472} = 2$, and $e^{1\cdot 0986123} = 3$, we have $e^{\cdot 6931472} \times e^{1\cdot 0986123} = 6 = e^{1\cdot 7917595}$,

or the log of 6 is obtained by adding together the logarithms of 2 and 3. Hence, instead of computing the logarithms of 4, 6, 8, 9, and 10, by the *series* above, they may be computed by simple addition, or by doubling, tripling, &c. when the number is the square, cube, &c. of a number whose logarithm has been already computed; or by subtraction or division by two, three, four, &c. in the converse cases.

$$\begin{aligned} \text{Thus, } \log 4 &= 2 \log 2. \\ \log 6 &= \log 2 + \log 3. \\ \log 8 &= 3 \log 2, \text{ or } \log 4 + \log 2. \\ \log 9 &= 2 \log 3. \\ \log 10 &= \log 2 + \log 5. \end{aligned}$$

3. *To compute the logarithms to any other base, as where $a = 10$.*

Here $M_{10} = \dfrac{\log_{10} N}{\log_e N}$. Put $N = 10$: then $M_{10} = \dfrac{\log_{10} 10}{\log_e 10} = \dfrac{1}{2\cdot 3025851} =$

lus .. $\cdot 43429448$.

$$\begin{aligned} \text{Hence } M_{10} \log_e 2 &= \log_{10} 2 = \cdot 43429448 \times \ \cdot 6931472 = \cdot 3010300 \\ M_{10} \log_e 3 &= \log_{10} 3 = \cdot 43429448 \times 1\cdot 0986123 = \cdot 4771213 \\ M_{10} \log_e 4 &= \log_{10} 4 = \cdot 43429448 \times 1\cdot 3862944 = \cdot 6020600 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ M_{10} \log_e 10 &= \log_{10} 10 = \cdot 43429448 \times 2\cdot 3025851 = 1\cdot 0000000, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

$$\begin{aligned} \text{That is, } 10^0 &= 1 \\ 10^{\cdot 3010300} &= 2 \\ 10^{\cdot 4771213} &= 3 \\ &\ldots\ldots \\ 10^1 &= 10 \\ &\ldots\ldots \end{aligned}$$

IV. ON TABLES OF LOGARITHMS.

To render logarithms really useful in computation, we must have them registered in tables. To compress them into the smallest possible space, and at the same time render them convenient for use, several contrivances have been adopted. The editors of such tables differ in the minutiæ of their arrangements; but the general principles of their construction are alike in all. Those in most general use are Dr. Hutton's, and hence this description will have reference especially to the last edition of that work. The great accuracy, too, of these tables, independent of the convenience of their arrangement, is a strong reason for this choice.

1. *Definitions.*

(1.) The *significant* figure of a number N is the figure which stands highest in the numerical scale. Thus 5 is the significant figure of 54·69, of 5·469, of ·5469, of ·05469, of ·005469, &c.

(2). The *distance of the significant figure* is the number of places in the decimal scale which it is distant from the units' figure : it is considered positive when to the left, and negative when to the right of the units' place. Thus in 5469, the significant figure is + 3, being in the third place to the left of the units' place 9: in 0·05469 it is — 2, (or as more conveniently written, $\bar{2}$) being in the second place to the right of the units' place 0·.

(3). The integer part of a logarithm is called the *characteristic* or *index* of that logarithm.

(4). The decimal part, which is always positive, is called the *mantissa* of the logarithm.

(5). The *arithmetical complement* of a logarithm is its defect from 10.

2. *Tabular theorems.*

(1). The removal of the unit-place, whilst the effective figures composing N remain the same, will alter the characteristic but not the mantissa.

For let $\log_{10} N = m + d$, m being any integer number whatever, positive or negative, and d the decimal part, always positive. Then the removal of the decimal point in N, p places will be the same as multiplying N by 10^p, where p is positive if the removal of the unit-place be to the right, and negative if to the left. In this case we have

$$\log_{10} 10^p N = \log_{10} 10^p + \log N = p + \log N = p + m + d;$$

and since p is an integer, $p + m$ is an integer. Whence the decimal d, or *mantissa,* remains the same, whilst the characteristic is increased or diminished by p, according as p is + or —.

This, of course, is to be understood in reference to the fourth definition ; that the mantissa is always to be taken positive.

(2). The characteristic of a logarithm is that number which expresses the distance of the significant figure from the unit-place.

Let the number N lie between 10^{p+1} and 10^p; then its logarithm lies between $p + 1$ and p ; or it is p + decimal. But the number N is composed of $p + 1$ places, or its significant figure is p places to the left of the unit-place.

If p be negative, then the number lies between 10^{-p+1} and 10^{-p}; and hence the logarithm is $-p$ + decimal. But the number 10^{-p} commences in the pth decimal place, and hence p places to the right of the unit-place.

(3). When we have to subtract a logarithm from another, we may add its arithmetical complement and subtract 10 from the sum.

For $\log a - \log b = \log a + (10 - \log b) - 10$.

(4). When m is very small with respect to N, we shall have, very nearly,

$$\log_a \left\{1 + \frac{m}{N}\right\} = m \log_a \left\{1 + \frac{1}{N}\right\}.$$

For developing these we have

$$\log_a \left\{1 + \frac{m}{N}\right\} = M_a \left\{\frac{m}{N} - \frac{m^2}{2N^2} + \frac{m^2}{3M^3} - \dots\right\}$$

$$m \log_a \left\{1 + \frac{1}{N}\right\} = M_a \left\{\frac{1}{N} - \frac{1}{2N^2} + \frac{1}{3M^3} - \dots\right\}$$

Now since $\frac{m}{N}$ is a very small fraction, the succeeding terms $\frac{m^2}{2N^2}, \frac{m^3}{3M^3}, \dots$ will be still smaller, and have their significant figures further and further removed from the unit-place : and if they be so taken that the significant figure fall more

remotely from that place than the extent to which we calculate our table, these terms may be rejected as insensible. In this case we have

$$\log_a\left\{1+\frac{m}{N}\right\} = M_a.\frac{m}{N}, \text{ and } m\log_a\left\{1+\frac{1}{N}\right\} = M_a.\frac{1}{N};$$

and since the second terms are equal, the first are so, or

$$\log_a\left\{1+\frac{m}{N}\right\} = m\log_a\left\{1+\frac{1}{N}\right\}.$$

3. *Description of the Tables.*

1. The *hyperbolic or napierean logarithms* are given for numbers from 1·01 to 10 to a *mantissa* of seven places with the proper *characteristics,* in Table V. pp. 219—223, Hutton's Tables. Then in Table VI. are given those from 1 to 1200 for every unit.

The number is given in the column headed "N," and the corresponding logarithm in the adjacent column headed "Logar."

These logarithms are only used in calculating integrals: those used for all other purposes being to the base $a = 10$.

2. The *common,* or *Briggs's logarithms*, are given, characteristics and mantissæ, for the numbers from 1 to 100 in Table 1, p. 2. These occupy the first two pairs of columns.

The numbers from 100 to 999 occupy to the bottom of p. 5; but the characteristics are not inserted, they being always determinable by inspection, theorems (1), (2).

The mantissæ of the logarithms of all numbers composed of *four* places follow these, forming the columns headed "0," from p. 6 to 185; and tabulated as the last, without the characteristics.

The three leading figures of the mantissæ are omitted from those of all the logarithms after the first in which they occur, and the places they would occupy left blank. These spaces are, therefore, to be understood as occupied by the three figures which occur above them. Thus, mantissa of log. 1091 (p. 7) is to be read ·0378248.

The mantissæ of those for numbers of *five* places, as far as 10799, are given in the same manner from p. 186 to 201.

The mantissæ of the logarithms of all numbers of five places are given on the same pages as those of four. This method has been adopted on account of the three leading figures of these mantissæ being the same for several succeeding logarithms; and thereby rendering it only requisite to repeat in the table the last four.

Thus mantissa of log. 10500 is ·0211893,
.... 10501 is ·0212307,
.... 10502 is ·0212720,

and so on. Hence the first four figures 1050 are given on the left margin, and the mantissa of its logarithm (which is the same with that of 10500, theor. 1) is given in the column under the fifth figure 0 at the head. The first four figures of the mantissa of 10501 being the same, and the mantissa itself being the same in the first three places, these figures ·021 are taken from the first column headed "0," and the remaining four, viz. 2307, from that headed "1": thus giving mantissa of log. 10501 equal to ·0212307, as above.

Whenever there is a change of the third figure of the mantissa in any of the columns not headed "0", the circumstance is indicated by a small line drawn

over the fourth figure of that mantissa, and in this case the first three figures will be taken from column 0 in the horizontal line immediately below. Thus mant. log. 10544 is ·0230054, and not ·0220054. In like manner, mant. log. 10545 is 0230466, and so on to mant. log. 10549 : and at 10550 the first three figures take their regular position in the horizontal lines.

The small tables on the right hand of the page, marked "Dif. and pro. pts.," enable us to obtain the mantissæ of logarithms to numbers of six places of figures. They are constructed on the principle of theorem 4, p. 253.

It will be seen, that the differences between the logarithms of two consecutive numbers of five places vary very slowly, or are nearly the same for several numbers together. It amounts to this, that of several consecutive (and therefore in arithmetical progression) numbers of five figures, the logarithms are also nearly in arithmetical progression; and hence also must the logarithms of numbers in arithmetical progression, and lying between any two consecutive numbers, be also in arithmetical progression But to appeal to theorem 4, we have

$$\log_{10} (\mathrm{N} + m) = \log_{10} \mathrm{N} + \log_{10} \left(1 + \frac{m}{\mathrm{N}}\right) = \log_{10} \mathrm{N} + \mathrm{M}_{10} \cdot \frac{m}{\mathrm{N}},$$

$$\text{or } \log_{10} (\mathrm{N} + m) - \log_{10} \mathrm{N} = \mathrm{M}_{10} \cdot \frac{m}{\mathrm{N}} = m \cdot \frac{\mathrm{M}_{10}}{\mathrm{N}}.$$

The tablets referred to are composed of the values of this expression for different values of N and m, in the manner of the following example.

Let $\mathrm{N} = 10000$, and $m = 1$. Then $\mathrm{M}_{10} = \cdot 43429448$.

Hence $\log_{\cdot 10} 10001 - \log_{\cdot 10} 10000 = \cdot 000043429 \ldots.$

or to seven places taking the nearest number, it is ·0000434, the effective figures of which are put down at the head of the tablet-column at p. 6 of the tables. In the same manner are the headings of all the tablets calculable. They were, however, not found in this manner; but by subtracting the calculated logarithms, that of each number from that of its next higher consecutive number differing by 1 in the unit-place.

The parts against the numbers 1, 2, 3 .... 9, in the tablets are the values of $\log_{10} (\mathrm{N} + m) - \log_{10} \mathrm{N}$ for the several values ·1, ·2, ·3, . . . ·9, or for the numbers 1, 2, 3, in the number whose sixth figures are 1, 2, ... 9 in the unit-place. These are called proportional parts of the logarithm for the sixth figure, and are inserted for the purpose of being taken out by inspection, instead of having to compute them in each individual case. These corrections are additive to the logarithm if taken to the first five figures of N, and subtractive if to the first five figures of N + 1. The former is the most convenient, and most generally adopted.

Thus, to recur to the first tablet, and dropping the ciphers, we have

$434 \times \cdot 1 = 43$, giving	1	43,
$434 \times \cdot 2 = 87$	2	87,
$434 \times \cdot 3 = 130$	3	130, and so on.

As for the smaller values of N to six places, especially under 107999, the values of $\frac{m}{\mathrm{N}} \cdot \mathrm{M}_{10}$ vary more rapidly than in other higher numbers of the same local extent, the mantissa for six places have been given in the tables from p. 186 to 201, with their tablets of proportional parts for the seventh figure in the unit-place. These are to be used where great accuracy of approximation is sought, (which, however, is rarely necessary) and their structure is the same as already described.

Of the tables to twenty places, it is unnecessary to say anything here, as well

as of those that follow: since they are well described in the introductory matter of the volume.

4. *The usage of the Tables.*

The direct use, viz. taking out the logarithms of numbers, is mainly implied in what has been said on the structure of the tables. It will, however, be well to recapitulate briefly in a didactic form the processes.

I. *The characteristic.* Count how many places to the right or left of the unit-place the significant figure stands. This number is the characteristic (theor. 2, p. 253); and is marked *minus* if to the right, and considered *plus* if to the left.

II. *The mantissa.* 1. If the effective figures be not more than four, the mantissæ of their logarithms will be found in juxta-position with them in the tables, and may be taken out at once *.

2. If the effective figures be five, find the first four in the column marked N, and the fifth in the horizontal line at the top. The last four figures of the mantissa are found at the angle formed by the horizontal and vertical lines in which the first four and the fifth figures are situated, meet: and the first three figures of the mantissa adjacent to the first four figures in the horizontal line, or in that immediately below, according to the explanation already given.

Thus, to find the mantissa of log. 74695, look for 7469 (p. 135, Tables) in the column N, and for 5 in the horizontal line at the top. We find at the angle of the lines in which 7469 and 5 stand, the last four figures of the mantissa, viz. 2915, and adjacent (the blank expressing the number above) to it, the first three, viz. 873. So that the mantissa is ·8732915.

Or again, had we sought the logarithm of 74819, the last four figures are $\bar{0}119$; and the dash over the 0 signifying that instead of 873 we must take 874 from the line immediately *below* that which contains 7481, for the first three figures of the mantissa. Hence the mantissa of 74819 is ·8740119.

3. If the number be composed of six effective places of figures, find for the first five as just directed. In the marginal tablet marked "pro," look for the sixth figure, and place the adjacent number below the number already found: add them together: then the sum is the mantissa of the six figures. This is obvious from theor. 4, and from what is there said on the subject.

4. If the given number be a vulgar fraction or mixed number, the fractional part may be reduced to a decimal, and the logarithm of the expression then taken. But if the decimal be of many places, it will be better reduced to a vulgar fraction, and the process adapted to division by logarithms followed.

Note. In actual practice, it is better to write down the first three figures before looking out the remaining four; though, for convenience of explanation, they have been spoken of in a reverse order.

The inverse use, that of finding the number when its logarithm is given, will, obviously, be equally simple and easy.

5. If the mantissa appear exactly in the table, we have but to write it down, and assign the unit-place in conformity with the rule already laid down according to the characteristic.

---

* In the logarithms of the numbers from 1 to 99, the characteristics are given also, on the supposition of the number being entirely integer. When this is not the case, the characteristic must be assigned according to the general rule.

6. If the mantissa be not found exactly in the table, take out that next less than the given one. Write this under the given one, at least as far as the figures are not common, and subtract it. With the difference thus found, enter the second column of the tablet marked " Pro." and the number adjacent to it is the sixth figure of N, and the first five those belonging to the next lower mantissa. The unit-place is, as before, to be assigned from the characteristic.

Thus, if $\log N = \bar{2}\cdot8730158$, and we require N; take out the mantissa, and write it under the given one, according to the following type.

$$\begin{array}{l} \bar{2}\cdot8730158 \\ \quad\quad\quad 24 = \text{mant. of } 74647 \\ \hline \quad\quad\quad 34 = \text{pp. 6 nearly, from the tablet;} \end{array}$$

and we have 746476 for the number, so far as the mantissa is concerned; and from the characteristic we have the first figure in the second place to the right of the unit-place. Hence the number sought is ·076476 very nearly.

V. LOGARITHMIC OPERATIONS.

1. *Multiplication by logarithms.* Add the logarithms of all the factors together, and the number whose logarithm is the sum will be the product. Theor. 1, p. 249.

Note 1. When some or all the characteristics are negative, the rules of algebraic addition must be employed.

Note 2. When the given numbers are of six places, it will not be necessary to do more to each of them in the way of correction separately, than to write the corrections down beneath the corresponding logarithms of five places, and at last add all the corrections and logarithms into one sum.

Ex. Find the product of ·002356, 47·2985, ·32986, 42·7579, and ·00004965.

$$\begin{array}{rrl} \log \cdot002356 = & \bar{3}\cdot3721753 & \text{p. 33, Tables.} \\ \log 47\cdot2980 = & 1\cdot6748428 & \rbrace \text{(to five places) p. 80.} \\ \text{prop. part for } 5 = & 47 & \\ \log \cdot32986 = & \bar{1}\cdot5183297 & \text{p. 51.} \\ \log 42\cdot7570 = & 1\cdot6310072 & \rbrace \text{(to five places) p. 71.} \\ \text{pp. for } 9 = & 91 & \\ \log \cdot00004965 = & \bar{5}\cdot6959193 & \text{p. 85.} \\ \hline & \bar{5}\cdot8922881 & \text{log of product.} \\ & 39 & = \text{mantissa of } 78034. \\ \hline & 42 & = \text{pp. for 8 nearly.} \end{array}$$

Hence, by the characteristic, the fifth decimal place is the significant figure, and the number composed of the digits 780348, the product itself is ·0000780348 nearly.

This example contains instances of every possible variety of case that can occur.

EXAMPLES FOR EXERCISE.

1. Multiply 498·256 by 41·2467. Ans. 20551·4.
2. Multiply 4·02674 by ·0123456. Ans. ·0497125.
3. Multiply 3·12567 by ·02868 by ·12379. Ans. ·01109705.
4. Find the product 2876·9 × ·10674 × ·098762 × ·0031598. Ans. ·095830.

2. *Division by logarithms.* Subtract the logarithm of the divisor from that of the dividend: the remainder is the logarithm of the quotient. See Theor. 2, p. 249.

Note. In accordance with what is shown at p. 253, we may use the arithmetical complement of the subtractive logarithms, which will often much facilitate the operation. To effect this, instead of taking the several figures of the logarithm from the table, write (which can be, with little practice, done by inspection) the complement of each figure of the logarithm from 9 except the last, and the complement of this from 10.

For example, find the arithmetical complement of log. 37·5 and of ·00375.

$$\begin{array}{rr|rr} & 10{\cdot}0000000 & & 10{\cdot}0000000 \\ \log 37{\cdot}5 = & 1{\cdot}5740313 & \log {\cdot}00375 = & \bar{3}{\cdot}5740313 \\ \hline \text{ac.} \log 37{\cdot}5 = & 8{\cdot}4259687 & \text{ac.} \log {\cdot}00375 = & 12{\cdot}4259687 \end{array}$$

in which, without writing down either of the lines, the arith. comp. may be written down from the inspection of the logarithms themselves in the table. Had there been six figures, the correction for the sixth might have been subtracted from the result of the addition, as in example 1. which follows.

This method of work should be early and regularly practised, on account of its almost constant occurrence in trigonometrical calculations.

When, however, there are several subtractive logarithms, it will be better for the most part to add them into one sum, and place the arithmetical complement of the whole under the column of additive ones, as in *Ex.* 2.

Ex. 1. Divide 1728·95 by ·110678.

$$\begin{array}{rl} \text{Log } 1728{\cdot}9 = & 3{\cdot}2377699 \\ \text{pp.} \quad 5 = & \quad\quad\quad 126 \\ \text{ac. log } {\cdot}11067 = & 10{\cdot}9559701 \\ \hline & 4{\cdot}1937526, \text{ subtracting } 10 \\ \text{pp.} \quad 8 = & \quad -317 \\ \hline & 4{\cdot}1937209 \\ & \quad\quad 088 = \text{mantissa } 15621 \\ \hline & \quad\quad 121 = \text{pp.} \quad\quad 4 \text{ nearly,} \end{array}$$

and characteristic 4 gives five places of integers: hence the quotient is 15621·4 nearly.

Ex. 2. Find the value of the expression $\dfrac{3{\cdot}1416 \times 82 \times \frac{73}{41}}{{\cdot}02912 \times 751{\cdot}3 \times \frac{6}{941}}$.

This, in a simplified state, is $\dfrac{3{\cdot}1416 \times 82 \times 73 \times 941}{{\cdot}02912 \times 751{\cdot}3 \times 6 \times 41}$

$$\begin{array}{rl} \log 3{\cdot}1416 = & 0{\cdot}4971509 \\ \log \quad 82 = & 1{\cdot}9138139 \\ \log \quad 73 = & 1{\cdot}8633229 \\ \log \quad 941 = & 2{\cdot}9735896 \\ \hline & 7{\cdot}2478773 = \text{log numerator.} \\ \hline \log {\cdot}02912 = & \bar{2}{\cdot}4641914 \\ \log \quad 751{\cdot}3 = & 2{\cdot}8758134 \\ \log \quad 6 = & 0{\cdot}7781513 \\ \log \quad 41 = & 1{\cdot}6127839 \\ \hline & 3{\cdot}7309400 = \text{log denominator.} \\ & 3{\cdot}5169373 = \text{differ.} = \text{log quotient.} \end{array}$$

Or thus, and better, by the arithmetical complements:

$$\begin{array}{rrcl}
\log & 3\cdot1416 & = & 0\cdot4971509 \\
\log & 82 & = & 1\cdot9138139 \\
\log & 73 & = & 1\cdot8633229 \\
\log & 941 & = & 2\cdot9735896 \\
\text{ac. log} & \cdot02912 & = & 11\cdot5358086\,^{*} \\
\text{ac. log} & 751\cdot3 & = & 7\cdot1241866 \\
\text{ac. log} & 6 & = & 9\cdot2218487 \\
\text{ac. log} & 41 & = & 8\cdot3872161 \\
\hline
 & & & 3\cdot5169373 = \text{log. quotient, as before.} \\
 & & & 18 = \text{mant. } 32880 \\
\hline
 & & & 55 = \text{pp.} \qquad 4 \text{ nearly.}
\end{array}$$

Hence the quotient is 3288·04 nearly.

EXAMPLES FOR EXERCISE.

Ex. 3. Divide ·06314 by ·007241. Ans. 8·71979.

4. Divide ·7438 by 12·9476. Ans. ·057447.

5. Divide ·102367 by 4·96523. Ans. ·0206168.

6. Divide ·06314 × ·7438 × ·102367 by ·007241 × 12·9476 × ·496523, and compare the result with the product 8·71979 × ·057447 × ·0206168.

Ans. They ought to be identical, or within a unit in the last place.

7. Divide ·0067859 by 123459. Ans. ·0000000549648.

3. *Proportion by logarithms.* This is only the application of logarithms to the operations of multiplication and division implied in finding the fourth term. Thus, if $a : b :: c : x$, then $x = \dfrac{bc}{a}$, and we have

$\log x = \log b + \log c + \text{ac. log } a - 10.$

EXAMPLES FOR PRACTICE.

Ex. 1. If 12·678 : 14·065 :: 100·979 : x, then x = 112·027.

2. If 1·9864 : ·4678 :: 50·4567 : x, then x = 11·8826.

3. If ·498621 : 2·9587 :: 2·9587 : x, then x = 17·5562.

4. *Involution by logarithms.* In conformity with what is shown in theor. 3, p. 249, we have $\log a^n = n \log a$; which gives the process:

Multiply the logarithm of the base by the index of the power: the product is the log. of the power.

EXAMPLES.

Raise ·09163 to the 4th power.

$$\begin{array}{rr}
\log \cdot09163 = & \bar{2}\cdot9620377 \\
\text{index} = & 4 \\
\hline
 & \bar{5}\cdot8481508 \\
 & 460 \\
\hline
 & 48
\end{array}$$

Hence $(\cdot09163)^4 = \cdot0000704938$.

Or, in a more illustrative form,

$$\begin{array}{rr}
\log \cdot09163 = & -2 + \cdot9620377 \\
 & 4 \\
\hline
 & -8 + 3\cdot8481508
\end{array}$$

which shows the process more at length, though obviously it need not be so put down.

* Here the 11 comes from $9 - \bar{2} = 9 + 2$.

Ex. 2. Find the value of $(1{\cdot}0045)^{365}$.

$$
\begin{array}{rl}
\log 1{\cdot}0045 = & {\cdot}00194994 \text{ tables, p. 186} \\
 & \quad 365 \text{ index} \\
\hline
 & 584982 \\
 & 1169964 \\
 & \quad 974970 \\
\hline
 & {\cdot}71172810 \\
 & \qquad 29 = \text{mant. } 51490 \\
\hline
 & \qquad 52 = \text{pp.} \qquad 6
\end{array}
$$

Hence the answer is $1{\cdot}0045^{365} = 5{\cdot}14906$ nearly.

EXAMPLES FOR EXERCISE.

Expressions,	$6{\cdot}05987^2$,	${\cdot}176546^3$,	${\cdot}076543^4$,	$1{\cdot}09684^7$,	$-1{\cdot}06524^3$.
Values	$36{\cdot}72203$,	${\cdot}00550267$,	${\cdot}0000343259$,	$1{\cdot}90986$,	$-1{\cdot}20877$.

5. *Evolution by logarithms.* Divide the log. of the number by the index of the root: the quotient is the log of the root: theor. 4, p. 249, where it is shown that $\frac{1}{n}\log a = \log a^{\frac{1}{n}}$.

The only difficulty that can present itself is where the characteristic is negative, and not divisible by the index of the root. To remove this, add a negative number to the characteristic sufficient to render it the *next higher multiple of the index*, and add the same number taken positively to the positive part of the logarithm, that is to the mantissa. The quotient of the characteristic will in this case be a negative integer, and the quotient of the positive part of the expression will be decimal, and form the mantissa of the required logarithm.

The following example will illustrate this.

Find the cube root of ${\cdot}000486296$.

$$
\begin{array}{rr}
\log {\cdot}00048629 = & \bar{4}{\cdot}6868953 \\
\text{pp. } 6 = & 53 \\
\hline
3 \,\big|\, & \bar{4}{\cdot}6869006 \\
\hline
 & \bar{2}{\cdot}8956335
\end{array}
$$

Here $\bar{4}{\cdot}6869006 = -4 + {\cdot}6869006 = -6 + 2{\cdot}6869006$;
and each of the terms divided by 3 gives

$$\log \text{root} = -2 + {\cdot}8956335 = \bar{2}\,8956335.$$

When the index is fractional, as $a^{\frac{m}{n}}$, we have

$$\log a^{\frac{m}{n}} = \frac{m}{n}\log a;$$

and the process obvious. In fact, involution and evolution by logarithms are the same rule, just as in common algebra under the same circumstances.

EXAMPLES FOR PRACTICE.

Expressions, $365{\cdot}567^{\frac{1}{2}}$, $2{\cdot}98763^{\frac{1}{3}}$, ${\cdot}967845^{\frac{1}{4}}$, ${\cdot}098674^{\frac{1}{7}}$ $\left(\frac{21}{373}\right)^{\frac{2}{3}}$, and $\left(\frac{112}{1727}\right)^{\frac{3}{5}}$.

Values 19·1198, 1·44027, ·9918624, ·718315, ·146895, and ·1937115.

MISCELLANEOUS EXERCISES ON LOGARITHMS.

1. Find the values of $3{\cdot}1416 \times 82 \times \frac{73}{41}$, and ${\cdot}02912 \times 751{\cdot}3 \times \frac{6}{941}$.

2. Find x in $7241 : 3{\cdot}58 :: 20{\cdot}46 : x$, and in $\sqrt{724} : \sqrt{\frac{5}{13}} :: 6{\cdot}927 : x$.

3. Find the square root of $\frac{2}{123}$, and the cube root of $\frac{1}{3{\cdot}14159}$.

4. Find the values of $\frac{(\frac{2}{3})^{\frac{1}{2}} \cdot (\frac{3}{4})^{\frac{1}{3}}}{17\frac{1}{3}}$, and $\frac{1}{7}\sqrt{\frac{5}{8}} \times {\cdot}012\sqrt[3]{\frac{7}{11}}$.

5. Assign the values of $\frac{\frac{1}{9}\sqrt{\frac{11}{21}} \times {\cdot}03\sqrt[3]{15\frac{1}{3}}}{7^{\frac{1}{3}} \cdot \sqrt[3]{12\frac{1}{3}} \times {\cdot}19\sqrt[4]{17\frac{1}{8}}}$ and $127^{\frac{1}{4}} \cdot \frac{\frac{5}{6}\sqrt{19} + \frac{4}{7}\sqrt[3]{35\frac{1}{3}}}{14\frac{7}{19} - \frac{1}{11}\sqrt[5]{28\frac{2}{3}}}$.

6. Find the fifth root of ${\cdot}00065$, the tenth root of $-{\cdot}001$, and the third root of $-{\cdot}00006$: and show whether they be all real or not.

7. Find the value of $\sqrt[5]{\frac{2^{\frac{1}{2}}}{3^{-\frac{1}{3}}}} \times 7^{\frac{1}{3}}$, and the 5th root of $\frac{3172 \cdot 3^{\frac{1}{2}} \cdot 5^{\frac{1}{3}}}{251}$.

8. Find the value of $-\left(\frac{60}{59}\right)^{-1} \times \frac{\sqrt{-3 \times -53}}{\sqrt[3]{-100}}$, and of $\frac{1}{7}\sqrt[3]{\frac{5}{8}} \times {\cdot}034\sqrt{\frac{8}{13}}$.

9. Of how many figures does the number represented by 2^{24} consist? And of how many does 9^9 consist?

10. Which is the greatest and which the least of the three numbers, 10^{10}, 9^{11}, or 11^9? And show how to determine generally which is the greater, a^b or b^a, supposing a greater than b.

11. Find the logarithm of $22{\cdot}5$, having given the logs of 2 and 3.

12. Having given the logs of 6 and 15 to find those of 8 and 9.

13. Given the logs of 2 and 3 to find those of $\frac{9}{16}$ and $\frac{2}{375}$.

14. Given the logs of 2, 3, 13, to find those of $\left(\frac{24}{135}\right)^{\frac{1}{3}}$ and $\sqrt[4]{1{\cdot}625}$.

15. Given $\log_e 15 = 2{\cdot}7080502$, $\log_e 5 = 1{\cdot}6094379$, to find $\log_3 25$.

16. Given $\log_e 2 = {\cdot}6931472$, $\log_e 5 = 1{\cdot}6094379$, and $\log_{10} 1{\cdot}9 = {\cdot}2787536$, to find $\log_e {\cdot}0019$.

THE SOLUTION OF EXPONENTIAL EQUATIONS.

An exponential equation is one in which the unknown appears in the form of an exponent or index. When in this form, unmixed with other combinations, the solution is readily obtained by means of logarithms. Thus, if $a^x = N$, then $x \log a = \log N$, and $x = \frac{\log N}{\log a}$, which is the general form of solution.

Moreover, it makes no analytical difference how complexly the base or the index be given, the same method of solution applying to these as to the simpler form given above, provided the combinations be by multiplication, division, involution, or evolution only.

When, however, the unknown appears as an index, and in any other character, for instance, as a base, or as an addend; or if two exponentials be connected by other signs than those of multiplication or division: then the solution can only be attained, in general, by means of trial and error, or some mode of approximation tantamount to it, having previously simplified the expressions by means of the logarithmic formulæ.

The following examples will illustrate the processes to be employed.

Ex. 1. Given $2^x = 769$, to find x. Here we have, as in formula above,

$$x = \frac{\log 769}{\log 2} = \frac{2{\cdot}8859263}{0{\cdot}3010300} = 9{\cdot}586839.$$

Ex. 2. Find x from the equation $a^{mx}\, b^{nx} = c$.

In this $mx \log a + nx \log b = \log c$, or $x = \dfrac{\log c}{m \log a + n \log b}$.

Ex. 3. Given $\left(\frac{5}{4}\right)^x = 54\frac{1}{2}$, to find x.

Here $x \log \frac{5}{4} = \log \frac{109}{2}$, or $x = \dfrac{\log 54{\cdot}5}{\log 1{\cdot}25} = 17{\cdot}9177$.

Ex. 4. Given $a^{b^x} = c$ to find x *.

Put $b^x = z$: then $a^z = c$, and hence $z = \dfrac{\log c}{\log a}$.

Again, since $b^x = z$, x † $= \dfrac{\log z}{\log b} = \dfrac{\log\left(\dfrac{\log c}{\log a}\right)}{\log b}$.

Ex. 5. Given $a^x\, b^z = k$, and $x : z :: r : s$, to find x and z.

These give $x \log a + z \log b = \log k$, and $z = \dfrac{sx}{r}$.

Insert the value of z from the second of these results in the first: then $\left\{\log a + \frac{s}{r} \log b\right\}x = \log k$, and resolving, we get

$$x = \frac{r \log k}{r \log a + s \log b}\text{; and hence } z = \frac{sx}{r} = \frac{s \log k}{r \log a + s \log b}.$$

Ex. 6. Given $a^x \div b^z = k$, and $x : z :: r : s$, to find x and z.

This gives, in the same way as before,

$$x = \frac{r \log k}{r \log a - s \log b} \text{ and } z = \frac{s \log k}{r \log a - s \log b}.$$

Ex. 7. Given $m^{x^2} . n^{\frac{1}{2}z^2} = a$, and $x : z :: r : s$, to find x^2 and z^2.

---

* In reading expressions of this form, it will be very important to keep in mind the signification of the notation. In the present case, it signifies a raised to the power of b^x; and however many, n, successive exponents there be, the valuation is supposed by raising the $(n-1)$th exponent to the power denoted by the nth: then raising the $(n-2)$ exponent to the power denoted by the last-mentioned: and so on, till we come to the base, which is raised to a power denoted by the result of all the previous involutions.

† In numerical solutions of such quantities, it will be *convenient* to calculate, in all cases, the value of z before proceeding to the final equation which gives the value of x: but in certain cases it will be essential to do so, as it may happen that c and a, being numbers less than unity, the real values of their logs. being then negative, we cannot express $\log {}^2a$ and $\log {}^2c$ without the introduction of the imaginary symbol. The solution is, however, generally expressible symbolically by $x = \dfrac{\log^2 c - \log^2 a}{\log b}$.

Here, taking the log of the first equation, and reducing the second, we get, after substitution,

$$2r^2 x^2 \log m + s^2 x^2 \log n = 2r^2 \log a, \text{ or } x^2 = \frac{2r^2 \log a}{2r^2 \log m + s^2 \log n}$$

$$\text{and } z^2 = \frac{s^2x^2}{r^2} = \frac{2s^2 \log a}{2r^2 \log m + s^2 \log n}.$$

Ex. 8. Given $m^{x^2} n^{z^2} = a$, and $x : z :: r : s$, to find x and z.

Here we have $x^2 \log m + z^2 \log n = \log a$, and $z = \frac{sx}{r}$.

By substitution $r^2 x^2 \log m + s^2 x^2 \log n = r^2 \log a$, or

$$x = \pm \sqrt{\frac{r^2 \log a}{r^2 \log m + s^2 \log n}}\text{; and from this again,}$$

$$z = \frac{sx}{r} = \pm \sqrt{\frac{s^2 \log a}{r^2 \log m + s^2 \log n}}.$$

Ex. 9. Find the value of x in the equation $x^x = 100$.

As an initial experiment, take $x_1 = 3$, and $x_2 = 4$. Then $3^3 = 27$, and $4^4 = 256$, one of which being considerably too little, and the other considerably too great, we may take for x_1 or x_2 a number midway between 3 and 4, with a prospect of a near approximation; and from the result, judge of the other value whether greater or less than 3·5. Also taking logs. we have x log. $x =$ log. $100 = 2$.

First let $x_1 = 3·5$; then,

$3·5 \log 3·5 = 1·9042380$
true no. $= 2·0000000$

too little by ·095762

Second, let $x_2 = 3·6$; then,

$3·6 \log 3·6 = 2·002689$
true no. $= 2·000000$

too great by ·002689

Hence, pp. 203, 204, $x = 3·59727$ nearly; the extent of the approximation, however, being less clear in equations of this class, than in purely algebraical ones. It has been often the case, that approximations have been trusted to too far; as in the example above given, for instance, though a rather favourable one for rapidity of approximation. By forming the value of the expression for $x = 3·5973$, we find it too great by ·0000149, and for $x = 3·5972$, too little by ·0000841. From these values we may obtain a further correction by a repetition of the use of the formula.

This method may, for practical purposes, be a little improved. For since $x^x = a$, we have, as before, $x \log x = \log a$. Put now $\log x = y$, and $\log a = b$: then we have $xy = b$; and taking the logs we get $\log x + \log y = \log b$, or since $\log x = y$, this is $y + \log y = \log b$. This may be solved for y by Trial and Error, and the value thus found will be $\log x$, and x becomes known. The following example is given in illustration.

Ex. 10. Given $x^x = 123456789$ to find the value of x.

Here $x \log x = xy = \log 123456789 = 8·0915148 = b$; and $y + \log y = \log b = ·9080298$.

Taking $y_2 = 1$, we have $1 + \log 1 = 1$, too great by ·0919702.

Taking $y_1 = ·9$, we have $·9 + \log ·9 = ·8542425$, too little by ·0537873.

Hence by the rule we have $y = ·93$ nearly; and we may proceed to a second approximation.

Taking $y_1 = ·93$, we have $y + \log y = ·8984829$, too little by ·0095469.

Taking $y_2 = ·94$, we have $y + \log y = ·9131279$, too great by ·0050981.

Hence again by the rule we have $y = ·93652$ nearly; and we may again proceed to approximate still more closely.

The next step gives $y = ·93651503$; and from this we have $x = 8·640026$, which is true in the last figure.

This method of solution becomes inapplicable when the equation is $x^x = \frac{1}{a}$, and a not less than unity; but it is easy to transform it so as to find the reciprocal of x, and thence x itself.

Put $x = \frac{1}{z}$: then we get $\left(\frac{1}{z}\right)^{\frac{1}{z}} = \frac{1}{a}$, and thence $a^z = z$. Take the logs putting $\log z = u$, and we get $z \log a = \log z = u$; and taking the logs again, we find $\log z + \log^2 a = \log u$, or $\log u - u = u \log^2 a$; an equation of the same nature as that of the last example, and soluble by a similar process *.

EXERCISES IN EXPONENTIAL EQUATIONS.

1. Find the value of x in the equation $\left(\frac{\sqrt{7}}{\sqrt[3]{8}}\right)^{2x} = 2^{\frac{1}{3}} . 6^{\frac{1}{2}}$. Ans. $x = 2·01374$.

2. Find y from $2^{3y} = 10$, and x from $(2^3)^x . 9^x = 4\frac{9}{10}$.
Ans. $y = 1·1073093$, and $x = ·371606$.

3. Resolve the equation $2^{3^x} = 2^4$. Ans. $x = 1·26186$.

4. Given $\frac{x}{y} = \frac{3}{4}$ and $3^x = (4^2)^{y^2}$. Ans. $x = ·222885$, $y = ·297181$.

5. If $v : z :: z : 4$, and $4^{2z} = 9^{3v}$, what are the values of v and z?
Ans. $v = 0$, $z = 0$; and $v = \frac{16}{9}\left(\frac{\log. 2}{\log. 3}\right)^2$, $z = \frac{8 \log. 2}{3 \log. 3}$.

6. In $\left(\frac{3}{673}\right)^{-\frac{1}{x}} = 1·75$, show that $x = 9·677291$.

7. Find x and y from $a^{(x+y)^2} = b^{(x+y)^3}$, and $a_1^{(x-y)^2} = b_1^{(x-y)}$.
Ans. $x = \frac{1}{2}\left\{\frac{\log. a}{\log. b} + \frac{\log. b_1}{\log. a_1}\right\}$, and $y = \frac{1}{2}\left\{\frac{\log. a}{\log. b} - \frac{\log. b_1}{\log. a_1}\right\}$; and $x = 0$, $y = 0$.

8. Find x from $x^x = 5$. Ans. $2·129372$.

9. Solve the equation $z^z = 2000$. Ans. $4·827822$.

10. Solve the equation $x^x = 123456789$. Ans. $8·640027$.

11. Given $x^y = 10^{y^2}$, and $x^{-\frac{1}{y}} = 5^{y^{\frac{1}{3}}}$, to find x and y.
Ans. $y = 0$ and x indeterminate; and $x = ·00117937$, $y = -2·92836$.

12. Given $z^{z^z} = 1000$; and $z^{-z^{-z}} = ·5$, to find z in both cases.
Ans. $z = 2·384917$ in the former; and z is imaginary in the latter.

SIMPLE INTEREST.

The interest of any sum for any time being proportional to that sum and the time, the interest of $1l.$ for 1 year, being multiplied by the principal and time, will give the interest for that time at the specified rate per cent.

* An elegant method of solving such equations may also be seen in Mr. Charles Bonnycastle's Appendix to his Father's Algebra, published in 1823. A very elegant, though practically more laborious method of solution, may also be seen in foreign elementary works on algebra, by means of *continued fractions*.

For the sake of expressing this algebraically, put p for the principal lent at interest, t the time of its continuance, r the rate of interest, or periodical interest upon 1*l.*, and a the amount of the principal and interest at the end of the given time, and lastly, i the interest itself.

Then, obviously, we have the fundamental theorems, $prt = i$, and $p + prt = p(1 + rt) = a$.

From these equations we may find any one of the values in terms of the other three: and taking all together, we have

$$a = p + prt \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$p = \frac{a}{1 + rt} \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$r = \frac{a - p}{pt} \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$t = \frac{a - p}{rp} \quad \ldots\ldots\ldots\ldots \quad (4)$$

Ex. Find in what time any principal will double itself at any given rate of simple interest.

Here by equation (1) $2p = a = p + prt$, or $p = prt$, and $t = \frac{1}{r}$;

or, by equation (4), $t = \frac{2p - p}{rp} = \frac{1}{r}$, as before.

COMPOUND INTEREST.

If r be the interest of 1*l.* for a given period, the amount at the end of that period will be $1 + r$; and this put out for an equal period at the same rate, will amount at the end of this period to $(1 + r) + (1 + r)r$, or $(1 + r)^2$; and this again put out for a third period equal to each of the former, becomes $(1 + r)^2 + (1 + r)^2 r$, or $(1 + r)^3$; and so on, till after t periods, the amount is $(1 + r)^t$.

As this is the amount of 1*l.* for t terms, the amount of p£ will be p times as much, viz. $p(1 + r)^t$; since each of the p pounds produces the same final amount. Hence, generally, adopting the notations of simple interest so far as they are common to both, and putting $R = 1 + r$,

$$a = pR^t \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$p = \frac{a}{R^t} \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$R = \sqrt[t]{\frac{a}{p}} \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$t = \frac{\log a - \log p}{\log R} \quad \ldots\ldots \quad (4)$$

The equations (2, 3, 4) being obtained from (1) by mere common processes.

It is usual to call R the *ratio*, meaning the ratio of increase at compound interest.

Example. Suppose it be required to find in how many years any principal sum will double itself, at any proposed rate of compound interest.

In this case the 4th theorem must be employed, making $a = 2p$; and then it is

$$t = \frac{\log a - \log p}{\log R} = \frac{\log 2p - \log p}{\log R} = \frac{\log 2}{\log R}.$$

Thus, if the rate of interest be 5 per cent. per annum; then $R = 1{\cdot}05$; and hence,

$$t = \frac{\log 2}{\log 1{\cdot}05} = \frac{{\cdot}301030}{{\cdot}021189} = 14{\cdot}2067 \text{ nearly};$$

that is, any sum doubles itself in less than $14\frac{1}{4}$ years, at the rate of 5 per cent. per annum compound interest.

The following Table will very much facilitate calculations of compound

interest on any sum, for any number of years, not exceeding 38, at various rates of interest: it being the value of $(1 + r)^t$ for various values of r and t.

The Amount of 1*l* per annum in any number of Years.

Yrs.	$2\frac{1}{2}$ perCent.	3 per Cent.	$3\frac{1}{2}$ perCent.	4 per Cent.	$4\frac{1}{2}$ perCent.	5 per Cent.	6 per Cent.
1	1·02500	1·03000	1·03500	1·04000	1·04500	1·05000	1·06000
2	1·05063	1·06090	1·07123	1·08160	1·09203	1·10250	1·12360
3	1·07689	1·09273	1·10872	1·12486	1·14117	1·15763	1·19102
4	1·10381	1·12551	1·14752	1·16986	1·19252	1·21551	1·26248
5	1·13141	1·15927	1·18769	1·21665	1·24618	1·27628	1·33823
6	1·15969	1·19405	1·22926	1·26532	1·30226	1·34010	1·41852
7	1·18869	1·22987	1·27228	1·31593	1·36086	1·40710	1·50363
8	1·21840	1·26677	1·31681	1·36857	1·42210	1·47746	1·59385
9	1·24886	1·30477	1·36290	1·42331	1·48610	1·55133	1·68948
10	1·28008	1·34392	1·41060	1·48024	1·55297	1·62889	1·79085
11	1·31209	1·38423	1·45997	1·53945	1·62285	1·71034	1·89830
12	1·34489	1·42576	1·51107	1·60103	1·69588	1·79586	2·01220
13	1·37851	1·46853	1·56396	1·66507	1·77220	1·88565	2·13293
14	1·41297	1·51259	1·61869	1·73168	1·85194	1·97993	2·26090
15	1·44830	1·55797	1·67535	1·80094	1·93528	2·07893	2 39656
16	1·48451	1·60471	1·73399	1·87298	2.02237	2·18287	2·54035
17	1·52162	1·65285	1·79468	1·94790	2·11338	2·29202	2·69277
18	1·55966	1·70243	1·85749	2·02582	2·20848	2·40662	2·85434
19	1·59865	1·75351	1·92250	2·10685	2·30786	2·52695	3 02560
20	1·63862	1·80611	1·98979	2·19112	2·41171	2·65330	3·20714
21	1·67958	1·86029	2·05943	2·27877	2·52024	2·78596	3·39956
22	1·72157	1·91610	2·13151	2·36992	2·63365	2·92526	3·60354
23	1·76461	1·97359	2·20611	2·46472	2·75217	3·07152	3·81975
24	1·80873	2·03279	2·28333	2·56330	2·87601	3 22510	4·04893
25	1·85394	2·09378	2·36324	2·66584	3·00543	3·38635	4·29187
26	1·90029	2·15659	2·44596	2·77247	3·14068	3·55567	4·54938
27	1·94780	2·22129	2·53157	2·88337	3·28201	3·73346	4·82235
28	1·99650	2·28793	2·62017	2·99870	3·42970	3·92013	5·11169
29	2·04641	2·35657	2·71188	3·11865	3·58404	4·11614	5·41839
30	2·09757	2·42726	2·80679	3·24340	3·74532	4·32194	5·74349
31	2·15001	2·50008	2·90503	3·37313	3·91386	4·53804	6·08810
32	2·20376	2·57508	3·00671	3·50806	4·08998	4·76494	6·45339
33	2·25885	2·65234	3·11194	3·64838	4·27403	5·00319	6·84059
34	2·31532	2·73191	3·22086	3·79432	4·46636	5·25335	7·25103
35	2·37321	2·81386	3·33359	3·94609	4·66735	5·51602	7·68609
36	2·43254	2·89828	3·45027	4·10393	4·87738	5·79182	8·14725
37	2·49335	2·98523	3·57103	4·26809	5·09686	6·08141	8·63609
38	2·55568	3·07478	3 69601	4·43881	5·32622	6·38548	9·15425

For example, let it be required to find, to how much 523*l*. will amount in 15 years, at the rate of 5 per cent. per annum compound interest.

In the table, on the line 15, and in the column 5 per cent. is the amount of 1*l*, viz. 2·0789; and this multiplied by the principal 523, gives the amount 1087·2647, or 1087*l* 5*s* $3\frac{1}{4}d$, and therefore the interest 564*l* 5*s* $3\frac{1}{4}d$.

Note 1. When the rate of interest is to be determined to any other time than a year; as, suppose to $\frac{1}{2}$ a year, or $\frac{1}{4}$ of a year; the rules are still the same: but then t will express that time, and R must be taken the amount for that time also.

Note 2. When the compound interest, or amount, of any sum, is required for the parts of a year; it may be determined in the following manner:

1*st*. *For any time which is some aliquot part of a year.* Find the amount of

$1l$ for 1 year, as before; then that root of it which is denoted by the aliquot part, will be the amount of $1l$. This amount being multiplied by the principal sum, will produce the amount of the given sum as required.

2*d*. *When the time is not an aliquot part of a year.* Reduce the time into days, and take the 365th root of the amount of $1l$ for 1 year, which will give the amount of the same for 1 day. Then raise this amount to that power whose index is equal to the number of days, and it will be the amount for that time. Which amount, being multiplied by the principal sum, will produce the amount of that sum, as in the former cases.

ANNUITIES.

Annuity is a term used for any periodical income, arising from money lent, or from houses, lands, salaries, pensions, &c. payable from time to time, but mostly by annual payments.

Annuities are divided into those that are in *possession*, and those in *reversion:* the former meaning such as have already commenced; and the latter such as will not begin till some particular event has happened, or till after some certain time has elapsed.

When an annuity is forborne for some years, or the payments not made for that time, the annuity is said to be in *arrears*, or in *reversion*.

An annuity may also be for a certain number of years; or it may be without any limit, and then it is called a *perpetuity*.

The *amount* of an annuity, forborne for any number of years, is the sum arising from the addition of all the annuities for that number of years, together with the interest due upon each after it becomes due.

The *present worth, or value, of an annuity,* is the price or sum which ought to be given for it, supposing it to be bought off, or paid all at once.

Let $a =$ the annuity, pension, or yearly rent;

$n =$ the number of years forborne, or lent for;

$R =$ the amount of $1l$ for 1 year;

$m =$ the amount of the annuity;

$v =$ its value, or its present worth.

Now by compound interest, (theor. 2,) we have $p = \frac{a}{R^t}$. Hence giving to t the successive values 1, 2, 3, ... n, we get $\frac{a}{R}$, $\frac{a}{R^2}$, $\frac{a}{R^3}$, .. $\frac{a}{R^n}$, as the present values of a due at the end of 1, 2, 3, .. n, years respectively. Therefore, the sum of all these will be the present value of the n years' annuities; and if n be infinite, it will be the present value of a perpetual annuity of a £ per term.

Now by summing this geometrical series, we have $v = \frac{a}{R} + \frac{a}{R^2} + \frac{a}{R^3} + \ldots + \frac{a}{R^n} = \frac{a}{R^n} \cdot \frac{R^n - 1}{R - 1}$, the present value of the annuity which is to terminate in n years; and if n be infinite, $v = \frac{a}{R - 1}$, for the value of the annuity in perpetuity.

Again, because the amount of £1 in n years is R^n, the increase in that time is $R^n - 1$; but its amount in one year, or the annuity answering to that increase

is R — 1: and as these are in the ratio of a to m, we have $m = \frac{R^n - 1}{R - 1} . a$, and the several cases relating to annuities in reversion are easily found to be as follow:

$$m = \frac{R^n - 1}{R - 1} . a = vR^n \dots (1)$$

$$= \frac{R^n - 1}{R - 1} . \frac{a}{R^n} = \frac{m}{R^n} \dots (2)$$

$$r = \left\{\frac{1}{R^p} - \frac{1}{R^n}\right\} . \frac{a}{R - 1} \dots (3)$$

$$a = \frac{R - 1}{R^n - 1} . m = \frac{R - 1}{R^n - 1} . vR^n \dots (4)$$

$$n = \frac{\log m - \log v}{\log R} = \frac{\log \frac{m(R-1)+a}{a}}{\log R} \dots (5)$$

$$\log. R = \frac{\log m - \log v}{n} \dots (6)$$

In theorem (3), r denotes the present value of an annuity in reversion, after p years, or not commencing till after the first p years; and it is found by taking the difference between the two values $\frac{R^n - 1}{R - 1} . \frac{a}{R^n}$ and $\frac{R^p - 1}{R - 1} . \frac{a}{R^p}$, for n years and p years. The other formulæ are derived from those in compound interest taken in connexion with the fundamental theorem deduced above.

However, for practical purposes the amount and present value of any annuity for any number of years, up to 21, will be most readily found by the two following tables. In works professedly devoted to the subject, these tables are carried to a much greater extent.

TABLE I.

The Amount of an Annuity of 1*l* at Compound Interest.

Yrs.	At 3 per C.	$3\frac{1}{2}$ per C.	4 per C.	$4\frac{1}{2}$ per C.	5 per C.	6 per C.
1	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
2	2·0300	2·0350	2·0400	2·0450	2·0500	2·0600
3	3·0909	3·1062	3·1216	3·1370	3·1525	3·1836
4	4·1836	4·2149	4·2465	4·2782	4·3101	4·3746
5	5·3091	5·3625	5·4163	5·4707	5·5256	5·6371
6	6·4684	6·5502	6·6330	6·7169	6·8019	6·9753
7	7·6625	7·7794	7·8983	8·0192	8·1420	8·3938
8	8·8923	9·0517	9·2142	9·3800	9·5491	9·8975
9	10·1591	10·3685	10·5828	10·8021	11·0266	11·4913
10	11·4639	11·7314	12·0061	12·2882	12·5779	13·1808
11	12·8078	13·1420	13·4864	13·8412	14·2068	14·9716
12	14·1920	14·6020	15·0258	15·4640	15·9171	16·8699
13	15·6178	16·1130	16·6268	17·1599	17·7130	18·8821
14	17·0863	17·6770	18·2919	18·9321	19·5986	21·0151
15	18·5989	19·2957	20·3236	20·7841	21·5786	23·2760
16	20·1569	20·9710	21·8245	22·7193	23·6575	25·6725
17	21·7616	22·7050	23·6975	24·7417	25·8404	28·2129
18	23·4144	24·4997	25·6454	26·8551	28·1324	30·9057
19	25·1169	26·3572	27·6712	29·0636	30·5390	33·7600
20	26·8704	28·2797	29·7781	31·3714	33·0660	36·7856
21	28·6765	30·2695	31·9692	33·7831	35·7193	39·9927

TABLE II.

The Present Value of an Annuity of 1*l*.

Yrs.	At 3 per C.	3½ per C.	4 per C.	4½ per C.	5 per C.	6 per C.
1	0·9709	0·9662	0·9615	0·9569	0·9524	0·9524
2	1·9135	1·8997	1·8861	1·8727	1·8594	1·8334
3	2·8286	2·8016	2·7751	2·7490	2·7233	2·6730
4	3·7171	3·6731	3·6299	3·5875	3·5460	3·4651
5	4·5797	4·5151	4·4518	4·3900	4·3295	4·2124
6	5·4172	5·3286	5·2421	5·1579	5·0757	4·9173
7	6·2303	6·1145	6·0020	5·8927	5·7864	5·5824
8	7·0197	6·8740	7·7327	6·5959	6·4632	6·2098
9	7·7861	7·6077	7·4353	7·2688	7·1078	6·8017
10	8·5302	8·3166	8·1109	7·9127	7·7217	7·3601
11	9·5256	9·0016	8·7605	8·5889	8·3054	7·8869
12	9·9540	9·6633	9·3851	9·1186	8·8633	8·3838
13	10·6350	10·3027	9·9857	9·6829	9·3936	8·8527
14	11·2961	10·9205	10·5631	10·2228	9·8986	9·2950
15	11·9379	11·5174	11·1184	10·7396	10·3797	9·7123
16	12·5611	12·0941	11·6523	11·2340	10·8378	10·1059
17	13·1661	12·6513	12·1657	11·7072	11·2741	10·4773
18	13·7535	13·1897	12·6593	12·1600	11·6896	10·8276
19	14·3238	13·7098	13·1339	12·5933	12·0853	11·1581
20	14·8775	14·2124	13·5903	13·0079	12·4622	11·4699
21	15·4150	14·6980	14·0292	13·4047	12·8212	11·7641

To find the amount of any annuity forborne a certain number of years.

Take the amount of 1*l* from the first table, for the proposed rate and time; then multiply it by the given annuity; and the product will be the amount, for the same number of years, and rate of interest. Also, the converse to find either the rate or the time.

Ex. To find how much an annuity of 50*l* will amount to in 20 years, at 3½ per cent. compound interest.

On the line of 20 years, and in the column of 3½ per cent. stands 28·2797, which is the amount of an annuity of 1*l* for the 20 years. Then 28·2797 × 50, gives 1413·985*l* = 1413*l* 19*s* 8*d* for the answer required.

To find the present value of any annuity for any number of years.

Proceed here by the second table, in the same manner as above for the first table, and the present worth required will be found.

Ex. 1. To find the present value of an annuity of 50*l*, which is to continue 20 years, at 3½ per cent. By the table, the present value of 1*l* for the given rate and time, is 14·2124; therefore 14·2124 × 50 = 710·62*l*, or 710*l* 12*s* 4*d*, is the present value required.

Ex. 2. To find the present value of an annuity of 20*l*, to commence 10 years hence, and then to continue for 11 years longer, or to terminate 21 years hence, at 4 per cent. interest. In such cases as this, we have to find the difference between the present values of two equal annuities, for the two given times; which, therefore, will be done by subtracting the tabular value of the one period from that of the other, and then multiplying by the given annuity. Thus, the tabular value for 21 years is 14·0292, and that for 10 years is 8·1109. Then, the difference 5·9183 multiplied by 20 gives 118·366*l*, or 118*l* 7*s* 3½*d*, the answer.

SERIES BY SUBTRACTION.

THIS method is most readily applicable to the cases where the several terms of the series are the differences (or the same multiple of the differences) between two equi-distant corresponding terms of some other series. A few simple examples will sufficiently illustrate the practice, whilst the principle of these processes is self-evident. The only difficulty is to find the series whose differences are the terms of the given one; and for this no general and simple rule exists.

Ex. 1. Let $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots\text{ ad inf.}=s$

then $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots\text{ ad inf.}=s-1$

by sub. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\dots\text{ ad inf.}=1$

Ex. 2. Let $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots=s.$

then $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\dots=s-\frac{3}{2}.$

by sub. $\frac{2}{1.3}+\frac{2}{2.4}+\frac{2}{3.5}+\frac{2}{4.6}+\dots=\frac{3}{2}.$

or $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+\dots=\frac{3}{4}.$

Ex. 3. Let $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots=s$

then $\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\dots=s-\frac{1}{2}.$

by sub. $\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+\dots=\frac{1}{2}.$

or $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\dots=\frac{1}{4}.$

Ex. 4. Find the sum of the series $\frac{1}{2.4.6}+\frac{1}{4.6.8}+\frac{1}{6.8.10}+\dots$

Take away the last factor out of each denominator, and assume

$$\frac{1}{2.4}+\frac{1}{4.6}+\frac{1}{6.8}+\dots=s$$

then $\frac{1}{4.6}+\frac{1}{6.8}+\frac{1}{8.10}+\dots=s-\frac{1}{8}.$

by sub. $\frac{4}{2.4.6}+\frac{4}{4.6.8}+\frac{4}{6.8.10}+\dots=\frac{1}{8}.$

Hence $\frac{1}{2.4.6}+\frac{1}{4.6.8}+\frac{1}{6.8.10}+\dots=\frac{1}{8}\times\frac{1}{4}=\frac{1}{32}.$

Ex. 5. Find the sum of $\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\dots$ *ad inf.*

assume $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots=s$, then transposing $\frac{1}{1}-\frac{1}{2}$,

we have $\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots=s-\frac{1}{2}$;

hence by subtraction $\frac{2}{1.3}-\frac{2}{2.4}+\frac{2}{3.5}-\dots=\frac{1}{2}$, or $s=\frac{1}{4}$.

Ex. 6. Find $\frac{1}{2.4.6.8}+\frac{1}{4.6.8.10}+\frac{1}{6.8.10.12}+\dots$ *ad inf.* Ans. $\frac{1}{288}$.

Ex. 7. Also of $\frac{1}{2.5.8.11} + \frac{1}{5.8.11.14} + \ldots$ *ad inf.* Ans. $\frac{1}{720}$.

Ex. 8. Sum the series $\frac{1}{1.5} - \frac{1}{3.7} + \frac{1}{5.9} - \ldots$ Ans. $\frac{1}{6}$.

Note. The upper line (carried to any extent) contains one term at the *end*, under which there is no term in the lower line. But in the above examples this circumstance creates no difference, since the terms being infinitely distant, and continually converging towards 0, that last term itself is virtually 0. However, if the law of the series be such, that the terms converge towards a limit different in value from 0, then this value being that of the last term of the upper line, *must be added to the sum obtained as above.*

Ex. 9. Thus, if $\frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots$ were purposed to be summed; the process would be, if performed according to the preceding type,

assume $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots = s$; then, transposing,

we have $\frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots\ldots\ldots = s - \frac{1}{2}$;

hence by subtraction $\frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots\ldots = -\frac{1}{2}$.

But as the values of the several terms of the series, whose value is s, converge towards 1, the uncompensated term itself is 1, which added to the value already found, gives $1 - \frac{1}{2} = \frac{1}{2}$ for the true sum of the series. This final term, in such cases, may be called the *correction of the sum.*

Ex. 10. Find the sum of $\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \frac{1}{9.11} + \ldots$ Ans. $\frac{1}{6}$.

MISCELLANEOUS EXAMPLES.

Ex. 1. Find the sum of $\frac{5}{1.4} - \frac{7}{2.5} + \frac{9}{3.6} - \frac{11}{4.7} + \frac{13}{5.8} - \ldots$ Ans. $\frac{5}{6}$.

Ex. 2. Show that $\left\{\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\right\} \left\{\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots\right\}$

is equal to the sum of $\frac{5}{1.2.1.3} + \frac{9}{2.3.3.5} + \frac{13}{3.4.5.7} + \ldots$.

Ex. 3. The sum of the series $\frac{1}{m(m+r)} + \frac{1}{(m+r)(m+2r)} + \frac{1}{(m+2r)(m+3r)} + \ldots$ *ad inf.* $= \frac{1}{mr}$.

Ex. 4. The sum of n terms* of $a + 2ar + 3ar^2 + 4ar^3 + \ldots$ is $S_n = \frac{a - ar^n}{(1-r)^2} - \frac{nar^n}{1-r}$, and the sum to infinity is $S_{\infty} = \frac{a}{(1-r)^2}$.

Ex. 5. Sum n terms of $\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \ldots$ and of $\frac{a}{m} + \frac{a+b}{mr} + \frac{a+2b}{mr^2} + \ldots$ and likewise the sum of each series to infinity.

---

* This and such questions differ from the preceding in no respect but this: that the expression for the nth term must be taken as the last of the assumed series instead of the infinitely distant term towards which in that case the succeeding terms more directly point. The *correction* for this must be made as directed in the note above.

REVERSION OF SERIES.

WHEN the powers of an unknown quantity are contained in the terms of a series, the finding the value of the unknown quantity in another series, which involves the powers of the quantity to which the given series is equal, and known quantities only, is called reverting the series *.

RULE 1. Assume a series for the value of the unknown quantity, of the same form with the series which is required to be reverted.

2. Substitute this series and its powers, for the unknown quantity and its powers, in the given series.

3. Make the resulting coefficients equal to the corresponding coefficients of the given series, whence the values of the assumed coefficients will be obtained.

4. When the series is expressed by means of another, as $ax + bx^2 + cx^3 + \ldots = ay + by^2 + cy^3 + \ldots.$ the value is to be obtained in the same manner, by assuming $x = \text{A}y + \text{B}y^2 + \text{C}y^3 + \ldots$

EXAMPLES.

Ex. 1. Let $z = ax + bx^2 + cx^3 + dx^4 + \ldots$ be given, to find the value of x in terms of z and known quantities.

Assume $x = \text{A}z + \text{B}z^2 + \text{C}z^3 + \ldots$, and substitute for the powers of x in the given series, the powers of this assumed series. Then we shall have

$$z = a\text{A}z + \left.\begin{matrix} a\text{B} \\ + b\text{A}^2 \\ \end{matrix}\right\} z^2 + \left.\begin{matrix} a\text{C} \\ + 2b\text{AB} \\ + c\text{A}^3 \end{matrix}\right\} z^3 + \left.\begin{matrix} a\text{D} \\ + 2b\text{AC} \\ + b\text{B}^2 \\ + 3c\text{A}^2\text{B} \\ + d\text{A}^4 \end{matrix}\right\} z^4 + \ldots.$$

By equating the coefficients of the homologous terms of z, we shall have

$$a\text{A} = 1, \text{ or } \text{A} = \frac{1}{a};$$

$$a\text{B} + b\text{A}^2 = 0, \text{ or } \text{B} = -\frac{b\text{A}^2}{a} = -\frac{b}{a^3};$$

$$a\text{C} + 2b\text{AB} + c\text{A}^3 = 0, \text{ or } \text{C} = \frac{2b^2 - ac}{a^5};$$

$$\text{and similarly, } \text{D} = \frac{5abc - 5b^3 - a^2d}{a^7};$$

and so on to any extent. Hence,

$$x = \frac{1}{a}z - \frac{b}{a^3}z^2 + \frac{2b^2 - ac}{a^5}z^3 - \frac{5b^3 - 5abc + a^2d}{a^7}z^4 + \ldots.$$

This conclusion forms a general theorem for every similar series, involving the like powers of the unknown quantity.

---

* Other methods of reversion are given by different mathematicians. The above is selected for its simplicity, most of the others depending for their evidence on principles more recondite than have yet been laid before the student, or being more difficult of application, or more confined as to generality. This is, evidently, only an application of the *Method of Indeterminate Coefficients*.

Ex. 2. Let the series $z = x \pm x^2 + x^3 \pm x^4 + \ldots$ be proposed for reversion.

Here $a = 1$, $b = \pm 1$, $c = 1$, $d = \pm 1$, and so on; these values being substituted in the theorem derived from the preceding example, we shall obtain $x = z \mp z^2 + z^3 \mp z^4 + \ldots$, the answer required.

Ex. 3. Let $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ be given for reversion.

Substituting as before, we have $a = 1$, $b = -\frac{1}{2}$, $c = \frac{1}{3}$, $d = -\frac{1}{4}$, and so on. These values being substituted, we shall have $x = y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + \ldots$; from which if y be given, and sufficiently small for the series to converge, the value of x will be known.

Ex. 4. Given the series $y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$ to find the value of x in terms of y. Ans. $x = y + \frac{1}{3}y^3 + \frac{2}{15}y^5 + \frac{17}{315}y^7 + \ldots$

Ex. 5. Given the series $y = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \ldots$ to find x in terms of y. Ans. $x = (y-1) - \frac{(y-1)^2}{2} + \frac{(y-1)^3}{3} - \frac{(y-1)^4}{4} + \ldots$

Ex. 6. It is required to find x and y from the two following equations:

$$\left.\begin{aligned} 30y &= x + \frac{x}{1.3} + \frac{x}{2.3} + \frac{x}{2.5} + \frac{x}{3.5} + \frac{x}{3.7} + \ldots \\ \frac{9x}{10} &= \frac{1}{3} + y + 2y^2 + \frac{10}{3}y^3 + 5y^4 + \ldots \end{aligned}\right\} \text{Ans. } x = 10,\ y = \frac{2}{3}.$$

Ex. 7. Given $y = x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{725678} + \ldots$ to find x.

Ans.* $x = y - \frac{y^3}{6} + \frac{y^5}{24} - \frac{61y^7}{5040} + \frac{277y^9}{725678} - \ldots$

THE METHOD OF FINITE DIFFERENCES.

I. *Definitions, notation, and principles.*

1. If the successive integers, 1, 2, 3, 4, ... be given to x in any expression, a series of numbers will be produced, which are called the *successive values* of that expression: and, conversely, that expression is called the *general term* of the series of numbers thus produced.

2. The general term or function of x from which the series of numbers is formed, is sometimes written, as in algebraic equations, $f(x)$; but more commonly u_x or v_x, the letter u or v being called the characteristic of the function. Thus, if $x^3 + 6x^2 - 5x + 10$ were the general term, it would be written u_x: and the values of this function, when 1, 2, 3, 4, .. are written for x, (viz. 12, 32, 76, 150, ...) are written $u_1, u_2, u_3, u_4, \ldots$

Also, if $-1, -2, -3, \ldots$ were put for x, the several results, 20, 36, 52, ... are written $u_{-1}, u_{-2}, u_{-3}, \ldots$

---

* In this equation y expresses the meridional parts corresponding to the latitude x: and it is remarkable that the numeral coefficients are the same in the direct and reverted series, and differ only in the signs of the even-numbered terms. *Leybourn's Rep.* II. 44.

by Mr Wallace R M College

3. If a series of n terms be given, the $(n+1)^{th}$ is called the *increment of the series*, or the *increment of the sum of the series*. The increment is the same function of $n+1$ that the last term of the series is of n; or in symbols, if u_n be the last term, u_{n+1} is the increment.

4. If the first term of a series be subtracted from the second, the second from the third, the third from the fourth, and so on; the several remainders constitute a new series, which is called *the first order of differences*. Taking the differences of the successive terms of this series, we obtain *the second order of differences;* and from this again, *the third order of differences;* and so on, as long as remainders result from such operations.

5. The general expression for the difference of the two consecutive terms, the $(x+1)^{th}$ and the x^{th}, will, according to the preceding notation, be written $u_{x+1} - u_x$: but it is often convenient to write it simply Δu_x, where Δ is called the *sign of differencing*. The second difference, or $\Delta(u_{x+1} - u_x)$, is written $\Delta^2 u_x$, the third $\Delta^3 u_x$, and so on, as long as any differences exist.

6. The symbol Σ prefixed to an expression u_x, signifies the operation of finding another expression v_x such that $v_{x+1} - v_x = u_x$. In other words, it expresses an operation directly the reverse of taking the difference of a function; and hence Δ and Σ are indicative of operations each of which neutralises the effect of the other. The symbol Σ is called the *sign of integration*.

7. A *factorial* is an expression composed of factors in arithmetical progression, as $x(x \pm a)(x \pm 2a) \ldots (x \pm na)$; where every factor differs from the preceding by the common quantity $\pm a$.

In respect of notation, the following has the advantage of concentrated writing, viz. $x^{n+1|\pm a}$: where the first factor of the series is written down; then the index of the number of factors in the factorial, in the manner of the binomial index; and lastly, separated from the index by a vertical line, the common difference of the several successive factors *.

II. *To find the general term of the successive orders of differences of a given function u_x.*

1. The expression for the x^{th} term of the n^{th} order of differences is

$$\Delta^n u_x = u_{x+n} - \frac{n}{1} \cdot u_{x+n-1} + \frac{n(n-1)}{1.2} \cdot u_{x+n-2} - \frac{n(n-1)(n-2)}{1.2.3} \cdot u_{x+n-3} + \ldots$$

the coefficients being those of the expanded binomial $(1-1)^n$.

For $\Delta u_x = u_{x+1} - u_x$, by def. 5

$$\begin{aligned}
\Delta^2 u_x &= \Delta\left\{u_{x+1} - u_x\right\} = \left\{u_{x+2} - u_{x+1}\right\} - \left\{u_{x+1} - u_x\right\} \\
&= u_{x+2} - 2u_{x+1} + u_x \\
\Delta^3 u_x &= \Delta\left\{u_{x+2} - 2u_{x+1} + u_x\right\} \\
&= \left\{u_{x+3} - 2u_{x+2} + u_{x+1}\right\} - \left\{u_{x+2} - 2u_{x+1} + u_x\right\} \\
&= u_{x+3} - 3u_{x+2} + 3u_{x+1} - u_x.
\end{aligned}$$

Proceeding thus we find the fourth, fifth, and subsequent differences, to any extent, to retain the assigned form: but to complete the proof, we must establish the necessary continuity of the law.

---

* This very elegant notation was invented by M. Kramp, Professor of Mathematics at Strasburgh. See his *Elémens d'Arith. Un.* p. 347, a work of great originality and value.

Now this will be done, if, supposing it true for the m^{th} difference, we prove that it will be so for the $(m+1)^{th}$ difference. In this case we have

$$\Delta^m u_x = u_{x+m} - \frac{m}{1} \cdot u_{x+m-1} + \frac{m(m-1)}{1.2} u_{x+m-2} - \frac{m(m-1)(m-2)}{1.2.3} u_{x+m-3} + \ldots$$

And writing $x+1$ for x, and subtracting $\Delta^m u_x$ from the result, we have

$$\begin{aligned}\Delta^{m+1} u_x = u_{x+m+1} - \frac{m}{1} \cdot u_{x+m} &+ \frac{m(m-1)}{1.2} u_{x+m-1} - \frac{m(m-1)(m-2)}{1.2.3} u_{x+m-2} + \ldots \\ - u_{x+m} &+ \frac{m}{1} \cdot u_{x+m-1} - \frac{m(m-1)}{1.2} u_{x+m-2} + \ldots \\ = u_{x+m+1} - \frac{(m+1)}{1} u_{x+m} &+ \frac{m(m+1)}{1\ 2} u_{x+m-1} - \frac{(m-1)(m)(m+1)}{1.2.3} u_{x+m-2} + \ldots\end{aligned}$$

which establishes the general and necessary continuity of the law of the series.

For example, if the form of the function were $u_x = ax^3$, then

$\Delta u_x = u_{x+1} - u_x = a(x+1)^3 - ax^3$

$\Delta^2 u_x = u_{x+2} - 2u_{x+1} + u_x = a(x+2)^3 - 2a(x+1)^3 + ax^3.$

$\Delta^3 u_x = u_{x+3} - 3u_{x+2} + 3u_{x+1} - u_x = a(x+3)^3 - 3a(x+2)^3 + 3a(x+1)^3 - ax^3$

and so on, to any required extent.

2. The preceding formula applies to any form of the function u_x: but when that function is one of an algebraic form, the actual process is simpler when the reductions are made, *pari passu*, from one step to another of the differencing.

Thus, if the function were $u_x = ax^3 - bx^2 + cx + d$, we should have

$$\begin{aligned}\Delta u_x &= a\left\{(x+1)^3 - x^3\right\} - b\left\{(x+1)^2 - x^2\right\} + c\left\{(x+1) - x\right\} + \left\{d - d\right\} \\ &= 3ax^2 + (3a - 2b)\,x + a - b + c \\ \Delta^2 u_x &= 3a\left\{(x+1)^2 - x^2\right\} + (3a - 2b)\left\{(x+1) - x\right\} + \left\{(a - b + c) - (a - b + c)\right\} \\ &= 6ax + (6a - 2b) \\ \Delta^3 u_x &= 6a\left\{(x+1) - x\right\} + \left\{(6a - 2b) - (6a - 2b)\right\} \\ &= 6a\end{aligned}$$

$\Delta^4 u_x = 0$, and so on for all higher orders of differences.

It may also be remarked, that though $d - d$, $(a - b + c) - (a - b + c)$, and so on, are written in the expressions above, they are unnecessary in practice, as the absolute term is always cancelled in differencing.

Scholium.

It is very clear, since $(x+1)^n - x^n$ is an expression of the $(n-1)^{th}$ degree, that the difference of an algebraic function is one degree lower than the function itself. In like manner, the second difference is one degree lower than the first difference, or two degrees lower than the given function. Proceeding thus, we shall find the n^{th} difference constant, and the $(n+1)^{th}$, $(n+2)^{th}$, and all subsequent differences, severally equal to 0.

EXAMPLES.

Ex. 1. Find the general term of the several orders of differences of

(1). $x^{10} + 6x^9 + 3x^2 - 10$;

(2). $ax^{\alpha} + bx^{\beta} + cx^{\gamma} + \ldots$;

(3). $3x^{-3} + 4x^{-2} + 6x^{-1} - 3x^0$;

(4). $\frac{3}{4}x^{\frac{2}{3}} - \cdot 9x^{\cdot 5} + \cdot 001x^{0-2}$.

Ex. 2. Show that $\Delta a^x = (a^{\Delta x} - 1)\, a^x$, and $\Delta a^{-x} = \frac{1}{a^x}\left(\frac{1}{a^{\Delta x}} - 1\right)$.

Ex. 3. The first differences of $u_x v_x$, of $\frac{u_x}{v_x}$, and of $\frac{1}{u_x v_x}$; where u_x and v_x express different, but given, functions of x.

The solution of the first of these examples is subjoined.

$$\begin{aligned}\Delta\ (u_x\, v_x) &= u_{x+1}\, v_{x+1} - u_x\, v_x \\ &= (u_x + \Delta u_x)\,(v_x + \Delta v_x) - u_x\, v_x \\ &= v_x\,.\,\Delta u_x + u_x\,.\,\Delta v_x + \Delta u_x\,.\,\Delta v_x.\end{aligned}$$

[To facilitate the student's comprehension of the signification of these results, let him apply them to the following particular examples:

Let $(x^2 - 4x + 6)\,(3x^3 - 2x) = u_x\, v_x$. Then

$\Delta u_x = 2x - 4$, and $\Delta v_x = 9x^2 - 2$. Hence,

$\Delta\ (u_x\, v_x) = (3x^3-2x)\,(2x-4) + (x^2-4x+6)\,(9x^2-2) + (2x-4)\,(9x^2-2)$.

Reduce this, and then actually multiply the *values* of u_x and v_x together, and take the difference: they will be found identical.

Let again $u_x = \frac{1}{10^x}$ and $v_x = 4^{x^2}$; and let $\cdot 3^{\frac{x}{10}}$ be divided by 10^{-x}, and the first difference found.]

Similarly, we find $\Delta\,\frac{u_x}{v_x} = \frac{v_x\,\Delta u_x - u_x\,\Delta v_x}{v_x\{v_x + \Delta v_x\}}$; and

$$\Delta\,\frac{1}{u_x\, v_x} = -\,\frac{u_x\,\Delta v_x + v_x\,\Delta u_x + \Delta u_x\,\Delta v_x}{u_x\, v_x\{u_x + \Delta u_x\}\,\{v_x + \Delta v_x\}}.$$

Ex. 4. Let a series of factors in arithmetical progression be given,

$$u_x = x(x + a)\,(x + 2a)\,(x + 3a)\ \ldots\ldots\ (x + na).$$

For x write its succeeding value, $x + a$, and subtract u_x from it:

$$u_{x+1} - u_x = (x + a)\,(x + 2a)\ldots\{x + (n + 1)a\} - x(x + a)\ \ldots\ (x + na),$$

$$\text{or } \Delta u_x = (x + a)\,(x + 2a)\ \ldots\ldots\ (x + na)\,\{x + (n + 1)\,a - x\}$$
$$= (n + 1)\,(x + a)\,(x + 2a)\ \ldots\ldots\ (x + na)\,a\,^*$$

That is, *the difference sought is the product of all the factors except the first, by the common difference and the number of factors.*

Ex. 5. Let $u_x = \frac{1}{(x + a)\,(x + 2a)\ \ldots\ldots\ (x + na)}$: then we have

$$\Delta u_x = \frac{-\,(n + 1)a\,^*}{x(x + a)\,(x + 2a)\ \ldots\ldots\ (x + na)}.$$

That is, *the difference is found by multiplying the denominator, the preceding value of the function, and the numerator, by the common difference, into the number of factors in the denominator so increased.*

Ex. 6. It is required to determine the first differences of xa^x, xa^{-x}, and $x^n a^{-x^m}$.

Ex. 7. Find the numerical values of the first ten terms of the first order of differences in the functions whose coefficients are numerical in Ex. 1.

Ex. 8. Find the second, third, and fourth differences of the functions given in Ex. 6.

---

* Since $a = \Delta x$, this latter expression may be written for it in the answers above. The factors are also often written $x x_1\ x_2\ x_3\ \ldots\ x_n$; and hence the results obtained in the text may be put $(n + 1)\ x_1\ x_2 \ldots x_n\ \Delta x$ and $-\,\frac{(n + 1)\,\Delta x}{x x_1\, x_2\, \ldots\, x_n}$, respectively.

III. *Having given an adequate number of terms of a series, to find the general term of the series.*

Suppose the general form of the term to be

$$ax^m + a_1x^{m-1} + a_2x^{m-2} + \ldots\ldots + a_{m-1}x + a_m;$$

then there is required the index m and coefficients $a, a_1, a_2, \ldots a_m, a_{m+1}$.

1. In the first place, we have seen (II. *Schol.*) that in an expression of the m^{th} degree, the m^{th} differences are all equal, and that all higher orders of differences become 0. Hence to find m, we have only to take the successive orders of differences, till we find one order all whose terms are equal. The number of these operations which are thus performed, gives the value of m.

2. Let $u_1, u_2, u_3, \ldots$ be the several given terms of the series: then, as these are the values of the general term when m is 1, 2, 3, ... we have

$$\begin{aligned}
1^m a + 1^{m-1} a_1 + 1^{m-2} a_2 + \ldots + 1a_{m-1} + a_m &= u_1 \\
2^m a + 2^{m-1} a_1 + 2^{m-2} a_2 + \ldots + 2a_{m-1} + a_m &= u_2 \\
3^m a + 3^{m-1} a_1 + 3^{m-2} a_2 + \ldots + 3a_{m-1} + a_m &= u_3 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(m+1)^m a + (m+1)^{m-1} a_1 + \ldots\ldots (m+1)\, a_{m-1} + a_m &= u_{m+1};
\end{aligned}$$

in which there are as many equations as there are unknown coefficients $a, a_1, a_2, \ldots a_m$, viz. $m + 1$. All these equations are, with respect to the unknowns, of the first degree; and the method most readily applicable to the process of solution, is that pointed out at pp. 179, 180, of this work.

Ex. 1. Let the series 7, 33, 79, 145, 231, be given to find its general term.

7	33	79	145	231	... given series
	26	46	66	86	... first differences
		20	20	20	... second differences.

Hence, as the second differences are all equal, we have $m = 2$, and the general term is of the form $ax^2 + a_1x + a_2$. Hence, substituting 1, 2, 3, in this for x, we have

$$\begin{aligned}
a + a_1 + a_2 &= 7 \\
4a + 2a_1 + a_2 &= 33 \\
9a + 3a_1 + a_2 &= 79
\end{aligned}$$

and, as at p. 180, we get $a = 10,\ a_1 = -4,\ a_2 = 1$.

The required expression is, therefore, $u_x = 10x^2 - 4x + 1$.

Ex. 2. Find the general term of the series 2, 24, 108, 320. Also ascertain whether any of the terms 1512, 4668, 7290, and 11011, belong to the series; and if so, assign their places.

Ex. 3. Given 2, 14, 66, to find the general term, and hence the next term *.

* This example was actually formed from the expression $x^4 - x^3 + x^2 + x$; that is, one of the fourth degree: but as the terms, *so far* as they are actually given, can be formed from one of the second degree, this latter ought to be considered the *determinate* solution of the question, in contradistinction to the *indeterminate* ones, which the solutions of the third, fourth, ... degrees do really become. These considerations suggest, that when a series of m terms is given, such that the m^{th} difference is not constant, then we may fulfil the condition to which these given terms are subject, by taking $\Delta^{m+1} u = 0$, and therefore also $\Delta^m u_2 = 0$: though at the same time the general term which results is only a particular case of a more general solution which would have been obtained by supposing the dimension of the general term to be higher. The assumption of the dimension of the general term is, therefore, in fact altogether arbitrary,

Ex. 4. Given 2, 14, 66, 212, ... and likewise 2, 14, 66, 212, 530 ...., to find the general term.

IV. *Having given a series of terms, to find the several orders of differences.*

1. Subtract the first term from the second, the second from the third, and so on, to get the entire first order of differences; employ the same process upon the first order to obtain the second; upon the second to obtain the third. The first terms of these several orders are those sought; as follows from *definition* 4.

2. Generally, however, it will be more convenient to employ the formula deduced in II. p. 274, making in all the expressions $x = 1$, and $n = 1, 2, 3, \ldots$ in succession. The expression so modified becomes

$$\Delta^n u_1 = u_{n+1} - \frac{n}{1} u_n + \frac{n(n-1)}{1.2} u_{n-1} - \ldots\ldots \pm \frac{n}{1} u_2 \mp u_1$$

Scholium.

It is evident from both the methods, that there must be given one term more than the number indicated by the order of the difference sought, in order to render the problem determinate.

EXAMPLES.

Ex. 1. To find the first term of the third order of differences of 1, 4, 8, 13, 19, ...

Here $u_1 = 1$, $u_2 = 4$, $u_3 = 8$, $u_4 = 13$, $u_5 = 19, \ldots$

Hence $\Delta^3 u_1 = u_4 - 3u_3 + 3u_2 - u_1 = 13 - 24 + 12 - 1 = 0$.

Ex. 2. The first * term of the seventh order of differences of 1, 4, 8, 16, 32, 64, 128, ...

Here $\Delta^7 u_1 = u_8 - 7u_7 + 21u_6 - 35u_5 + 35u_4 - 21u_3 + 7u_2 - u_1$;

in which, inserting the given values of $u_1, u_2, u_3, \ldots$, we have

$$\Delta^7 u_1 = 256 - 896 + 1344 - 1120 + 560 - 168 + 28 - 1 = 3.$$

Ex. 3. Given 1, 2, 4, 8, 16, ... to find the first term of the seventh order of differences. Ans. 1.

Ex. 4. Find the several orders of differences of 1, 2, 3, 4, 5, ...

Ex. 5. Find the several orders of differences of 1, 4, 9, 16, 25, ...

Ex. 6. Find the several orders of differences of 1, 8, 27, 64, 125, ...

Ex. 7. Find the first four orders of differences of the logarithms of 101, 102, 103, 104, 105, ...

Ex. 8. Given 1, 6, 20, 50, 105, ... to find the first four orders of differences.

---

so that the index be at least equal to the number of terms; these terms being always understood to be consecutive in the scale, beginning at unity.

When, however, the calculation is of *one single term*, however distant, it will be effected more easily by the following process, without determining the *general term* of the series. If several be required, the general term, (that is, the *most simple general term* which can be found from the given numbers,) is indispensable.

* When only the first term of a single order of differences is required, as in the first three examples, the second method is the preferable, as in the text; but when all are required in succession, it will be more convenient to employ the former method. Thus, if the several orders of this example were sought, the work would stand as below:

Ex. 9. Given 1, 4, 8, 16, 32, 64, 128, 256, ... to find the several orders of differences.

1	4	8	16	32	64	128	256	....the given series
	3	4	8	16	32	64	128	....first differences
		1	4	8	16	32	64	....second differences
			3	4	8	16	32	....third differences
				1	4	8	16	....fourth differences
					3	4	8	....fifth differences
						1	4	....sixth differences
							3	....seventh differences.

V. *Having given the first terms of the first n orders of differences, to find the $(n+1)^{\text{th}}$ term of the series: or in symbols, given $u_1, \Delta u_1, \Delta^2 u_1, \ldots. \Delta^n u_1$, to find u_{n+1}.*

The solution is $u_{n+1} = u_1 + \frac{n}{1}\Delta u_1 + \frac{n(n-1)}{1.2}\Delta^2 u_1 + \ldots$ carried to the term involving Δ^n

For by II. we have, putting $x = 1$, in all cases,

$$\Delta u_1 = u_2 - u_1, \text{ or } u_2 = u_1 + \Delta u_1.$$

$$u_3 = u_2 + \Delta u_2 = u_1 + \Delta u_1 + \Delta\{u_1 + \Delta u_1\}$$
$$= u_1 + 2\Delta u_1 + \Delta^2 u_1$$
$$u_4 = u_3 + \Delta u_3 = u_1 + 2\Delta u_1 + \Delta^2 u_1 + \Delta\{u_1 + 2\Delta u_1 + \Delta^2 u_1\}$$
$$= u_1 + 3\Delta u_1 + 3\Delta^2 u_1 + \Delta^3 u\ ;$$

and so on to any extent required, the coefficients being those arising from the expansion of the binomial $(1+1)^n$, and the necessary continuity of the law being capable of establishment nearly as in II.

EXAMPLES.

Ex. 1. To find the twentieth term of the series, 2, 6, 12, 20, 30, ...

u_1	u_2	u_3	u_4	u_5	.... the functions
2	6	12	20	30	....their given values
	4	6	8	10	....first order of differences
		2	2	2	....second order of differences
			0	0	.... third order of differences.

Hence $u_1 = 2$, $\Delta u_1 = 4$, $\Delta^2 u_1 = 2$, $\Delta^3 u_1 = 0$, and so on, all the subsequent orders of differences being zero. Hence

$$u_{20} = u_1 + \frac{19}{1}\Delta u_1 + \frac{19.18}{1.2}\Delta^2 u_1 = 2 + 76 + 342 = 420,$$

which is the twentieth term of the series sought.

Ex. 2. Find the tenth term of the series, 2, 5, 9, 14, 20, .. Ans. 65.

Ex. 3. Required the fifth term of 1, 3, 6, 10, ... Ans. 15.

Ex. 4. To find the tenth term of the series, 1, 4, 8, 13, 19, .. Ans. 64.

Ex. 5. Required the twentieth term of 1, 8, 27, 64, 125, ... Ans. 8000.

Ex. 6. Required the sixth term of 101, 108½, 118, 129½, ... Ans. 158½.

VI. *To convert a given function in powers of x to one which shall have every term composed of factors $x \pm a$, $x \pm b$, and so on.*

Divide synthetically by $x \pm a$; the last coefficient is the coefficient of $(x \pm a)^0$ of the transformed expression: divide again in the same manner by $x \pm b$, stopping one step sooner; the last coefficient is that of $(x \pm a)$ in the trans-

formed expression: proceed similarly with $x \pm c$, stopping one step sooner than in the preceding; then the last coefficient is that of $(x \pm a)$ $(x \pm b)$ in the transformed expression. Proceed thus with all the factors, then the coefficients of the transformed expression will be all determined.

For let $Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots. + Lx^2 + Mx + N$ be the function given in powers of x: then dividing by the several given factors, we have in succession

$$Ax^{n-1} + B_1x^{n-2} + C_1x^{n-2} + \ldots\ldots + L_1x + M_1 + \frac{N_1}{x \pm a}$$

$$Ax^{n-2} + B_2x^{n-3} + C_2x^{n-4} + \ldots\ldots + L_2 + \frac{M_2}{x \pm b} + \frac{N_1}{(x \pm a)\,(x \pm b)}$$

$$Ax^{n-3} + B_3x^{n-4} + C_3x^{n-5} + \ldots\ldots \frac{L_3}{x \pm c} + \frac{M_2}{(x \pm b)(x \pm c)} + \frac{N_1}{(x \pm a)\,(x \pm b)\,(x \pm c)},$$

and so on, till we arrive at

$$A + \frac{B_{n-1}}{x \pm m} + \frac{C_{n-2}}{(x \pm l)\,(x \pm m)} + \ldots.$$

Multiplying now by $(x \pm a)$ $(x \pm b)$ $(x \pm c)$ .... $(x \pm l)$ $(x \pm m)$, we have $A(x \pm a)\,(x \pm b)$ .... $(x \pm m) + B_{n-1}\,(x \pm a)\,(x \pm b) \ldots (x \pm l) + \ldots. + N_1$ for the transformed expression.

If these factors be taken in *arithmetical progression*, the result is a transformation into an expression of *factorials*. See def. 7, p. 274.

EXAMPLES.

Ex. 1. Transform the function $3x^4 - 6x^3 + 2x^2 - 5x - 9$ into factorials involving $x + 1$, $x + 2$, $x + 3$, $x + 4$.

$$\begin{array}{r|l}
 & 3 - 6 + 2 - 5 - 9 \\
-1 & \quad - 3 + 9 - 11 + 16 \\ \hline
 & 3 - 9 + 11 - 16 + 7 \\
-2 & \quad - 6 + 30 - 82 \\ \hline
 & 3 - 15 + 41 - 98 \\
-3 & \quad - 9 + 72 \\ \hline
 & 3 - 24 + 113 \\
-4 & \quad - 12 \\ \hline
 & 3 - 36
\end{array}$$

and the transformed function becomes, in the notation of Kramp, p. 274.

$$3(x + 1)^{4|1} - 36\,(x + 1)^{3|1} + 113\,(x + 1)^{2|1} - 98\,(x + 1)^{1|1} + 7.$$

Or, in the common notation,

$3\,(x + 1)\,(x + 2)\,(x + 3)\,(x + 4) - 36\,(x + 1)\,(x + 2)\,(x + 3) + 113\,(x + 1)\,(x + 2) - 98\,(x + 1) + 7.$

When the given expression itself is a combination of binomial factors, and it is required to transform it into some other combination, as a factorial one, the given expression may be first reduced to powers, and then transformed by the general rule; as in the next example.

Ex. 2. Given $(x + 1)\,(x - 3)\,(x + 3)\,(x + 5)$ to be converted into factors involving $x + 1$, $x + 2$, $x + 3$, $x + 4$.

Since $x + 1$ is a factor of the given and the sought expressions, it need not be attended to, as it multiplies all the terms in both. However, for illustration

we shall work out as though no two factors in the two expressions agreed with each other.

$$
\begin{array}{l}
1 + 1\ (-3 \\
\quad -3 - 3 \\
\hline
1 - 2 - 3\ (3 \\
\qquad 3 - 6 - 9 \\
\hline
1 + 1 - 9 - 9\ (5 \\
\qquad 5 + 5 - 45 - 45 \\
\hline
1 + 6 - 4 - 54 - 45
\end{array}
\qquad\qquad
\begin{array}{r|l}
 & 1 + 6 - \ \ 4 - 54 - 45 \\
-1 & \quad -1 - \ \ 5 + \ \ 9 + 45 \\
\hline
 & 1 + 5 - \ \ 9 - 45 + 0 \\
-2 & \quad -2 - \ \ 6 + 30 \\
\hline
 & 1 + 3 - 15 - 15 \\
-3 & \quad -3 - \ \ 0 \\
\hline
 & 1 + 0 - 15 \\
-4 & \quad -4 \\
\hline
 & 1 - 4
\end{array}
$$

Hence the given expression is $x^4 + 6x^3 - 4x^2 - 54x - 45$, and the factorial is $(x+1)^{4|1} - 4(x+1)^{3|1} - 15(x+1)^{2|1} - 15(x+1)$.

Ex. 3. Show that $x^3 = (x-1)\,x\,(x+1) + x$ by this method.

Ex. 4. Transform x^4 to factorials involving $x-4$, $x-3$, $x-2$, $x-1$.

VII. *To integrate the general term of a series, or to find the expression whose first difference constitutes that general term.*

1. *When the expression is composed of factors in arithmetical progression.*

Multiply the increment (or given general term) by the preceding value of the first factor, and divide the result by the number of terms thus obtained, and by the common difference of the factors. This result, when *corrected*, gives the sum required.

2. *When the expression to be integrated is the reciprocal of such a series of factors in arithmetical progression.*

Expunge the last factor from the denominator; divide the resulting fraction by the number of factors remaining, and the common difference of the factors. Then this result, written minus, will be the integral sought.

These being precisely the reverse processes by which the differences or increments were found, their truth is evident.

The *correction* arises from this cause: that $\Delta\,(z \pm a) = \Delta z$, and hence it cannot *à priori* be ascertained whether the integral is $z \pm a$, or simply z.

The correction is found from this consideration. If from any circumstance we can find what the aggregate value of a certain number of the terms is, and at the same time ascertain what value the integral gives of the same number; then the difference of these two results is the correction. Most frequently, putting $x = 0$ is the best method: but examples will render this process much plainer than precept could do.

The student will find little difficulty in reducing all expressions which involve only positive integer powers of x, that he commonly meets with, to one or more of these forms.

EXAMPLES.

Ex. 1. Integrate $x^{4|1}$ or $x\,(x+1)\,(x+2)\,(x+3)$.

The preceding value is $x-1$, the resulting number of factors is 5, and their difference 1: hence by the rule,

$$\Sigma u_x = \frac{1}{5}(x-1)\,x\,(x+1)\,(x+2)\,(x+3)+c = \frac{1}{5}(x-1)^{5|1}+c.$$

Ex. 2. Integrate $(5x+1)(5x+6.)$ Ans. $\frac{1}{15}(5x-4)(5x+1)(5x+6)+c.$

Ex. 3. Given $\left\{\frac{x}{2}-3\right\}\left\{\frac{x}{2}-\frac{5}{2}\right\}\left\{\frac{x}{2}-2\right\}.$ Ans. $\frac{1}{32}(x-7)^{4|1}+c.$

Ex. 4. Given $\left\{\frac{4x}{5}-1\right\}\left\{\frac{4x}{5}-\frac{1}{5}\right\}.$ Ans. $\frac{1}{300}(4x-9)(4x-5)(4x-1)+c.$

Ex. 5. Integrate the expression $(x+1)(x+3)$.

This expression not being composed of successive values, (the increment of x being generally taken in unity on account of the most frequent application of this method,) it must be reduced to such a form. We write, for this purpose, $(x+1)(x+3)=(x+2)(x+3)-(x+3)$ each term of which is integrable by the rule; and we have

$$\begin{aligned}\Sigma u_x &= \Sigma(x+2)(x+3)-\Sigma(x+3)\\ &= \frac{1}{3}(x+1)(x+2)(x+3)-\frac{1}{2}(x+2)(x+3)+c.\\ &= \frac{1}{6}(2x-1)(x+2)(x+3)+c.\end{aligned}$$

Ex. 6. Integrate $(2x+3)(2x+7)$; that is, $(2x+5)(2x+7)-2(2x+7)$.

Ans. $\Sigma u_x = \frac{1}{3}x(2x+5)(2x+7).$

Ex. 7. Prove (1) $\Sigma(2x+1)(2x+3)^2 = \frac{1}{24}(6x+7)(2x-1)(2x+1)(2x+3)+c.$

(2) $\Sigma x = \frac{1}{2}x(x-1)+c.$

(3) $\Sigma x^2 = \frac{1}{6}(x-1)\,x\,(2x-1)+c.$

(4) $\Sigma x^3 = \frac{1}{4}\{x(x-1)\}^2+c.$

(5) $\Sigma x^4 = \frac{1}{30}(6x^5-15x^4+10x^3-x)+c.$

Ex. 8. Let (1) $u_x = \frac{1}{(x+1)(x+2)}$; then $\Sigma u_x = \frac{-1}{x+1}+c.$

(2) $u_x = \frac{1}{x(x+1)(x+2)}$; then $\Sigma u_x = \frac{-1}{2x(x+1)}+c.$

(3) $u_x = \frac{1}{(3x+2)(3x+5)(3x+8)}$; then $\Sigma u_x = \frac{-1}{6(3x+2)(3x+5)}+c.$

Ex. 9. Integrate $\frac{x}{(2x+1)(2x+3)(2x+5)}$. Here we have, as before,

$$u_x = \frac{\frac{1}{2}(2x+1)-\frac{1}{2}}{(2x+1)(2x+3)(2x+5)} = \frac{\frac{1}{2}}{(2x+3)(2x+5)} - \frac{\frac{1}{2}}{(2x+1)(2x+3)(2x+5)}$$

Hence $\Sigma u_x = -\frac{4x+1}{8(2x+1)(2x+3)}+c.$

Ex. 10. Integrate $\frac{4x+3h}{x(x+h)(x+2h)}$, where $\Delta x = h$.

Ex. 11. Show that the integral of $\frac{1}{(x+1)(x+3)}$ is $c - \frac{2x+3}{2(x+1)(x+2)}$:

and that $\Sigma \frac{1}{x^2-1} = c + \frac{1-2x}{2x(x-1)}$.

3. *The integration of the exponentials a^x and a^{-x}.*

By II. Ex. 2, $\Delta a^x = a^x(a^{\Delta x}-1)$; and hence $a^x = \frac{\Delta a^x}{a^{\Delta x}-1}$; whence

taking the integrals, $\Sigma a^x = \frac{a^x}{a^{\Delta x}-1} + c.$

Similarly $\Delta a^{-x} = \frac{1-a^{\Delta x}}{a^{\Delta x}} a^{-x}$; and $a^{-x} = \frac{a^{\Delta x}}{1-a^{\Delta x}} \Delta a^{-x}$; and, integrating,

$$\Sigma a^{-x} = \frac{a^{\Delta x}}{1-a^{\Delta x}} a^{-x} + c.$$

For in both cases Δx is constant. When $\Delta x = 1$, we have simply,

$$\Sigma a^x = \frac{a^x}{a-1} + c, \text{ and } \Sigma a^{-x} = \frac{a}{1-a} a^{-x} + c.$$

VIII. *To find the sum of a series, whose general term is given.*

Write $n+1$ for n in the general term: then the integral is the sum of the series.

For let the series be $u_1 + u_2 + u_3 + \ldots . + u_n = S_n$: then $u_1 + u_2 + u_3 + \ldots . + u_n + u_{n+1} = S_{n+1}$. Hence by subtraction $\Delta S_n = S_{n+1} - S_n = u_{n+1}$; and integrating, $S_n = \Sigma u_{n+1}$.

EXAMPLES.

Ex. 1. Find the sum of the series $1 + 2 + 3 + \ldots + n$.

Here $u_n = n$, and $u_{n+1} = n+1$. Hence $\Delta S_n = n+1$.

By integration $S_n = \Sigma(n+1) = \frac{n(n+1)}{2} + c.$

To find c, which is the same value whatever n may be, put $n = 0$; then $S_n = 0$ also; and we have $0 = 0 + c$, or $c = 0$. Whence the sum of the series of n terms is $\frac{n(n+1)}{2}$, as at p. 161.

Ex. 2. Find the sum of the series $1^2 + 2^2 + 3^2 + \ldots . + n^2$.

Here $u_n = n^2$ and $u_{n+1} = (n+1)^2$. Whence as before

$\Delta S_n = (n+1)^2$, and integrating, $S_n = \Sigma(n+1)^2 = \frac{n(n+1)(2n+1)}{2.3}$,

the correction being found 0 as in the last example.

Ex. 3. Sum the series of cubes $1^3 + 2^3 + 3^3 + \ldots . n^3$.

Here $\Delta S_n = (n+1)^3$, and S_n* $= \left\{\frac{n(n+1)}{2}\right\}^2$.

Ex. 4. Find the sum of n terms of the series $1.2 + 2.5 + 3.8 + 4.11 + \ldots .$

Here the general term of the first factor of the several terms is obviously n, and the second (found by III.) is $3n-1$. Hence the general term of the series

---

* By comparing the solutions of examples 1 and 3 we see that the sum of n terms of the cubes of the natural numbers is equal to the square of the sum of those numbers themselves.

is $u_n = n(3n-1)$; and the increment or $(n+1)^{\text{th}}$ term is $u_{n+1} = (n+1)(3n+2) = 3n(n+1) + 2(n+1)$. Whence integrating, $S_n = n^2(n+1)$.

Ex. 5. Show that $1^2 + 3^2 + 5^2 + 7^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(4n^2-1)$.

Ex. 6. Find the sum of n terms of each of the following series:

(1.) $1 + 3 + 5 + 7 + \ldots$
(2.) $1 + 4 + 7 + 10 + \ldots$
(3.) $1 + 5 + 9 + 13 + \ldots$
(4.) $1 + 6 + 11 + 16 + \ldots$

Ex. 7. Find the sum of the 2nd, 3rd, and 4th powers of the preceding series of numbers.

Ex. 8. Sum the series of n terms of $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots$

$$\text{Here } S_n = \Sigma \frac{1}{(n+1)(n+2)(n+3)} = c - \frac{1}{2(n+1)(n+2)}.$$

To correct, put $n = 0$, then $S_0 = c - \frac{1}{2(0+1)(0+2)} = c - \frac{1}{4}$.

But $S_0 = 0$; whence $c - \frac{1}{4} = 0$, or $c = \frac{1}{4}$; and the corrected integral is

$$S_n = \frac{1}{4} - \frac{1}{2(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}.$$

Ex. 9. Sum $\frac{1}{3.5.9} + \frac{2}{5.7.9} + \frac{3}{7.9.11} + \ldots$ to n terms.

$$\text{Here } u_n = \frac{n}{(2n+1)(2n+3)(2n+5)};\text{ and } u_{n+1} = \frac{n+1}{(2n+3)(2n+5)(2n+7)}$$

$$= \frac{\frac{1}{2}(2n+3) - \frac{1}{2}}{(2n+3)(2n+5)(2n+7)} = \frac{1}{2(2n+5)(2n+7)} - \frac{1}{2(2n+3)(2n+5)(2n+7)}$$

Whence integrating u_{n+1} and correcting, we get $S_n = \frac{n(n+1)}{6(2n+3)(2n+5)}$.

Ex. 10. Find the sum of 21 terms of $\frac{5}{1.2.3.4} + \frac{9}{3.4.5.6} + \frac{13}{5.6.7.8} + \ldots$

$$\text{Here } u_n = \frac{4n+1}{(2n-1)\,2n(2n+1)(2n+2)} \text{ and } u_{n+1} = \frac{4n+5}{(2n+1)(2n+2)(2n+3)(2n+4)}.$$

Now the denominator of the function u_{n+1} not being composed of consecutive integer values of n, it must be so transformed as to answer that condition, or else into some other form which can be integrated. The latter plan is adopted here.

$$\left.\begin{array}{l}\text{Put } 2n+1 = u, \text{ then } 2n+3 = u_1 \\ \text{and } 2n+2 = v, \text{ then } 2n+4 = v_1\end{array}\right\}\ldots\ldots(1)$$

where u_1 and v_1 are the next values of u and v.

Now $\Delta\frac{1}{uv} = -\frac{u\Delta v + v\Delta u + \Delta u\,\Delta v}{uu_1\,vv_1}$; and if we find the values of u, v, u_1, v_1, Δu and Δv in (1) and insert them in this equation, we get

$$\Delta\frac{1}{uv} = -\frac{2(4n+5)}{uu_1\,vv_1} = -2\Delta S_n.$$

Hence integrating, restoring the values of u, v, u_1, v_1, and correcting we finally obtain the sum of the given series, viz.:—

$$S_n = \frac{n(2n+3)}{2(2n+1)(2n+2)}.$$

Ex. 11. Let the several series below be summed to n terms:

$$\begin{array}{l}
1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \ldots. \\
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \ldots. \\
1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + \ldots. \\
1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + \ldots. \\
1 + 5 + 15 + 35 + 70 + 126 + 210 + 330 + \ldots. \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}$$

Ans. Generally the m^{th} series is $S_n = \dfrac{n(n+1)(n+2)\ldots\{n+(m-1)\}}{1.2.3\ldots m}$.*

OTHER EXAMPLES FOR PRACTICE.

Ex. 1. Sum the series $a + ar + ar^2 + \ldots. + ar^n$ by integration; and likewise $3 + 6 + 12 + 24 + \ldots.$ to ten terms.

Ex. 2. Fifty terms of the series $\dfrac{6}{2.7} + \dfrac{6}{7.12} + \dfrac{6}{12.17} + \ldots. = \dfrac{25}{42}$.

Ex. 3. Sum the series $\dfrac{10}{1.2.3.4} + \dfrac{18}{3.4.5.6} + \dfrac{26}{5.6.7.8} + \ldots$ to fifteen terms, and to infinity. Ans. $\dfrac{495}{992}$ and $\dfrac{1}{2}$.

Ex. 4. Sum $\dfrac{1}{2.6} + \dfrac{1}{4.8} + \dfrac{1}{6.10} + \ldots.$ to n terms and to infinity.

Ex. 5. Find the sum of fifteen terms of each of the following series:—

(1) $\dfrac{5}{1.2.3} + \dfrac{6}{2.3.4} + \dfrac{7}{3.4.5} + \ldots.$

(2) $\dfrac{1}{1.4} + \dfrac{1}{2.5} + \dfrac{1}{3.6} + \dfrac{1}{4.7} + \ldots.$

(3) $\dfrac{1}{5^2-1} + \dfrac{1}{6^2-1} + \dfrac{1}{7^2-1} + \ldots.$

(4) $\dfrac{1}{3^2-2^2} + \dfrac{1}{7^2-2^2} + \dfrac{1}{11^2-2^2} + \ldots.$

IX. *The summation of series whose general term is not known.*

The sum is obtained from the following expression, by the insertion of the values of n and the first terms of the several orders of differences, found as in IV.

$$S_n = nu_1 + \frac{n(n-1)}{1.2}\Delta u_1 + \frac{n(n-1)(n-2)}{1.2.3}\Delta^2 u_1 + \ldots\ldots$$

For we have seen in V. p. 279, that in all cases

$$\begin{aligned}
u_1 &= u_1 \\
u_2 &= u_1 + \Delta u_1 \\
u_3 &= u_1 + 2\Delta u_1 + \Delta^2 u_1 \\
u_4 &= u_1 + 3\Delta u_1 + 3\Delta^2 u_1 + \Delta^3 u_1 \\
u_5 &= u_1 + 4\Delta u_1 + 6\Delta^2 u_1 + 4\Delta^3 u_1 + \Delta^4 u_1 \\
u_6 &= u_1 + 5\Delta u_1 + 10\Delta^2 u_1 + 10\Delta^3 u_1 + 5\Delta^4 u_1 + \Delta^5 u_1 \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

---

* The several series in this example have, in connection with each other, very remarkable properties, and have been much used in mathematical research. They are called the *figurate series*, or *series of figurate numbers*; and are thus derived from each other by successive additions. The first row is a series of units, and the others are formed in succession by adding each term of the m^{th} row to the $(m+1)^{\text{th}}$ term of the $(m-1)^{\text{th}}$ row. The summation shows that S_n in the successive series is expressed by the successive terms of the expanded binomial $(1+1)^n$ or $(1-1)^{-n}$; though only that of the m^{th} series is put down. See also p. 289.

Whence, adding the several vertical columns, (which are the same with the series in Ex. 11, VIII. p. 285,) we have

$$S_n = u_1 + u_2 + u_3 + \ldots\ldots + u_n$$
$$= nu_1 + \frac{n(n-1)}{1.2}\Delta u_1 + \frac{n(n-1)(n-2)}{1.2.3}\Delta^2 u_1 + \ldots.$$

Ex. 1. To find the sum of n terms of $1 + 2 + 3 + 4 \ldots.$

$$\begin{array}{r|r|l} 1 & 2 & 3 \quad 4 \ \ldots.\ \text{given series} \\ & 1 & 1 \quad 1 \ \ldots.\ \text{first differences} \\ & & 0 \quad 0 \ \ldots.\ \text{second and all higher differences.} \end{array}$$

Hence $u_1 = 1$, $\Delta u_1 = 1$, $\Delta^2 u_1 = 0$, and so on: and we have

$$S_n = nu_1 + \frac{n(n-1)}{1.2}\Delta u_1 = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2},$$

as was obtained by integration at p. 283.

Ex. 2. Find the sum of $1^2 + 2^2 + 3^2 + \ldots + 10^2$.

$$\begin{array}{rr|r|r|l} \text{Here} & 1 & 4 & 9 & 16 \quad 25 \ldots.\text{the given terms,} \\ & & 3 & 5 & 7 \quad 9 \ldots.\text{first differences,} \\ & & & 2 & 2 \quad 2 \ldots\text{second differences,} \\ & & & & 0 \quad 0 \ldots\text{third differences.} \end{array}$$

Hence $u_1 = 1$, $\Delta u_1 = 3$, $\Delta^2 u_1 = 2$, $\Delta^3 u_1 = 0$, and so on; and

$$S_n = n + \frac{n(n-1)}{1.2}.3 + \frac{(n-1)(n-2)}{1.2.3}.2 = \frac{n(n+1)(2n+1)}{1.2.3}.$$

Put $n = 10$, then $S_{10} = \frac{10.11.21}{6} = 385.$

Ex. 3. Find the sums of 50 terms of the series in IV. Examples 2, 3, 6.

It has been seen (p. 275) that an expression of the n^{th} degree has its n^{th} differences equal, and all the higher orders in succession become 0: but when an expression is of any other form, (as a^x or log. x for instance,) the successive differences never vanish; though in many of them, these differences become very small, and the assumption of their becoming 0, leads to a very small amount of error in the final numerical result. When the sum of n terms of such a series, therefore, whose differences in successive orders become very small, is required, we are at liberty to assume those differences as actually vanishing.

Ex. 4. Required the sum of all the logarithms on the fifteen pages of Hutton's Tables, from p. 186 to p. 201, inclusive, the entire numbers being integer.

Here n=8000, u_1=5, Δu_1=·00000435, $\Delta^2 u_1$=·00000000005 *. Hence S_{8000} =

$$8000.5 + \frac{8000.7999}{1.2}.\ {\cdot}00000437 + \frac{8000.7999.7998}{1.2.3}.\ {\cdot}00000000005$$

= 40143·4476668 ; which is, probably, true to five decimal places.

X. *The interpolation of series.*

Let $u_1, u_2, u_3, \ldots\ldots u_n$, be the given values of an unknown function u_x when equidistant values of x are substituted, and any number, p, of them be absent: it is required to supply the absent terms, and to find the value of the function u_x for any value of x intermediate between its extreme given values.

---

* The second difference is taken at $\left(\frac{1}{200}\right)^{th}$ of the interval ·00000001, this being the difference at the 220^{th}, 210 , 200^{th}, 190^{th}, and 180^{th}, terms: or nearly a mean of all.

1. *To supply the deficient terms of the regular series.*

Suppose m of the terms given, then the m^{th} difference, as derived from these, must be taken equal to zero, in conformity with the principle of the last solution. But in general

$$\Delta^{m+1}u_x = u_{x+m} - \frac{m}{1} \cdot u_{x+m-1} + \frac{m(m-1)}{1.2} u_{x+m-2} - \dots \pm u_1$$

and if in this we make x equal to 1, 2, ... $(n - m)$ we shall have, putting $\Delta^m u_1 = 0$, $\Delta^m u_2 = 0$, ... $\Delta^m u_{n-m} = 0$, the equations requisite for the determination of the $n - m$ unknown terms, viz.:

$$\Delta^m u_1 = u_{m+1} - \frac{m}{1} u_m + \frac{m(m-1)}{1.2} u_{m-1} - \dots \pm u_1 = 0$$

$$\Delta^m u_2 = u_{m+2} - \frac{m}{1} u_{m+1} + \frac{m(m-1)}{1.2} u_m - \dots \pm u_2 = 0$$

$$\Delta^m u_3 = u_{m+3} - \frac{m}{1} \cdot u_{m+2} + \frac{m(m-1)}{1.2} u_{m+1} - \dots \pm u_3 = 0,$$

and so on, till we obtain as many equations as there are unknown or absent terms of the series.

For illustration, suppose u_1 and u_3 were given to find u_2. Then, as there is but one deficient term, we have simply

$$\Delta^2 u_1 = u_3 - 2u_2 + u_1 = 0, \text{ or } u_2 = \frac{1}{2}\left\{u_3 + u_1\right\}.$$

Again, if u_1, u_2, u_4, u_5, were given to find u_3: then

$$\Delta^4 u_1 = u_5 - 4u_4 + 6u_3 - 4u_2 + u_1 = 0, \text{ or } u_3 = \frac{1}{6}\left\{4(u_2 + u_4) - (u_1 + u_5)\right\}.$$

Thirdly, suppose u_1, u_2, u_5, u_6, were given to find u_3 and u_4.

Here $\Delta^4 u_1 = 0$ and $\Delta^4 u_2 = 0$; or putting their values,

$$u_5 - 4u_4 + 6u_3 - 4u_2 + u_1 = 0; \quad u_6 - 4u_5 + 6u_4 - 4u_3 + u_2 = 0,$$

which two equations resolved for u_3 and u_4, we obtain

$$u_3 = \frac{1}{10}\left\{-3u_1 + 10u_2 + 5u_5 - 2u_6\right\}; \quad u_4 = \frac{1}{10}\left\{-2u_1 + 5u_2 + 10u_5 - 3u_6\right\}.$$

EXAMPLES.

Ex. 1. Given the logarithms of 101, 102, 104, 105, to find that of 103. In this, $u_1 = 2{\cdot}0043214$, $u_2 = 2{\cdot}0086002$, $u_4 = 2{\cdot}0170333$, $u_5 = 2{\cdot}0211893$. By the method explained we have

$$u_3 = \frac{1}{6}\left\{4(u_2 + u_4) - (u_1 + u_5)\right\} = 2{\cdot}0128372.$$

Ex. 2. Given the logs. of 510, 511, 513, 514, to find that of 512.

Ex. 3. Given the square roots of 1, 2, 3, 5, 6, 7, 8, to find the square roots of 4 and 9. Likewise determine the cube of 10 from those of 7, 8, 9, 11, 12; and from those of 8, 9, 11, 12, 13.

Ex. 4. Given the logs. of 10, 11, 12, 14, 16, and 19, to interpolate the logs. of 13, 15, 17, 18.

Ex. 5. The expression which gave the following values has its terms at the asterisks deficient: it is required to supply them, as nearly as can be done, by interpolation. They are 3·9956352, 3·9956396, *, *, 3·9956527, 3·9956571, *, 3·9956659, 3·9956703, 3·9956659.

2. *When the terms to be inserted are not those belonging to the equidistant values of x.*

The value will be approximately given by the general formula

$$u_{x+1} = u_1 + \frac{x}{1}\Delta u_1 + \frac{x(x-1)}{1.2}\Delta^2 u_1 + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 u_1 + \ldots.$$

the series being continued till $\Delta^m u_1 = 0$ occurs as before.

For this is only assuming that the series preserves the same law in passing through the intermediate stages between any two terms, that it does in passing from term to term by single steps.

EXAMPLES.

Ex. 1. Given log sines of 3°4′, 3°5′, 3°6′, 3°7′, 3°8′, to find that of 3°6′15″.

Angles.	log. sines.	first diffs.	second diffs.	third diffs.
3°4′	8·7283366			
		·0023516		
3 5	8·7306882		— ·0000126	
		·0023390		— ·0000001
3 6	8·7330272		— ·0000127	
		·0023263		+ ·0000004
3 7	8·7353535		— ·0000123	
		·0023140		
3 8	8·7376675			

Whence, $u_1 = 8{\cdot}7283366$, $\Delta u_1 = {\cdot}0023516$, $\Delta^2 u_1 = -{\cdot}0000126$ and $\Delta^3 u_1 = -{\cdot}0000001$. Also $x = 3°6'15'' - 3°4' = 2'15'' = \frac{9}{4}$, the equidistant interval being 1′. Consequently, by the formula, $u_{\frac{9}{4}} = u_1 + \frac{9}{4}\Delta u_1 + \frac{45}{32}\Delta^2 u_1 + \frac{15}{128}\Delta^3 u_1 =$ $8{\cdot}733609993 = \log \sin 3°6'15''$ nearly.

Ex. 2. Given log sines of 1°, 1°1′, 1°2′, 1°3′, to find that of 1°1′40″.

Ex. 3. From the series $\frac{1}{50}, \frac{1}{51}, \frac{1}{52}, \frac{1}{53}, \frac{1}{54}$, find the term which lies in the middle between $\frac{1}{52}$ and $\frac{1}{53}$. Ans. $\frac{38552503}{2024006400}$.

Ex. 4. Given the log tangents of 68°54′, 68°55′, 68°56′, 68°57′, to find that of 68°56′20″.

Ex. 5. Given the natural tangents of the same arcs, to find the natural tangent of 68°56′20″: and compare its logarithm with the answer to the last example.

Ex. 6. Given the sines of 7°34′, 7°35′, 7°36′, 7°37′, 7°38′, 7°39′, to find the sine of 7°37′30″.

Ex. 7. Find also the same sine, supposing, first, that the sine of 7°34′ had not been given in the data; and secondly, that sine 7°40′ had been given in addition to the data.

3. *When the first differences of a series of n equidistant terms are very small,* any intermediate term may be interpolated by the following formula:

$$u_1 - \frac{n}{1}u_2 + \frac{n(n-1)}{1.2}u_3 - \frac{n(n-1)(n-2)}{1.2.3}u_4 + \ldots\ldots = 0.$$

For $(1-1)^n = 1 - \frac{n}{1} + \frac{n(n-1)}{1\,2} - \frac{n(n-1)(n-2)}{1.2.3} + \ldots. = 0$;

and as $u_1, u_2, u_3, \ldots.$ are by hypothesis *nearly equal*, we have

$$u_1 - \frac{n}{1}u_2 + \frac{n(n-1)}{1.2}u_3 - \ldots. = u_1 - \frac{n}{1}u_1 + \frac{n(n-1)}{1.2}u_1 - \ldots = 0 \text{ nearly}$$

EXAMPLES.

Ex. 1. Given the square roots of 10, 11, 12, 13, 15, to find that of 14. Denote them by u_1, u_2, u_3, u_4, u_5, and u_6, of which u_5 is the term sought, and $n = 5$ the number of terms given. Hence we have

$$u_1 - 5u_2 + 10u_3 - 10u_4 + 5u_5 - u_6 = 0,$$

and hence $u_5 = \frac{1}{5}\left\{u_6 + 10u_4 - 10u_3 + 5u_2 - u_1\right\}$, or

$$\sqrt{14} = \frac{1}{5}\left\{\sqrt{15} + 10\sqrt{13} - 10\sqrt{12} + 5\sqrt{11} - \sqrt{10}\right\}$$

$$= \frac{18{\cdot}7083257}{5} = 3{\cdot}74166514 \text{ nearly.}$$

Ex. 2. Given the square roots of 37, 38, 39, 41, and 42, to find that of 40.

Ex. 3. Given the cube roots of 45, 46, 47, 48, 49, to find that of 50.

Scholium.

In Ex. 11, p. 285, the general expression for the sum of n terms of the m^{th} order of *figurate numbers* is given. These numbers are so called, from the circumstance of their capability of being arranged so as to form equilateral triangles, squares, regular pentagons, hexagons, ... ; and they are accordingly called triangular, square, pentagonal, hexagonal, . . . numbers. When the progression is $1 + 2 + 3 + 4 + \ldots n$, a triangle of that number of balls n in each side may be formed of the sum of the series: when it is $1 + 3 + 5 + 7 \ldots (2n - 1)$, then a square of the side n may be formed : when $1 + 4 + 7 + 10 + \ldots (3n - 2)$ constitutes the series, it will form a pentagon of n balls in each side : when $1 + 5 + 9 + 13 + \ldots (4n - 3)$, a hexagon : and generally, when the general term is $\{qn - (q - 1)\}$, then a $(q + 2)$-gonal figure will result for every integer value of q.

If, now, in any one case q receive all possible integer values from 1 to any specified numbers, a series of polygons, of $q + 2$ sides each will be formed ; and if they were balls, laid stratum on stratum, they would altogether form a pyramid on a $(q + 2)$-gonal base ; and the integration of the general term of the $(q + 2)$-gonal polygonal series would give the number of balls in the pyramid.

Amongst all these there are only two forms that are capable of being used for the piling of balls, on account of the balls pressing unequally on the contiguous ones, and strictly being incapable of a stable or permanent position : these are the *pyramid on the triangular base*, and the *pyramid on the square base*, p. 240.

First. Here (p. 283) $1 + 2 + 3 + \ldots n = \frac{n(n+1)}{1.2}$ = number in the base of the triangular pile ; and $\Sigma \frac{n(n+1)}{1.2} = \frac{n(n+1)(n+2)}{1.2.3}$ = number in the *triangular pile ;* agreeing with pages 161 and 283.

Second. For the square pile, we have n^2 for the general term, or number in the base ; and (p. 283) the sum of all these strata, or courses, from 1 to n, is $\frac{n(n+1)(2n+1)}{1.2.3}$ = number of balls in the *square pile ;* again, as in pages 162 and 283.

GEOMETRY.

DEFINITIONS AND PRINCIPLES.

1. GEOMETRY treats of the *forms, magnitudes,* and *positions* of bodies produced according to any specified method of construction. All the other qualities of body are left out of consideration, and the attention confined exclusively to these, either singly or in combination. In this case the body is called a *figure.*

2. The object of geometrical inquiry is twofold:—first, theoretical or speculative; and secondly, practical or operative. In one case it is *a science;* in the other it is *an art founded upon science.*

3. As a body of knowledge it is divided into sections, each of which is called a *proposition;* and every proposition is either a *theorem,* a *problem,* or a *porism.*

4. A *theorem* is a proposition in which some statement is made concerning a specified figure, in addition to the conditions of its construction, but inevitably flowing from those conditions. In order to its obtaining reception as a truth, it must be *proved* or *demonstrated.*

5. A *problem* is a proposition in which when certain figures are given, or already exhibited in construction, some other figure dependent upon these is to be found which shall fulfil some assigned conditions. It requires for its completion a discovery of the method of constructing the figure sought, and a demonstration that the method does effect the proposed object *.

6. The *words* in which a proposition is stated constitute the *enunciation;* it is called the *general enunciation* when there is no reference made to an exhibited figure; and the *particular enunciation* when such a figure is referred to.

7. The general term *solution* is often applied to the whole series of constructions and reasonings that are necessary to complete a proposition after the general enunciation is given.

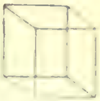
---

* The peculiar shade which distinguishes a *porism* from a problem and a theorem (of the nature of each of which it in some degree partakes) cannot here be explained to the student intelligibly. See, however, a note to Prob. I. of this work; and for a history of the methods devised for investigating them, the article *Porisms* in the Penny Cyclopædia, by J. O. Halliwell, Esq. F.R.S.

To these may be added, that a *lemma* is a proposition which is premised, or solved beforehand, in order to render what follows more simple and perspicuous.

A *corollary* is a consequent truth, gained immediately from some preceding truth, or demonstration.

A *scholium* is a remark or observation made upon something going before it: generally of a collateral rather than of a direct application.



8. Every kind of body that can be supposed actually to exist has three dimensions, *length*, *breadth*, and *thickness;* and the branch of geometry which takes all these dimensions simultaneously into consideration is hence called the *geometry of three dimensions.* It is also very frequently called *solid geometry ;* and often again, though not in strict propriety, the *geometry of planes and solids.*

9. The boundary of a body is the *superficies* or *surface.* It includes the dimensions, *length* and *breadth*, but not that of thickness. We may speak and think, and reason about the superficies, without taking note of the thickness of the body, whose superficies it is. Hence, for all the purposes of geometry, the figure which is called a superficies has no thickness.

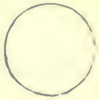
10. The boundaries of a surface are *lines.* The line has therefore no breadth nor thickness, but *length only.*

11. The extremities of lines, and their mutual intersections, are called *points.* A point, therefore, has no dimensions, but *position* only.

12. A *straight line,* or a *right line,* is that which preserves, at all its points, the same direction *.

13. A *curve line* is one which, at its successive points, changes its direction.

14. A *plane surface* is one in which *any* two points being taken, the straight line passing through them shall lie wholly in that plane. If this be not the case, the superficies is said to be a curved surface.



15. A *circle* is a figure lying wholly upon a plane superficies, and is composed of one curved line (called the *circumference)* such that all straight lines drawn from a certain point in that surface to the circumference are equal to one another. This point is called the *centre of the circle.*

The *circumference* itself is often called a circle, and also sometimes the periphery.

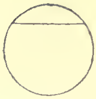
16. The *radius* of a circle is a line drawn from the centre to the circumference.

17. The *diameter* of a circle is a line drawn through the centre, and terminating at the circumference on both sides.

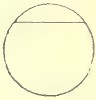
18. An *arc* of a circle is any part of the circumference.

* When the term *line* is used in the following treatise, it designates a straight line: and the curved line is called simply a *curve.* In more general reasoning, the generic term line is applied to all kinds of lines, and another mode of classification of them adopted, which will be explained in a future stage of the work.

19. A *chord* is a right line joining the extremities of an arc.

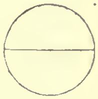


20. A *segment* is any part of a circle bounded by an arc and its chord.



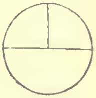
21. A *semicircle* is half the circle, or a segment cut off by a diameter.

The half circumference is sometimes called the Semicircle.

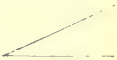


22. A *sector* is any part of a circle which is bounded by an arc, and two radii drawn to its extremities.

23. A *quadrant*, or quarter of a circle, is a sector having a quarter of the circumference for its arc, and its two radii are perpendicular to each other. A quarter of the circumference is sometimes called a quadrant.

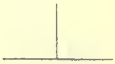


24. An *angle* is the inclination or opening of two lines, having different directions, and meeting in a point.

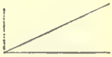


According to circumstances, they are distinguished into right or oblique; and the oblique are again distinguished into acute or obtuse.

25. When one line standing on another line makes the adjacent angles equal to one another, each of them is called a *right angle*, and the straight lines are said to be *perpendicular* to one another.



26. An *oblique angle* is that which is made by two oblique lines; and is either less or greater than a right angle.



Of oblique angles, that which is less than a right angle, is called *acute;* and that which is greater than a right angle, is called an *obtuse angle.*

27. *Plane figures* are bounded either by right lines or curves.

28. Plane figures that are bounded by right lines have names according to the number of their sides, or of their angles; for they have as many sides as angles; the least number being three.

29. A figure of three sides (and consequently three angles) is called a *triangle:* and it receives particular denominations from the relations of its sides and angles.

When defined according to its sides, it is *equilateral, isosceles,* or *scalene.*

30. An *equilateral triangle* is that whose three sides are all equal.

31. An *isosceles triangle* is that which has two equal sides

32. A *scalene triangle* is that whose three sides are all unequal.

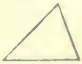
When defined according to its angles, it is either *right-angled, obtuse-angled,* or *acute-angled.*

33. A *right-angled triangle* is that which has one right angle.

All other triangles are oblique-angled, and are either obtuse or acute.

34. An *obtuse-angled triangle* has one obtuse angle.

35. An *acute-angled triangle* has all its three angles acute.



36. A figure of four sides and angles is called a *quadrangle*, or a *quadrilateral,* or a *trapezium.*

37. A *parallelogram* is a quadrilateral which has both its pairs of opposite sides parallel. And it takes the following particular names, viz. *rectangle, square, rhombus, rhomboid.*

38. A *rectangle* is a parallelogram, having a right angle.

39. A *square* is an equilateral rectangle; having its length and breadth equal, or all its sides equal, and all its angles equal.

40. A *rhomboid* is an oblique-angled parallelogram.

41. A *rhombus* is an equilateral rhomboid; having all its sides equal, but its angles oblique.

42. A *trapezium* is a quadrilateral which has not its opposite sides parallel.

43. A *trapezoid* has only one pair of opposite sides parallel.

44. A *diagonal* is a line joining any two opposite angles of a quadrilateral, or of any other right lined figure.

45. Plane figures that have more than four sides are, in general, called *polygons;* and they receive other particular names, according to the number of their sides or angles. Thus,

46. A *pentagon* is a polygon of five sides; a *hexagon*, of six sides; a *heptagon*, seven; an *octagon*, eight; a *nonagon*, nine; a *decagon*, ten; an *undecagon*, eleven; and a *dodecagon*, or *duodecagon*, twelve sides.

47. A *regular polygon* has all its sides and all its angles equal. If they are not both equal, the polygon is *irregular.*

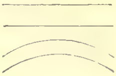
48. An *equilateral triangle* is also a regular figure of three sides, and the square is one of four: the former being also called a *trigon*, and the latter a *tetragon.*

49. Any figure is *equilateral* when all its sides are equal: and it is *equiangular*, when all its angles are equal. When both these are equal, it is called a *regular figure.*

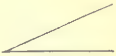
50. By the *distance of a point from a line* is meant the *shortest* line that can be drawn from the point to the line. It is shown that this is the *perpendicular*. See *th*. 21.

51. When two or more lines are considered in relation to one another, they take different names, either *parallel, oblique, perpendicular,* or *tangential.*

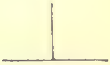
52. *Parallel lines* are always at the same distance; and they never meet, though ever so far produced.



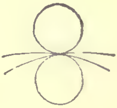
53. *Oblique lines* change their distance, and would meet, if produced on the side of the least distance.



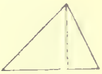
54. One line is *perpendicular* to another, when it inclines not more on the one side than the other, or when the angles on both sides of it are equal.



55. A line or circle is *tangential,* or is a tangent to a circle, or other curve, when it touches it, without cutting, although both are produced.

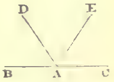


56. The height or *altitude* of a figure is a perpendicular let fall from an angle, or its vertex, to the opposite side, called the *base.*



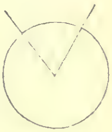
57. In a right-angled triangle, the side opposite the right angle is called the *hypothenuse*; and the other two sides are called the *legs,* and sometimes the *base* and *perpendicular.*

58. When an angle is denoted by three letters, of which one stands at the angular point, and the other two on the two sides, that which stands at the angular point is read in the middle: thus BAD signifies the angle contained by the lines BA and AD, and so of the other angles DAE and EAC.



59. For the purpose of *calculation* the circumference of every circle is supposed to be divided into 360 equal parts called *degrees;* and each degree into 60 minutes, each *minute* into 60 *seconds,* and so on. Hence a *semicircle* contains 180 degrees, and a *quadrant* 90 degrees.

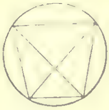
60. The *measure of an angle,* is an arc of any circle contained between the two lines which form that angle, the angular point being the centre; and it is estimated by the number of degrees contained in that arc, it being shown at prop. 4, that such a mode of admeasurement is consistent with the principles and truths of geometry.



61. Lines, or chords, are said to be equi-distant from the centre of a circle, when perpendiculars drawn to them from the centre are equal.

62. And the right line on which the greater perpendicular falls, is said to be farther from the centre.

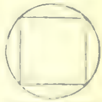
63. An *angle in a segment* is that which is contained by two lines, drawn from any point in the arc of the segment, to the two extremities of that arc.



64. An *angle on a segment*, or *an arc*, is that which is contained by two lines, drawn from any point in the opposite or supplemental part of the circumference, to the extremities of the arc, and containing the arc between them.

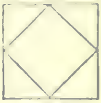
65. An *angle at the circumference*, is that whose angular point or summit is any where in the circumference: and an *angle at the centre*, is that whose angular point is at the centre.

66. A *right-lined figure* is *inscribed* in a circle, or the circle circumscribes it, when all the angular points of the figure are in the circumference of the circle.



67. A *right-lined figure circumscribes* a circle, or the circle is inscribed in it, when all the sides of the figure touch the circumference of the circle.

68. One right-lined figure is *inscribed* in another, or the latter *circumscribes* the former, when all the angular points of the former are placed in the sides of the latter.



69. A *secant* is a line that cuts a circle, lying partly within, and partly without it.

70. Two triangles, or other right-lined figures, are said to be mutually equilateral, when all the sides of the one are equal to the corresponding sides of the other, each to each: and they are said to be *mutually equiangular*, when the angles of the one are respectively equal to those of the other.

71. *Identical figures*, are such as are both mutually equilateral and equiangular; or that have all the sides and all the angles of the one, respectively equal to all the sides and all the angles of the other, each to each; so that if the one figure were applied to, or laid upon the other, all the sides of the one would exactly fall upon and cover all the sides of the other; the two becoming as it were but one and the same figure.

72. *Similar figures*, are those that have all the angles of the one equal to all the angles of the other, each to each, and the sides about the equal angles proportional.

73. The *perimeter* of a figure, is the sum of all its sides taken together.

74. The first part of geometry has respect to figures traced upon a plain superficies, or in which only two dimensions are concerned. It is called, therefore, the *geometry of two dimensions;* and often also *plane geometry*, though general usage has restricted the term, plane geometry, to a particular class of such figures as may be traced upon a plane. These figures are entirely composed of straight lines and circles, which are therefore called *geometrical lines;* whereas lines constructed any other way are called *mechanical curves*, or *lines of the higher orders.*

AXIOMS.

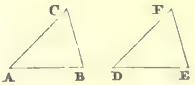
Axioms are those fundamental truths, which, from their simplicity, are evident to every mind, and are essential in a body of science as the foundation of a system of reasoning.

1. Things which are equal to the same thing are equal to each other.
2. When equals are added to equals, the wholes are equal.
3. When equals are taken from equals, the remainders are equal.
4. When equals are added to unequals, the wholes are unequal.
5. When equals are taken from unequals, the remainders are unequal.
6. Things which are double of the same thing, or equal things, are equal to each other.
7. Things which are halves of the same thing, are equal.
8. Every whole is equal to all its parts taken together, and greater than any of them.
9. Things which coincide, or fill the same space, are identical, or mutually equal in all their parts.
10. All right angles are equal to one another.
11. Angles that have equal measures, or arcs, are equal.

THEOREM I. *

If two triangles have two sides and the included angle in the one, equal to two sides and the included angle in the other, the triangles will be identical, or equal in all respects.

In the two triangles ABC, DEF, if the side AC be equal to the side DF, and the side BC equal to the side EF, and the angle C equal to the angle F; then will the two triangles be identical, or equal in all respects.



For conceive the triangle ABC to be applied to, or placed on, the triangle DEF, in such a manner that the point C may coincide with the point F, and the side AC with the side DF, which is equal to it.

Then, since the angle F is equal to the angle C (*hyp.*), the side BC will fall on

---

* In the complete process by which a theorem is enunciated and established, the following parts are to be always found.

I. The general enunciation, already explained, (see *def.* 6.) and which comprises

1. The *subject* spoken of, or the *hypothesis* to which the theorem is affirmed to be true.

2. The *predicate* or affirmation made respecting the hypothetical figure. These are alike to be found in the general and particular enunciations.

II. Sometimes a *preliminary construction* is required to connect the hypothetical with the predicated parts of the figure.

III. One or more *syllogisms*, by which the necessary dependence of the predicate upon the hypothesis is established.

The syllogism is composed of three propositions,

1. The *major premise* :—an axiom or previously demonstrated truth.

2. The *minor premise* :—a proposition which agrees in *subject* with this axiom or theorem, and in predicate with the enunciated theorem.

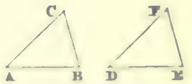
3. The *conclusion*, which is the theorem in question, and inferred from the premises.

the side EF. Also, because AC is equal to DF, and BC equal to EF (*hyp.*), the point A will coincide with the point D, and the point B with the point E; consequently the side AB will coincide with the side DE. Therefore the two triangles are identical, and have all their other corresponding parts equal (*ax.* 9), namely, the side AB equal to the side DE, the angle A to the angle D, and the angle B to the angle E.

THEOREM II.

When two triangles have two angles and the included side in the one, equal to two angles and the included side in the other, the triangles are identical, or have their other sides and angle equal.

Let the two triangles ABC, DEF, have the angle A equal to the angle D, the angle B equal to the angle E, and the side AB equal to the side DE; then these two triangles will be identical.



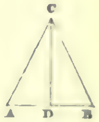
For, conceive the triangle ABC to be placed on the triangle DEF, in such manner that the side AB may fall exactly on the equal side DE. Then, since the angle A is equal to the angle D (*hyp.*), the side AC must fall on the side DF; and, in like manner, because the angle B is equal to the angle E, the side BC must fall on the side EF. Thus the three sides of the triangle ABC will be exactly placed on the three sides of the triangle DEF; consequently the two triangles are identical (*ax* 9), having the other two sides AC, BC, equal to the two DF, EF, and the remaining angle C equal to the remaining angle F.

THEOREM III.

In an isosceles triangle, the angles at the base are equal: or, if a triangle have two sides equal, their opposite angles will also be equal.

If the triangle ABC have the side AC equal to the side BC: then will the angle B be equal to the angle A.

For, conceive the angle C to be bisected, or divided into two equal parts, by the line CD, making the angle ACD equal to the angle BCD.



Then, the two triangles, ACD, BCD, have two sides and the contained angle of the one, equal to two sides and the contained angle of the other, viz. the side AC equal to BC, the angle ACD equal to BCD, and the side CD common; therefore these two triangles are identical, or equal in all respects (*th.* 1); and consequently the angle A equal to the angle B.

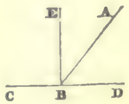
Cor. 1. Hence the line which bisects the vertical angle of an isosceles triangle, bisects the base, and is also perpendicular to it.

Cor. 2. Hence too it appears, that every equilateral triangle, is also equiangular, or has all its angles equal.

THEOREM IV.

When one line meets another, the angles which it makes on the same side of the other, are together equal to two right angles.

Let the line AB meet the line CD: then will the two angles ABC, ABD, taken together, be equal to two right angles.



For, first, when the two angles ABC, ABD, are equal to each other, they are both of them right angles (*def.* 25).

But when the angles are unequal, suppose BE drawn perpendicular to CD. Then, since the two angles EBC, EBD, are right angles (*def.* 25), and the angle EBD is equal to the two angles EBA, ABD, together (*ax.* 8), the three angles, EBC, EBA, and ABD, are equal to two right angles.

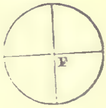
But the two angles EBC, EBA, are together equal to the angle ABC (*ax.* 8). Consequently the two angles ABC, ABD, are also equal to two right angles.

Cor. 1. Hence also, conversely, if the two angles ABC, ABD, on both sides of the line AB, make up together two right angles, then CB and BD form one continued right line CD.

Cor. 2. Hence, all the angles which can be made, at any point B, by any number of lines, on the same side of the right line CD, are, when taken all together, equal to two right angles.

Cor. 3. And, as all the angles that can be made on the other side of the line CD are also equal to two right angles; therefore all the angles that can be made quite round a point B, by any number of lines, are equal to four right angles.

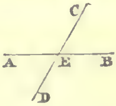
Cor. 4. Hence also the whole circumference of a circle, being the sum of the measures of all the angles that can be made about the centre F (*def.* 57), is the measure of four right angles. Consequently, a semicircle, or 180 degrees, is the measure of two right angles; and a quadrant, or 90 degrees, the measure of one right angle.



THEOREM V.

When two lines intersect each other, the opposite angles are equal.

Let the two lines AB, CD, intersect in the point E; then will the angle AEC be equal to the angle BED, and the angle AED equal to the angle CEB.



For, since the line CE meets the line AB, the two angles AEC, BEC, taken together, are equal to the two right angles (*th.* 4).

In like manner, the line BE, meetiug the line CD, makes the two angles BEC, BED, equal to two right angles.

Therefore the sum of the two angles AEC, BEC, is equal to the sum of the two BEC, BED (*ax.* 1).

And if the angle BEC, which is common, be taken away from both these, the remainiug angle AEC will be equal to the remaining angle BED (*ax.* 3).

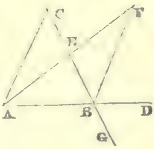
And in like manner it may be shown, that the angle AED is equal to the opposite angle BEC.

THEOREM VI.

When one side of a triangle is produced, the outward angle is greater than either of the two inward opposite angles.

Let ABC be a triangle, having the side AB produced to D; then will the outward angle CBD be greater than either of the inward opposite angles A or C.

For, conceive the side BC to be bisected in the point E, and draw the line AE, producing it till EF be equal to AE; and join BF.

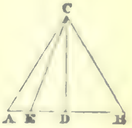


Then, since the two triangles AEC, BEF, have the side AE = the side EF, and the side CE = the side BD (*suppos.*), and the included or opposite angles at E also equal (*th.* 5), therefore those two triangles are equal in all respects (*th.* 1), and have the angle C = the corresponding angle EBF. But the angle CBD is greater than the angle EBF; consequently, the said outward angle CBD is also greater than the angle C.

In like manner, if CB be produced to G, and AB be bisected, it may be shown that the outward angle ABG, or its equal CBD, is greater than the other angle A.

THEOREM VII.

When a triangle has two of its angles equal, the sides opposite to them are also equal.

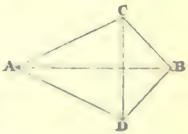


For draw CD perpendicular to the base; and conceive the triangle BCD to revolve about CD till the plane of it coincides with that of CDA. Then since CDA and CDB are right angles (*def.* 25), the line BD will coincide with DA. Now if B coincide with A, the line CB will coincide with CA, and the angles CBD, CAD coinciding, they will be equal: but if it be denied that B will coincide with A, let it coincide with some other point E in the line DA, and join CE, and hence (*th.* 6) the angle CED (that is CBD) is greater than CAD, which is contrary to the hypothesis, and hence cannot be true of the figure respecting which the hypothesis is made. Hence B cannot but coincide with A, and CA cannot but be equal to CB.

Cor. Hence every equiangular triangle is also equilateral.

THEOREM VIII.

When two triangles have all the three sides in the one, equal to all the three sides in the other, the triangles are identical, or have also their three angles equal, each to each.



Let the two triangles ABC, ABD, have their three sides respectively equal, viz. the side AB equal to AB, AC to AD, and BC to BD; then shall the two triangles be identical, or have their angles equal, viz. those angles that are opposite to the equal sides; namely, the angle BAC to the angle BAD, the angle ABC to the angle ABD, and the angle C to the angle D.

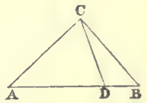
For, conceive the two triangles to be joined together by their longest equal sides, and draw the line CD.

Then, in the triangle ACD, because the side AC is equal to AD (*hyp.*), the angle ACD is equal to the angle ADC (*th.* 3). In like manner, in the triangle BCD, the angle BCD is equal to the angle BDC, because the side BC is equal to BD. Hence then, the angle ACD being equal to the angle ADC, and the angle BCD to the angle BDC, by equal additions the sum of the two angles ACD, BCD, is equal to the sum of the two ADC, BDC (*ax.* 2), that is the whole angle ACB equal to the whole angle ADB.

Since then, the two sides AC, CB, are equal to the two sides AD, DB, each to each (*hyp.*), and their contained angles ACB, ADB, also equal, the two triangles ABC, ABD, are identical (*th.* 1), and have the other angles equal, viz. the angle BAC to the angle BAD, and the angle ABC to the angle ABD

THEOREM IX.

The greater side of every triangle is opposite to the greater angle; and the greater angle opposite to the greater side.



Let ABC be a triangle, having the side AB greater than the side AC; then will the angle ACB, opposite the greater side AB, be greater than the angle B, opposite the less side AC.

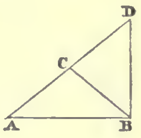
For, on the greater side AB, take the part AD equal to the less side AC, and join CD. Then, since BCD is a triangle, the outward angle ADC is greater than the inward opposite angle B (*th.* 6). But the angle ACD is equal to the said outward angle ADC, because AD is equal to AC (*th.* 3). Consequently, the angle ACD also is greater than the angle B. And since the angle ACD is only a part of ACB, much more must the whole angle ACB be greater than the angle B.

Again, conversely, if the angle C be greater than the angle B, then will the side AB, opposite the former, be greater than the side AC, opposite the latter.

For, if AB be not greater than AC, it must be either equal to it, or less than it. But it cannot be equal, for then the angle C would be equal to the angle B (*th.* 3), which it is not, by the supposition. Neither can it be less, for then the angle C would be less than the angle B, by the former part of this; which is also contrary to the supposition. The side AB, then, being neither equal to AC, nor less than it, must necessarily be greater.

THEOREM X.

The sum of any two sides of a triangle is greater than the third side.



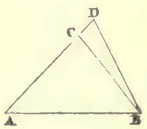
Let ABC be a triangle; then will the sum of any two of its sides be greater than the third side, as for instance, AC + CB greater than AB.

For, produce AC till CD be equal to CB, or AD equal to the sum of the two AC + CB; and join BD:—then, because CD is equal to CB (*constr.*), the angle D is equal to the angle CBD (*th.* 3). But the angle ABD is greater than the angle CBD, consequently it must also be greater than the angle D. And, since the greater side of any triangle is opposite to the greater angle (*th.* 9), the side AD (of the triangle ABD) is greater than the side AB. But AD is equal to AC and CD, or AC and CB, taken together (*constr.*); therefore AC + CB is also greater than AB.

Cor. The shortest distance between two points is a single right line drawn from the one point to the other.

THEOREM XI.

The difference of any two sides of a triangle is less than the third side.



Let ABC be a triangle; then will the difference of any two sides, as AB — AC, be less than the third side BC.

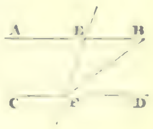
For, produce the less side AC to D, till AD be equal to the greater side AB, so that CD may be the difference of the two sides AB — AC; and join BD. Then, because AD is equal to AB (*constr.*) the opposite angles D and ABD are

equal (*th.* 3). But the angle CBD is less than the angle ABD, and consequently also less than the equal angle D. And since the greater side of any triangle is opposite to the greater angle (*th.* 9), the side CD (of the triangle BCD) is less than the side BC.

Otherwise. Set off upon AB, a distance AI, equal to AC. Then (*th.* 20) AC + CB is greater than AB, that is, greater than AI + IC. From these, take away the equal parts AC, AI, respectively; and there remains CB greater than IC. Consequently, IC is less than CB.

THEOREM XII.

When a line intersects two parallel lines, it makes the alternate angles equal to each other.



Let the line EF cut the two parallel lines AB, CD; then will the angle AEF be equal to the alternate angle EFD.

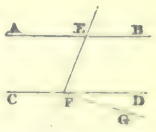
For if they are not equal, one of them must be greater than the other; let it be EFD for instance, which is the greater, if possible; and conceive the line FB to be drawn, cutting off the part or angle EFB equal to the angle AEF, and meeting the line AB in the point B.

Then, since the outward angle AEF, of the triangle BEF, is greater than the inward opposite angle EFB (*th.* 6); and since these two angles also are equal (*constr.*), it follows, that those angles are both equal and unequal at the same time: which is impossible. Therefore, the angle EFD is not unequal to the alternate angle AEF, that is, they are equal to each other.

Cor. Right lines which are perpendicular to one of two parallel lines, are also perpendicular to the other.

THEOREM XIII.

When a line, cutting two other lines, makes the alternate angles equal to each other, those two lines are parallel.



Let the line EF, cutting the two lines AB, CD, make the alternate angles AEF, DFE, equal to each other; then will AB be parallel to CB.

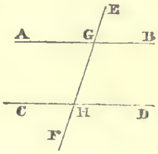
For if they be not parallel, let some other line, as FG, be parallel to AB. Then, because of these parallels, the angle AEF is equal to the alternate angle EFG (*th.* 12). But the angle AEF is equal to the angle EFD (*hyp.*) Therefore the angle EFD is equal to the angle EFG (*ax.* 1); that is, a part is equal to the whole: which is impossible. Therefore no line but CD can be parallel to AB.

Cor. Those lines which are perpendicular to the same line, are parallel to each other.

THEOREM XIV.

When a line cuts two parallel lines, the outward angle is equal to the inward opposite one, on the same side; and the two inward angles, on the same side, are together equal to two right angles.

Let the line EF cut the two parallel lines AB, CD; then will the outward angle EGB be equal to the inward opposite angle GHD, on the same side of the line EF; and the two inward angles BGH, GHD, taken together, will be equal to two right angles.



For since the two lines AB, CD, are parallel, the angle AGH is equal to the alternate angle GHD (*th.* 12). But the angle AGH is equal to the opposite angle EGB (*th.* 5.) Therefore the angle EGB is also equal to the angle GHD (*ax.* 1).

Again, because the two adjacent angles EGB, BGH, are together equal to two right angles (*th.* 4), of which the angle EGB has been shown to be equal to the angle GHD; therefore the two angles BGH, GHD, taken together, are also equal to two right angles.

Cor. 1. And, conversely, if one line meeting two other lines, make the angles on the same side of it equal, those two lines are parallels.

Cor. 2. If a line, cutting two other lines, make the sum of the two inward angles on the same side, less than two right angles, those two lines will not be parallel, but will meet each other when produced.

THEOREM XV.

Those lines which are parallel to the same line are parallel to each other.

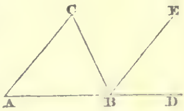
Let the lines AB, CD, be each of them parallel to the line EF; then shall the lines AB, CD, be parallel to each other.

For, let the line GI be perpendicular to EF. Then will this line be also perpendicular to both the lines AB, CD (*cor. th.* 12), and consequently the two lines AB, CD, are parallels (*cor. th.* 13).

THEOREM XVI.

When one side of a triangle is produced, the outward angle is equal to both the inward opposite angles taken together.

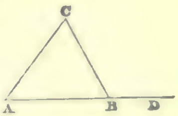
Let the side AB, of the triangle ABC, be produced to D; then will the outward angle CBD be equal to the sum of the two inward opposite angles A and C.



For, conceive BE to be drawn parallel to the side AC of the triangle. Then BC, meeting the two parallels AC, BE, makes the alternate angles C and CBE equal (*th.* 12). And AD, cutting the same two parallels AC, BE, makes the inward and outward angles on the same side, A and EBD, equal to each other (*th.* 14). Therefore, by equal additions, the sum of the two angles A and C, is equal to the sum of the two CBE and EBD, that is, to the whole angle CBD (*ax.* 2).

THEOREM XVII.

In any triangle, the sum of all the three angles is equal to two right angles.



Let ABC be any plane triangle; then the sum of the three angles, A + B + C, is equal to two right angles.

For, let the side AB be produced to D. Then the outward angle CBD is equal to the sum of the two inward opposite angles A + C (*th.* 16). To each of these equals add the inward angle B, then will the sum of the three inward angles, A + B + C, be equal to the sum of the two adjacent angles ABC + CBD (*ax.* 2). But the sum of these two last adjacent angles is equal to two right angles (*th.* 4). Therefore also the sum of the three angles of the triangle, A + B + C, is equal to two right angles (*ax.* 1).

Cor. 1. If two angles in one triangle be equal to two angles in another triangle, the third angles will also be equal (*ax.* 3), and the two triangles equiangular.

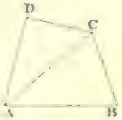
Cor. 2. If one angle in one triangle, be equal to one angle in another, the sums of the remaining angles will also be equal (*ax.* 3).

Cor. 3. If one angle of a triangle be right, the sum of the other two will also be equal to a right angle, and each of them singly will be acute, or less than a right angle.

Cor. 4. The two least angles of every triangle are acute, or each less than a right angle.

THEOREM XVIII.

In any quadrangle, the sum of all the four inward angles is equal to four right angles.



Let ABCD be a quadrangle; then the sum of the four inward angles, A + B + C + D, is equal to four right angles.

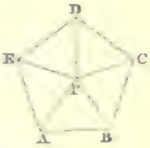
Let the diagonal AC be drawn, dividing the quadrangle into two triangles, ABC, ADC. Then, because the sum of the three angles of each of these triangles is equal to two right angles (*th.* 17); it follows, that the sum of all the angles of both triangles, which make up the four angles of the quadrangle, must be equal to four right angles (*ax.* 2).

Cor. 1. Hence, if three of the angles be right ones, the fourth will also be a right angle.

Cor. 2. And if the sum of two of the four angles be equal to two right angles, the sum of the remaining two will also be equal to two right angles.

THEOREM XIX.

In any figure whatever, the sum of all the inward angles, taken together, is equal to twice as many right angles, wanting four, as the figure has sides.



Let ABCDE be any figure; then the sum of all its inward angles, A + B + C + D + E, is equal to twice as many right angles, wanting four, as the figure has sides.

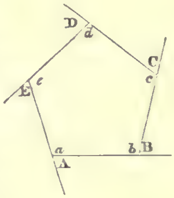
For, from any point P, within it, draw lines, PA, PB, PC, &c. to all the angles, dividing the polygon into as many triangles as it has sides. Now the sum of the three angles of each of these triangles, is equal to two right angles (*th.* 17);

therefore the sum of the angles of all the triangles is equal to twice as many right angles as the figure has sides. But the sum of all the angles about the point P, which are so many of the angles of the triangles, but no part of the inward angles of the polygon, is equal to four right angles, (*cor.* 3, *th.* 4,) and must be deducted out of the former sum. Hence it follows that the sum of all the inward angles of the polygon alone, A + B + C + D + E, is equal to twice as many right angles as the figure has sides, wanting the said four right angles.

THEOREM XX.

When every side of any figure is produced, the sum of all the outward angles thereby made is equal to four right angles.

Let A, B, C,.... be the outward angles of any polygon, made by producing all the sides; then will the sum, A + B + C + D + E, of all those outward angles, be equal to four right angles.

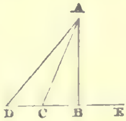


For every one of these outward angles, together with its adjacent inward angle, make up two right angles, as A + *a* equal to two right angles, being the two angles made by one line meeting another (*th.* 4). And there being as many outward, or inward angles, as the figure has sides; therefore the sum of all the inward and outward angles, is equal to twice as many right angles as the figure has sides. But the sum of all the inward angles, with four right angles, is equal to twice as many right angles as the figure has sides (*th.* 19). Therefore the sum of all the inward and all the outward angles is equal to the sum of all the inward angles and four right angles (*ax.* 1). From each of these take away all the inward angles, and there remain all the outward angles equal to four right angles (by *ax.* 3).

THEOREM XXI.

A perpendicular is the shortest line that can be drawn from a given point to an indefinite line: and, of any other lines drawn from the same point, those that are nearest the perpendicular are less than those more remote.

If AB, AC, AD,.... be lines drawn from the given point A, to the indefinite line DE, of which AB is perpendicular; then shall the perpendicular AB be less than AC, and AC less than AD, &c.



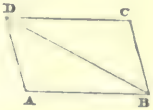
For, the angle B being a right one, the angle C is acute (*cor.* 3, *th.* 17), and therefore less than the angle B. But the less angle of a triangle is subtended by the less side (*th.* 9). Therefore the side AB is less than the side AC.

Again, the angle ACB being acute, as before, the adjacent angle ACD will be obtuse (*th.* 4); consequently the angle D is acute (*cor.* 3, *th.* 17), and therefore is less than the angle C. And since the less side is opposite to the less angle, therefore the side AC is less than the side AD.

Cor. A perpendicular is the least distance of a given point from a line.

THEOREM XXII.

The opposite sides and angles of any parallelogram are equal to each other; and the diagonal divides it into two equal triangles.



Let ABCD be a parallelogram, of which the diagonal is BD; then will its opposite sides and angles be equal to each other, and the diagonal BD will divide it into two equal parts, or triangles.

For, since the sides AB and DC are parallel, as also the sides AD and BC (*def.* 37), and the line BD meets them; therefore the alternate angles are equal (*th.* 12), namely, the angle ABD to the angle CDB, and the angle ADB to the angle CBD: hence the two triangles, having two angles in the one equal to two angles in the other, have also their third angles equal (*cor.* 1, *th.* 17), namely, the angle A equal to the angle C, which are two of the opposite angles of the parallelogram.

Also, if to the equal angles ABD, CDB, be added the equal angles CBD, ADB, the wholes will be equal (*ax.* 2), namely, the whole angle ABC to the whole ADC, which are the other two opposite angles of the parallelogram.

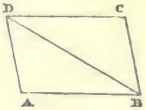
Again, since the two triangles are mutually equiangular and have a side in each equal, viz. the common side BD; therefore the two triangles are identical (*th.* 2), or equal in all respects, namely, the side AB equal to the opposite side DC, and AD equal to the opposite side BC, and the whole triangle ABD equal to the whole triangle BCD.

Cor. 1. Hence, if one angle of a parallelogram be a right angle, all the other three will also be right angles, and the parallelogram a rectangle.

Cor. 2. Hence also, the sum of any two adjacent angles of a parallelogram is equal to two right angles.

THEOREM XXIII.

Every quadrilateral, whose opposite sides are equal, is a parallelogram, or has its opposite sides parallel.



Let ABCD be a quadrangle, having the opposite sides equal, namely, the side AB equal to DC, and AD equal to BC; then shall these equal sides be also parallel, and the figure a parallelogram.

For, let the diagonal BD be drawn. Then, the triangles, ABD, CBD, being mutually equilateral (*hyp.*), they are also mutually equiangular (*th.* 8), or have their corresponding angles equal; consequently the opposite sides are parallel (*th.* 13); viz. the side AB parallel to DC, and AD parallel to BC, and the figure is a parallelogram.

THEOREM XXIV.

Those lines which join the corresponding extremes of two equal and parallel lines, are themselves equal and parallel.

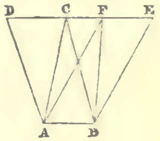
Let AB, DC, be two equal and parallel lines; then will the lines AD, BC, which join their extremes, be also equal and parallel. (*See the fig. above.*)

For, draw the diagonal BD. Then, because AB and DC are parallel, (*hyp.*)

the angle ABD is equal to the alternate angle BDC (*th.* 12): hence then, the two triangles having two sides and the contained angles equal, viz. the side AB equal to the side DC, and the side BD common, and the contained angle ABD equal to the contained angle BDC, they have the remaining sides and angles also respectively equal (*th.* 1); consequently AD is equal to BC, and also parallel to it (*th.* 12).

THEOREM XXV.

Parallelograms, as also triangles, standing on the same base, and between the same parallels, are equal to each other.



Let ABCD, ABEF, be two parallelograms, and ABC, ABF, two triangles, standing on the same base AB, and between the same parallels AB, DE; then will the parallelogram ABCD be equal to the parallelogram ABEF, and the triangle ABC equal to the triangle ABF.

For, since the line DE cuts the two parallels AF, BE, and the two AD, BC, it makes the angle E equal to the angle AFD, and the angle D equal to the angle BCE (*th.* 14); the two triangles ADF, BCE, are therefore equiangular (*cor.* 1, *th.* 17); and having the two corresponding sides AD, BC, equal (*th.* 22), being opposite sides of a parallelogram, these two triangles are identical, or equal in all respects (*th.* 2). If each of these equal triangles then be taken from the whole space ABED, there will remain the parallelogram ABEF in the one case, equal to the parallelogram ABCD in the other (*ax.* 3).

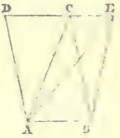
Also the triangles ABC, ABF, on the same base AB, and between the same parallels, are equal, being the halves of the said equal parallelograms (*th.* 22).

Cor. 1. Parallelograms, or triangles, having the same base and altitude, are equal. For the altitude is the same as the perpendicular or distance between the two parallels, which is every where equal, by the definition of parallels.

Cor. 2. Parallelograms, or triangles, having equal bases and altitudes, are equal. For, if the one figure be applied with its base on the other, the bases will coincide or be the same, because they are equal: and so the two figures, having the same base and altitude, are equal.

THEOREM XXVI.

If a parallelogram and a triangle stand on the same base, and between the same parallels, the parallelogram will be double the triangle, or the triangle half the parallelogram.



Let ABCD be a parallelogram, and ABE a triangle, on the same base AB, and between the same parallels AB, DE; then will the parallelogram ABCD be double the triangle ABE, or the triangle half the parallelogram.

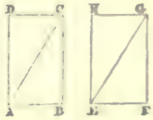
For, draw the diagonal AC of the parallelogram, dividing it into two equal parts (*th.* 22). Then because the triangles ABC, ABE, on the same base, and between the same parallels, are equal (*th.* 25), and because the one triangle ABC is half the parallelogram ABCD (*th.* 22), the other equal triangle ABE is also equal to half the same parallelogram ABCD.

Cor. 1. A triangle is equal to half a parallelogram of the same base and altitude, because the altitude is the perpendicular distance between the parallels, which is every where equal, by the definition of parallels.

Cor. 2. If the base of a parallelogram be half that of a triangle, of the same altitude, or the base of the triangle be double that of the parallelogram, the two figures will be equal to each other.

THEOREM XXVII.

Rectangles that are contained by equal lines, are equal to each other.



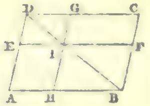
Let BD, FH, be two rectangles, having the sides AB, BC, equal to the sides EF, FG, each to each; then will the rectangle BD be equal to the rectangle FH.

For, draw the two diagonals AC, EG, dividing the two parallelograms each into two equal parts. Then the two triangles ABC, EFG, are equal to each other (*th.* 1), because they have the two sides AB, BC, and the contained angle B, equal to the two sides EF, FG, and the contained angle F. (*hyp.*) But these equal triangles are the halves of the respective rectangles: and because the halves, or the triangles, are equal, the wholes, or the rectangles DB, HF, are also equal (*ax.* 6).

Cor. The squares on equal lines are also equal; for every square is a species of rectangle.

THEOREM XXVIII.

The complements of the parallelograms, which are about the diagonal of any parallelogram, are equal to each other.

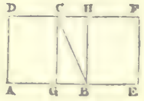


Let AC be a parallelogram, BD a diagonal, EIF parallel to AB or DC, and GIH parallel to AD or BC, making AI, IC, complements to the parallelograms EG, HF, which are about the diagonal DB: then will the complement AI be equal to the complement IC.

For, since the diagonal DB bisects the three parallelograms AC, EG, HG (*th.* 22); therefore, the whole triangle DAB being equal to the whole triangle DCB, and the parts DEI, IHB, respectively equal to the parts DGI, IFB, the remaining parts AI, IC, must also be equal (*ax.* 3).

THEOREM XXIX.

A trapezoid, or trapezium having two sides parallel, is equal to half a parallelogram, whose base is the sum of those two sides, and its altitude the perpendicular distance between them.



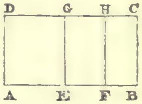
Let ABCD be the trapezoid, having its two sides AB, DC, parallel; and in AB produced take BE, equal to DC, so that AE may be the sum of the two parallel sides; produce DC also, and let EF, GC, BH, be all three parallel to AD. Then is AF a parallelogram of the same altitude with the trapezoid ABCD, having its base AE equal to the sum of the parallel sides of the trapezoid; and it is to be proved that the trapezoid ABCD is equal to half the parallelogram AF.

Now, since triangles, or parallelograms, of equal bases and altitude, are equal (*cor.* 2, *th.* 25), the parallelogram DG is equal to the parallelogram HE, and the triangle CGB is equal to the triangle CHB; consequently the line BC bisects, or equally divides, the parallelogram AF, and ABCD is the half of it.

THEOREM XXX.

The sum of all the rectangles contained under one whole line, and the several parts of another line, any way divided, is equal to the rectangle contained under the two whole lines.

Let AD be the one line, and AB the other, divided into the parts AE, EF, FB; then will the rectangle contained by AD and AB, be equal to the sum of the rectangles of AD and AE, and AD and EF, and AD and FB: thus expressed, $AD.AB = AD.AE + AD.EF + AD.FB$.



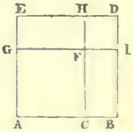
For, make the rectangle AC of the two whole lines AD, AB; and draw EG, FH, perpendicular to AB, or parallel to AD, to which they are equal (*th.* 22). Then the whole rectangle AC is made up of all the other rectangles AG, EH, FC: but these rectangles are contained by AD and AE, EG and EF, FH and FB; which are equal to the rectangles of AD and AE, AD and EF, AD and FB, because AD is equal to each of the two EG, FH. Therefore the rectangle AD.AB is equal to the sum of all the other rectangles AD.AE, AD.EF, AD.FB.

Cor. If a right line be divided into any two parts, the square on the whole line is equal to both the rectangles of the whole line and each of the parts.

THEOREM XXXI.

The square of the sum of two lines is greater than the sum of their squares, by twice the rectangle of the said lines. Or, the square of a whole line is equal to the squares of its two parts, together with twice the rectangle of those parts.

Let the line AB be the sum of any two lines AC, CB; then will the square of AB be equal to the squares of AC, CB, together with twice the rectangle of AC.CB. That is, $AB^2 = AC^2 + CB^2 + 2AC.CB$.



For, let ABDE be the square on the sum or whole line AB, and ACFG the square on the part AC. Produce CF and GF to the other sides at H and I.

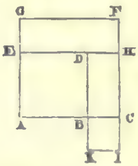
From the lines CH, GI, which are equal, being each equal to the sides of the square AB or BD (*th.* 22), take the parts CF, GF, which are also equal, being the sides of the square AF, and there remains FH equal to FI, which are also equal to DH, DI, being the opposite sides of the parallelogram. Hence the figure HI is equilateral: and it has all its angles right ones (*cor.* 1, *th.* 22); it is therefore a square on the line FI, or the square of its equal CB. Also the figures EF, FB, are equal to two rectangles under AC and CB, because GF is equal to AC, and FH or FI equal to CB: but the whole square AD is made up of the four figures, viz. the two squares AF, FD, and the two equal rectangles EF, FB; that is, the square of AB is equal to the squares of AC, CB, together with twice the rectangle of AC, CB.

Cor. Hence, if a line be divided into two equal parts; the square of the whole line will be equal to four times the square of half the line.

THEOREM XXXII.

The square of the difference of two lines is less than the sum of their squares, by twice the rectangle of the said lines.

Let AC, BC, be any two lines, and AB their difference: then will the square of AB be less than the squares of AC, BC, by twice the rectangle of AC and BC. Or $AB^2 = AC^2 + BC^2 - 2AC.BC$.



For, let ABDE be the square on the difference AB, and ACFG the square on the line AC. Produce ED to H; also produce DB and HC, and draw KI, making BI the square of the other line BC.

Now, it is obvious, that the square AD is less than the two squares AF, BI, by the two rectangles EF, DI: but GF is equal to the one line AC, and GE or FH is equal to the other line BC; consequently the rectangle EF, contained under EG and GF, is equal to the rectangle of AC and BC.

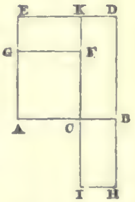
Again, FH being equal to CI or BC or DH, by adding the common part HC, the whole HI will be equal to the whole FC, or equal to AC; and consequently the figure DI is equal to the rectangle contained by AC and BC.

Hence the two figures EF, DI, are two rectangles of the two lines AC, BC; and consequently the square of AB is less than the squares of AC, BC, by twice the rectangle AC, BC.

THEOREM XXXIII.

*The rectangle under the sum and difference of two lines, is equal to the difference of the squares of those lines.**

Let AB, AC, be any two unequal lines; then will the difference of the squares of AB, AC, be equal to a rectangle under their sum and difference: that is, $AB^2 - AC^2 = (AB + AC)(AB - AC)$.



For, let ABDE be the square of AB, and ACFG the square of AC. Produce DB till BH be equal to AC; draw HI parallel to AB or ED, and produce FC both ways to I and K.

Then the difference of the two squares AD, AF, is evidently the two rectangles EF, KB. But the rectangles EF, BI, are equal, being

---

* This and the two preceding theorems are evinced algebraically, by the three expressions

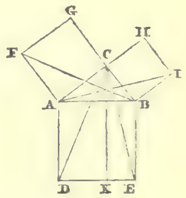
$$(a + b)^2 = a^2 + 2ab + b^2 = a^2 + b^2 + 2ab$$
$$(a - b)^2 = a^2 - 2ab + b^2 = a^2 + b^2 - 2ab$$
$$(a + b)(a - b) = a^2 - b^2.$$

Of course it is here assumed that an algebraic product corresponds to a geometrical rectangle, which is shown in the *Application of Algebra to Geometry*.

contained under equal lines; for EK and BH are each equal to AC, and GE is equal to CB, being equal to the difference between AB and AC, or their equals AE and AG. Therefore the two EF, KB, are equal to the two KB, BI, or to the whole KH; and consequently KH is equal to the difference of the squares AD, AF. But KH is a rectangle contained by DH, or the sum of AB and AC, and by KD, or the difference of AB, and AC: therefore the difference of the squares of AB, AC, is equal to the rectangle under their sum and difference.

THEOREM XXXIV.

In any right-angled triangle, the square of the hypothenuse is equal to the sum of the squares of the other two sides.



Let ABC be a right-angled triangle, having the right angle C; then will the square of the hypothenuse AB, be equal to the sum of the squares of the other two sides AC, CB. Or $AB^2 = AC^2 + BC^2$.

For, on AB describe the square AE, and on AC, CB, the squares AG, BH; then draw CK parallel to AD or BE; and join AI, BF, CD, CE.

Now, because the line AC meets the two CG, CB, so as to make two right angles, these two form one straight line GB (*cor.* 1, *th.* 6). And because the angle FAC is equal to the angle DAB, being each a right angle, or the angle of a square; to each of these angles add the common angle EAC, so will the whole angle or sum FAB, be equal to the whole angle or sum CAD: but the line FA is equal to the line AC, and the line AB to the line AD, being sides of the same square; so that the two sides FA, AB, and their included angle FAB, are equal to the two sides CA, AD, and the contained angle CAD, each to each: therefore the whole triangle AFB is equal to the whole triangle ACD (*th.* 1).

But the square AG is double the triangle AFB, on the same base FA, and between the same parallels FA, GB (*th.* 26); in like manner the parallelogram AK is double the triangle ACD, on the same base AD, and between the same parallels AD, CK: and since the doubles of equal things are equal (*ax.* 6); therefore the square AG is equal to the parallelogram AK.

In like manner, the other square BH is proved equal to the other parallelogram BK: consequently the two squares AG and BH together, are equal to the two parallelograms AK and BK together, or to the whole square AE; that is, the sum of the two squares on the two less sides is equal to the squares on the greatest side.

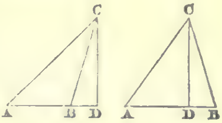
Cor. 1. Hence, the square of either of the two less sides, is equal to the difference of the squares of the hypothenuse and the other side (*ax.* 3); or equal to the rectangle contained by the sum and difference of the said hypothenuse and other side (*th.* 33).

Cor. 2. Hence, also, if two right-angled triangles have two sides of the one equal to two corresponding sides of the other; their third sides will also be equal, and the triangles identical.

THEOREM XXXV.

In any triangle, the difference of the squares of the two sides is equal to the difference of the squares of the segments of the base, or of the two lines, or distances, included between the extremes of the base and the perpendicular.

Let ABC be any triangle, having CD perpendicular to AB; then will the difference of the squares of AC, BC, be equal to the difference of the squares of AD, BD; that is, $AC^2 - BC^2 = AD^2 - BD^2$.



For, since (*th.* 34) $AC^2 = AD^2 + DC^2$ and $BC^2 = BD^2 + DC^2$, the differences of these are equal; that is, $AC^2 - BC^2 = AD^2 - BD^2$.

Cor. The rectangle of the sum and difference of the two sides of any triangle, is equal to the rectangle of the sum and difference of the distances between the perpendicular and the two extremes of the base, or equal to the rectangle of the base and the difference or sum of the segments, according as the perpendicular falls within or without the triangle.

That is, $(AC + BC)(AC - BC) = (AD + BD)(AD - BD)$:

or, $(AC + BC)(AC - BC) = AB(AD - BD)$ *in the 2nd fig.;*

and $(AC + BC)(AC - BC) = AB(AD + BD)$ *in the 1st fig.*

THEOREM XXXVI.

In any obtuse-angled triangle, the square of the side subtending the obtuse angle, is greater than the sum of the squares of the other two sides, by twice the rectangle of the base and the distance of the perpendicular from the obtuse angle.

Let ABC be a triangle, obtuse angled at B, and CD perpendicular to AB; then will the square of AC be greater than the squares of AB, BC, by twice the rectangle of AB, BD: that is, $AC^2 = AB^2 + BC^2 + 2AB \cdot BD$. *See the 1st fig. above, or below.*

For, $AD^2 = AB^2 + BD^2 + 2AB \cdot BD$ (*th.* 31),

and $AD^2 + CD^2 = AB^2 + BD^2 + CD^2 + 2AB \cdot BD$ (*ax.* 2):

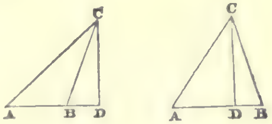
But $AD^2 + CD^2 = AC^2$, and $BD^2 + CD^2 = BC^2$ (*th.* 34);

therefore $AC^2 = AB^2 + BC^2 + 2AB \cdot BD$.

THEOREM XXXVII.

In any triangle, the square of the side subtending an acute angle, is less than the squares of the base and the other side, by twice the rectangle of the base and the distance of the perpendicular from the acute angle.

Let ABC be a triangle, having the angle A acute, and CD perpendicular to AB; then will the square of BC be less than the squares of AB, AC, by twice the rectangle of AB, AD. That is, $BC^2 + 2AD \cdot DB = AB^2 + AC^2$.



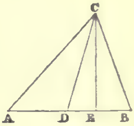
For $BD^2 = AD^2 + AB^2 - 2AD \cdot AB$ (*th.* 32),

and $BD^2 + DC^2 = AD^2 + DC^2 + AB^2 - 2AD \cdot AB$ (*ax.* 2);

therefore $BC^2 = AC^2 + AB^2 - 2AD \cdot AB$ (*th.* 34).

THEOREM XXXVIII.

In any triangle, the double of the square of a line drawn from the vertex to the middle of the base, together with double the square of the half base, is equal to the sum of the squares of the other two sides.

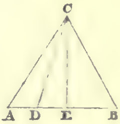


Let ABC be a triangle, and CD the line drawn from the vertex to the middle of the base AB, bisecting it into the two equal parts AD, DB; then will the sum of the squares of AC, CB, be equal to twice the sum of the squares of CD, AD; or $AC^2 + CB^2 = 2CD^2 + 2AD^2$.

$$\text{For } AC^2 = CD^2 + AD^2 + 2AD \cdot DE \ (th.\ 36),$$
$$\text{and } BC^2 = CD^2 + BD^2 - 2AD \cdot DE \ (th.\ 37);$$
$$\text{whence } AC^2 + BC^2 = 2CD^2 + AD^2 + BD^2 = 2CD^2 + 2AD^2 \ (ax.\ 2).$$

THEOREM XXXIX.

In an isosceles triangle, the square of a line drawn from the vertex to any point in the base, together with the rectangle of the segments of the base, is equal to the square of one of the equal sides of the triangle.

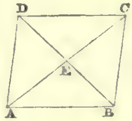


Let ABC be the isosceles triangle, and CD a line drawn from the vertex to any point D in the base: then will the square of AC be equal to the square of CD, together with the rectangle of AD and DB. That is, $AC^2 = CD^2 + AD \cdot DB$.

$$\text{For } AC^2 - CD^2 = AE^2 - DE^2 \ (th.\ 35) = AD \cdot DB \ (th.\ 33);$$
$$\text{Therefore } AC^2 = CD^2 + AD \cdot DB \ (ax.\ 2).$$

THEOREM XL.

In any parallelogram, the two diagonals bisect each other; and the sum of their squares is equal to the sum of the squares of all the four sides of the parallelogram.



Let ABCD be a parallelogram, whose diagonals intersect each other in E: then AE will be equal to EC, and BE to ED; and the sum of the squares of AC, BD, will be equal to the sum of the squares of AB, BC, CD, DA. That is,

$$AE = EC, \text{ and } BE = ED,$$
$$\text{and } AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2.$$

For, the triangles, AEB, DEC, are equiangular, because they have the opposite angles at E equal (*th.* 5), and the two lines AC, BD, meeting the parallels AB, DC, make the angle BAE equal to the angle DCE, and the angle ABE equal to the angle CDE, and the side AB equal to the side DC (*th.* 22); therefore these two triangles are identical, and have their corresponding sides equal (*th.* 2), viz. AE = EC, and BE = ED.

Again, since AC and BD are bisected in E, we have (*th.* 38).

$AD^2 + DC^2 = 2AE^2 + 2ED^2$ and $AB^2 + BC^2 = 2AE^2 + 2EB^2$;

hence (*ax.* 2) $AB^2 + BC^2 + CD^2 + DA^2 = 4AE^2 + 4ED^2 = AC^2 + BD^2$.

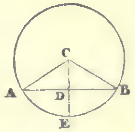
Cor. 1. If $AD = DC$, or the parallelogram be a rhombus; then $AD^2 = AE^2 + ED^2$, $CD^2 = DE^2 + CE^2$, &c.

Cor. 2. Hence, and by *th.* 34, the diagonals of a rhombus intersect at right angles.

THEOREM XLI.

If a line, drawn through or from the centre of a circle, bisect a chord, it will be perpendicular to it; or, if it be perpendicular to the chord, it will bisect both the chord and the arc of the chord.

Let AB be any chord in a circle, and CD a line drawn from the centre C to the chord. Then, if the chord be bisected in the point D, CD will be perpendicular to AB.



Draw the two radii CA, CB. Then the two triangles ACD, BCD, having CA equal to CB (*def.* 44), and CD common, also AD equal to DB (*hyp.*); they have all the three sides of the one, equal to all the three sides of the other, and so have their angles also equal (*th.* 8): hence then, the angle ADC being equal to the angle BDC, these angles are right angles, and the line CD is perpendicular to AB (*def.* 53).

Again, if CD be perpendicular to AB, then will the chord AB be bisected at the point D, or have AD equal to DB; and the arc AEB bisected in the point E, or have AE equal to EB.

For, having drawn CA, CB, as before: then, in the triangle ABC, because the side CA is equal to the side CB, their opposite angles A and B are also equal (*th.* 3): hence then, in the two triangles ACD, BCD, the angle A is equal to the angle B, and the angles at D are equal (*def.* 53); therefore their third angles are also equal (*cor.* 1, *th.* 17); and having the side CD common, they have also the side AD equal to the side DB (*th.* 2).

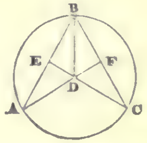
Also, since the angle ACE is equal to the angle BCE, the arc AE, which measures the former (*def.* 57), is equal to the arc BE, which measures the latter, since equal angles must have equal measures.

Cor. Hence a line bisecting any chord at right angles, passes through the centre of the circle.

THEOREM XLII.

If more than two equal lines can be drawn from any point within a circle to the circumference, that point will be the centre.

Let ABC be a circle, and D a point within it: then if any three lines, DA, DB, DC, drawn from the point D to the circumference, be equal to each other, the point D will be the centre.



Draw the chords AB, BC, which let be bisected in the points E, F, and join DE, DF.

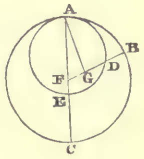
Then, the two triangles, DAE, DBE, have the side DA equal to the side DB by supposition, and the side AE equal to the side EB by hypothesis, also the side DE common: therefore these two triangles are iden-

tical, and have the angles at E equal to each other (*th.* 8); consequently, DE is perpendicular to the middle of the chord AB (*def.* 54), and therefore passes through the centre of the circle (*cor. th.* 41).

In like manner, it may be shown that DF passes through the centre: and consequently that the point D is the centre of the circle, and that the three equal lines DA, DB, DC, are radii.

THEOREM XLIII.

If two circles placed one within another, touch, the centres of the circle and the point of contact will be all in the same right line.



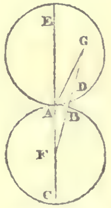
Let the two circles ABC, ADE, touch one another internally in the point A; then will the point A and the centres of those circles be all in the same right line.

Let F be the centre of the circle ABC, through which draw the diameter AFC. Then, if the centre of the other circle can be out of this line AC, let it be supposed in some other point as G; through which draw the line FG, cutting the two circles in B and D.

Now, in the triangle AFG, the sum of the two sides FG, GA, is greater than the third side AF (*th.* 10), or greater than its equal radius FB. From each of these take away the common part FG, and the remainder GA will be greater than the remainder GB: but the point G being supposed the centre of the inner circle, its two radii, GA, GD, are equal to each other; consequently GD will also be greater than GB. Again, ADE being the inner circle, GD is necessarily less than GB: so that GD is both greater and less than GB; which is absurd. To remove this absurdity we must abandon the supposition that produced it, which was that G might be out of the line AFC: and consequently the centre G cannot be out of the line AFC; that is, the line joining the centre FG, passes through the point of contact A.

THEOREM XLIV.

If two circles touch one another externally, the centres of the circles and the point of contact will be all in the same right line.



Let the two circles ABC, ADE, touch one another externally at the point A; then will the point of contact A and the centres of the two circles be all in the same right line.

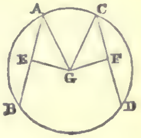
Let F be the centre of the circle ABC, through which draw the diameter AFC, and produce it to the other circle at E. Then, if the centre of the other circle ADE can be out of the line FE, let it, if possible, be supposed in some other point as G; and draw the lines AG, FBDG, cutting the two circles in B and D.

Then, in the triangle AFG, the sum of the two sides AF, AG, is greater than the third side FG (*th.* 10): but, if F and G being the centres of the two circles, the two radii GA, GD, are equal, as are also the two radii AF, FB. Hence the sum of GA, AF, is equal to the sum of GD, BF; and therefore this latter sum also, GD, BF, is greater than GF, which is absurd: and consequently, as in the former proposition, the centre G cannot be out of the line EF.

THEOREM XLV.

Any chords in a circle, which are equally distant from the centre, are equal to each other; or if they be equal to each other, they will be equally distant from the centre.

Let AB, CD, be any two chords at equal distances from the centre G: then will these two chords AB, CD, be equal to each other.



Draw the two radii GA, GC, and the two perpendiculars GE, GF, which are the equal distances from the centre G. Then the two right-angled triangles, GAE, GCF, have the side GA equal the side GC, and the side GE equal the side GF, and the angle at E equal to the angle at F, therefore those two triangles are identical (*cor.* 2. *th.* 34), and have the line AE equal to the line CF: but AB is the double of AE, and CD is the double of CF (*th.* 41), and therefore AB is equal to CD (*ax.* 6).

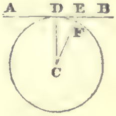
Again, if the chord AB be equal to the chord CD; then will their distances from the centre, GE, GF, also be equal to each other.

For, since AB is equal CD by supposition, the half AE is equal the half CF: and the radii GA, GC, being equal, as well as the right angles E and F, therefore the third sides are equal (*cor.* 2, *th.* 34), or the distance GE equal the distance GF.

THEOREM XLVI.

A line perpendicular to the extremity of a radius, is a tangent to the circle.

Let the line ADB be perpendicular to the radius CD of a circle; then shall AB touch the circle in the point D only.



From any other point E in the line AB draw CFE to the centre, cutting the circle in F.

Then because the angle D, of the triangle CDE, is a right angle, the angle at E is acute (*cor.* 3, *th.* 17), and consequently less than the angle D: but the greater side is always opposite to the greater angle (*th.* 9), and therefore the side CE is greater than the side CD, or greater than its equal CF. Hence the point E is without the circle; and the same for every other point in the line AB; and consequently the whole line is without the circle, and meets it in the point D only.

THEOREM XLVII.

When a line is tangent to a circle, a radius drawn to the point of contact is perpendicular to the tangent.

Let the line AB touch the circumference of a circle at the point D; then will the radius CD be perpendicular to the tangent AB. *See the last figure.*

For the line AB being wholly without the circumference except at the point D, every other line, as CE, drawn from the centre C to the line AB, must pass out of the circle to arrive at this line. The line CD is, therefore, the shortest that can be drawn from the point C to the line AB, and consequently (*th.* 21), it is perpendicular to that line.

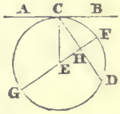
Cor. Hence, conversely, a line drawn perpendicular to a tangent, at the point of contact, passes through the centre of the circle.

THEOREM XLVIII.

The angle formed by a tangent and chord is measured by half the arc of that chord.

Let AB be a tangent to a circle, and CD a chord drawn from the point of contact C; then is the angle BCD measured by half the arc CFD, and the angle ACD measured by half the arc CGD.

Draw the radius EC to the point of contact, and the radius EF perpendicular to the chord at H.



Then the radius EF, being perpendicular to the chord CD, bisects the arc CFD (*th.* 41). Therefore CF is half the arc CFD.

In the triangle CEH, the angle H being a right one, the sum of the two remaining angles E and C is equal to a right angle (*cor.* 3, *th.* 17), which is equal to the angle BCE, because the radius CE is perpendicular to the tangent. From each of these equals take away the common part or angle C, and there remains the angle E equal to the angle BCD : but the angle E is measured by the arc CF (*def.* 60), which is the half of CFD; therefore the equal angle BCD must also have the same measure, namely, half the arc CFD of the chord CD.

Again, the line GEF, being perpendicular to the chord CD, bisects the arc CGD (*th.* 41); and therefore CG is half the arc CGD. Now, since the line CE, meeting FG, makes the sum of the two angles at E equal to two right angles (*th.* 6), and the line CD makes with AB the sum of the two angles at C equal to two right angles; if from these two equal sums there be taken away the parts or angles CEH and BCH, which have been proved equal, there remains the angle CEG equal to the angle ACH. Now the former of these, CFG, being an angle at the centre, is measured by the arc CG (*def.* 60); consequently the equal angle ACD must also have the same measure CG, which is half the arc CGD of the chord CD.

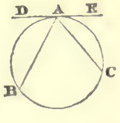
Cor. 1. The sum of two right angles is measured by half the circumference. For the two angles BCD, ACD, which makes up two right angles, are measured by the arcs CF, CG, which make up half the circumference, FG being a diameter.

Cor. 2. Hence also one right angle must have for its measure a quarter of the circumference, or 90 degrees.

THEOREM XLIX.

An angle at the circumference of a circle is measured by half the arc that subtends it.

Let BAC be an angle at the circumference; it has for its measure, half the arc BC which subtends it.

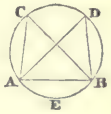


For, suppose the tangent DE to pass through the point of contact A : then, the angle DAC being measured by half the arc ABC, and the angle DAB by half the arc AB (*th.* 48); it follows, by equal subtraction, that the difference, or angle BAC, must be measured by half the arc BC, which it stands upon.

THEOREM L.

All angles in the same segment of a circle, or standing on the same arc, are equal to each other.

Let C and D be two angles in the same segment ACDB. or, which is the same thing, standing on the supplemental arc AEB; then will the angle C be equal to the angle D.

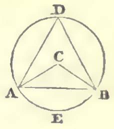


For each of these angles is measured by half the arc AEB; and thus, having equal measures, they are equal to each other (*ax.* 11).

THEOREM LI.

An angle at the centre of a circle is double the angle at the circumference, when both stand on the same arc.

Let C be an angle at the centre C, and D an angle at the circumference, both standing on the same arc or same chord AB: then will the angle C be double of the angle D, or the angle D equal to half the angle C.

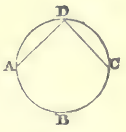


For, the angle at the centre C is measured by the whole arc AEB (*def.* 60), and the angle at the circumference D is measured by half the same arc AEB (*th.* 49); therefore the angle D is only half the angle C, or the angle C double the angle D.

THEOREM LII.

An angle in a semicircle, is a right angle.

If ABC or ADC be a semicircle; then any angle D in that semicircle, is a right angle.

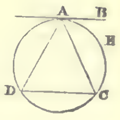


For, the angle D, at the circumference, is measured by half the arc ABC (*th.* 49), that is, by a quadrant of the circumference: and a quadrant is the measure of a right angle (*cor.* 4, *th.* 6; or *cor.* 2, *th.* 48). Therefore the angle D is a right angle.

THEOREM LIII.

The angle formed by a tangent to a circle, and a chord drawn from the point of contact, is equal to the angle in the alternate segment.

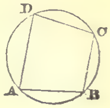
If AB be a tangent, and AC a chord, and D any angle in the alternate segment ADC; then will the angle D be equal to the angle BAC made by the tangent and chord of the arc AEC.



For the angle D, at the circumference, is measured by half the arc AEC (*th.* 49); and the angle BAC, made by the tangent and chord, is also measured by the same half arc AEC (*th.* 48); therefore, these two angles are equal (*ax.* 11).

THEOREM LIV.

The sum of any two opposite angles of a quadrangle inscribed in a circle, is equal to two right angles.

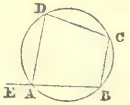


Let ABCD be any quadrilateral inscribed in a circle; then shall the sum of the two opposite angles A and C, or B and D, be equal to two right angles.

For the angle A is measured by half the arc DCB, which it stands upon, and the angle C by half the arc DAB, (*th.* 49); therefore the sum of the two angles A and C is measured by half the sum of these two arcs, that is, by half the circumference. But half the circumference is the measure of two right angles (*cor.* 4, *th.* 6); therefore the sum of the two opposite angles A and C is equal to two right angles. In like manner it is shown, that the sum of the other two opposite angles, D and B, is equal to two right angles.

THEOREM LV.

If any side of a quadrangle, inscribed in a circle, be produced out, the outward angle will be equal to the inward opposite angle.

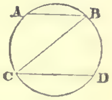


If the side AB, of the quadrilateral ABCD, inscribed in a circle, be produced to E; the outward angle DAE will be equal to the inward opposite angle C.

For, the sum of the two adjacent angles DAE and DAB is equal to two right angles (*th.* 4); and the sum of the two opposite angles C and DAB is also equal to two right angles (*th.* 54); therefore the former sum, of the two angles DAE and DAB, is equal to the latter sum, of the two C and DAB (*ax.* 1). From each of these equals taking away the common angle DAB, there remains the angle DAE equal the angle C.

THEOREM LVI.

Any two parallel chords intercept equal arcs.

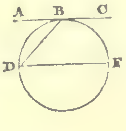


Let the two chords AB, CD, be parallel: then will the arcs AC, BD, be equal; or AC = BD.

Draw the line BC. Then, because the lines AE, CD, are parallel, the alternate angles B and C are equal (*th.* 12). But the angle at the circumference B, is measured by half the arc AC (*th.* 49); and the other equal angle at the circumference C is measured by half the arc BD; therefore the halves of the arcs AC, BD, and consequently the arcs themselves, are also equal.

THEOREM LVII.

When a tangent and chord are parallel to each other, they intercept equal arcs.



Let the tangent ABC be parallel to the chord DF; then are the arcs BD, BF, equal; that is, BD = BF.

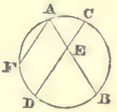
Draw the chord BD. Then, because the lines AB, DF, are parallel, the alternate angles D and B are equal (*th.* 12), But the angle B, formed by a tangent and chord, is measured by half the arc BD (*th.* 48); and the other angle at

the circumference D is measured by half the arc BF (*th.* 49); therefore the arcs BD, BF, are equal.

THEOREM LVIII.

The angle formed, within a circle, by the intersection of two chords, is measured by half the sum of the two intercepted arcs.

Let the two chords AB, CD, intersect at the point E: then the angle AEC, or DEB, is measured by half the sum of the two arcs AC, DB.



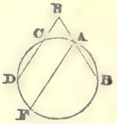
Draw the chord AF parallel to CD. Then, because the lines AF, CD, are parallel, and AB cuts them, the angles on the same side A and DEB are equal (*th.* 14): but the angle at the circumference A is measured by half the arc BF, or of the sum of FD and DB (*th.* 49); therefore the angle E is also measured by half the sum of FD and DB.

Again, because the chords AF, CD, are parallel, the arcs AC, FD, are equal (*th.* 56); therefore the sum of the two arcs AC, DB, is equal to the sum of the two FD, DB; and consequently the angle E, which is measured by half the latter sum, is also measured by half the former.

THEOREM LIX.

The angle formed, out of a circle, by two secants, is measured by half the difference of the intercepted arcs.

Let the angle E be formed by two secants EAB and ECD; this angle is measured by half the difference of the two arcs AC, DB, intercepted by the two secants.



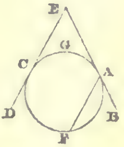
Draw the chord AF parallel to CD. Then, because the lines AF, CD, are parallel, and AB cuts them, the angles on the same side A and BED are equal (*th.* 14): but the angle A, at the circumference, is measured by half the arc BF (*th.* 49), or of the difference of DF and DB: therefore the equal angle E is also measured by half the difference of DF, DB.

Again, because the chords AF, CD, are parallel, the arcs AC, FD, are equal (*th.* 56); therefore the difference of the two arcs AC, DB, is equal to the difference of the two DF, BD; and consequently the angle E, which is measured by half the latter difference, is also measured by half the former.

THEOREM LX.

The angle formed by two tangents, is measured by half the difference of the two intercepted arcs.

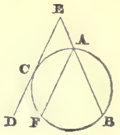
Let EB, ED, be two tangents to a circle at the points A, C; then the angle E is measured by half the difference of the two arcs CFA, CGA.



Draw the chord AF parallel to ED. Then, because the lines AF, ED, are parallel, and EB meets them, the angles on the same side A and E are equal (*th.* 14): but the angle A, formed by the chord AF and tangent AB, is measured

by half the arc AF, (*th.* 48); therefore the equal angle E is also measured by half the same arc AF, or half the difference of the arcs CFA and CF, or CGA (*th.* 57).

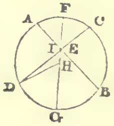
Cor. In like manner it is proved, that the angle E, formed by a tangent ECD, and a secant EAB, is measured by half the difference of the two intercepted arcs CA and CFB.



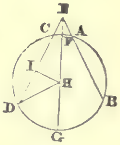
THEOREM LXI.

When two lines, meeting a circle each in two points, cut one another, either within it or without it; the rectangle of the parts of the one, is equal to the rectangle of the parts of the other; the parts of each being measured from the point of meeting to the two intersections with the circumference.

Let the two lines AB, CD, meet each other in E; then the rectangle of AE, EB, will be equal to the rectangle of CE, ED. Or, AE . EB = CE . ED.



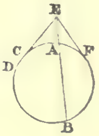
For, through the point E draw the diameter FG; also, from the centre H draw the radius DH, and draw HI perpendicular to CD.



Then, since DEH is a triangle, and the perp. HI bisects the chord CD (*th.* 41), the line CE is equal to the difference of the segments DI, EI, the sum of them being DE: and because H is the centre of the circle, and the radii DH, FH, GH, are all equal, the line EG is equal to the sum of the sides DH, HE; and EF is equal to their difference.

But the rectangle of the sum and difference of the two sides of a triangle is equal to the rectangle of the sum and difference of the segments of the base (*th.* 35); therefore the rectangle of FE, EG, is equal to the rectangle of CE, ED. In like manner it is proved, that the same rectangle of FE, EG, is equal to the rectangle of AE, EB: and consequently, the rectangle of AE, EB, is also equal to the rectangle of CE, ED (*ax.* 1).

Cor. 1. When one of the lines in the second case, as DE, by revolving about the point E, comes into the position of the tangent EC or ED, the two points C and D running into one; then the rectangle of CE, ED, becomes the square of CE, because CE and DE are then equal. Consequently, the rectangle of the parts of the secant AE . EB, is equal to CE^2, the square of the tangent.

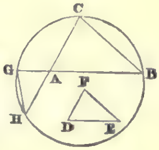


Cor. 2. Hence both the tangents EC, EF, drawn from the same point E, are equal; since the square of each is equal to the same rectangle or quantity AE . EB.

THEOREM LXII.

In equiangular triangles the rectangles of the corresponding or like sides, taken alternately, are equal.

Let ABC, DEF, be two equiangular triangles, having the angle A equal to the angle D, the angle B to the angle E, and the angle C to the angle F; also the like sides AB, DE, and AC, DF, being those opposite the equal angles: then will the rectangle of AB, DF, be equal to the rectangle of AC, DE.



In BA produced take AG equal to DF; and through the three points B, C, G, conceive a circle BCGH to be described, meeting CA produced at H, and join GH.

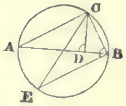
Then the angle G is equal to the angle C on the same arc BH, and the angle H equal to the angle B on the same arc CG (*th.* 50); also the opposite angles at A are equal (*th.* 7): therefore the triangle AGH is equiangular to the triangle ACB, and consequently to the triangle DFE also. But the two like sides AG, DF, are also equal by supposition; consequently the two triangles AGH, DFE, are identical (*th.* 2), having the two sides AG, AH, equal to the two DF, DE, each to each.

But GA . AB = HA . AC (*th.* 61): consequently, DF . AB = DE . AC.

THEOREM LXIII.

The rectangle of the two sides of any triangle, is equal to the rectangle of the perpendicular on the third side and the diameter of the circumscribing circle.

Let CD be the perpendicular, and CE the diameter of the circle about the triangle ABC; then CA . CB = CD . CE.

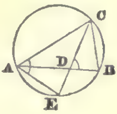


For, join BE: then in the two triangles ACD, ECB, the angles A and E are equal, standing on the same arc BC (*th.* 50); also the right angle D is equal the angle B, which is also a right angle, being in a semicircle (*th.* 52): therefore these two triangles have also their third angles equal, and are equiangular. Hence, AC, CE, and CD, CB, being like sides, subtending the equal angles, the rectangle AC . CB, of the first and last of them, is equal to the rectangle CE . CD, of the other two (*th.* 62).

THEOREM LXIV.

The square of a line bisecting any angle of a triangle, together with the rectangle of the two segments of the opposite side, is equal to the rectangle of the two other sides including the bisected angle.

Let CD bisect the angle C of the triangle ABC; then we shall have CD^2 + AD . DB = AC . CB.



For, let CD be produced to meet the circumscribing circle at E, and join AE.

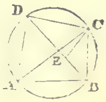
Then the two triangles ACE, BCD, are equiangular: for the angles at C are equal by supposition, and the angles B and E are equal, standing on the same arc AC (*th.* 50);

consequently the third angles at A and D are equal (*cor.* 1, *th.* 17): also AC, CD, and CE, CB, are like or corresponding sides, being opposite to equal angles: therefore AC . CB = CD . CE (*th.* 62). But CD . CE = CD^2 + CD . DE (*th.* 30); therefore AC . CB = CD^2 + CD . DE, CD^2 + AD , DB, since CD . DE = AD . DB (*th.* 61).

THEOREM LXV.

The rectangle of the two diagonals of any quadrilateral inscribed in a circle, is equal to the sum of the two rectangles of the opposite sides.

Let ABCD be any quadrilateral inscribed in a circle, and AC, BD, its two diagonals: then AC . BD = AB . DC + AD . BC.



For, let CE be drawn, making the angle BCE equal to the angle DCA. Then the two triangles ACD, BCE, are equiangular; for the angles A and B are equal, standing on the same arc DC; and the angles DCA, BCE, are equal by construction; consequently, the third angles ADC, BEC, are also equal; also AC, BC, and AD, BE, are like or corresponding sides, being opposite to the equal angles: therefore the rectangle AC.BE is equal to the rectangle AD.BC (*th.* 62).

Again, the two triangles ABC, DEC, are equiangular: for the angles BAC, BDC, are equal, standing on the same arc BC; and the angle DCE is equal to the angle BCA, by adding the common angle ACE to the two equal angles DCA, BCE; therefore the third angles E and ABC are also equal: but AC, DC, and AB, DE, are the like sides: therefore AC.DE = AB.DC (*th.* 62).

Hence, by equal additions, AC.BE + AC.DE = AD.BC + AB.DC. But AC.BE + AC.DE = AC.BD (*th.* 30): therefore AC.BD = AD.BC + AB.DC (*ax.* 1).

Cor. Hence, if ABD be an equilateral triangle, and C any point in the arc BCD of the circumscribing circle, we have AC = BC + DC. For AC.BD = AD.BC + AB.DC; and dividing by BD = AB = AD, there results AC = BC + DC.

RATIOS AND PROPORTIONS.

DEFINITIONS.

76. Ratio is the relation subsisting between two magnitudes of the *same kind,* in respect of quantity.

Of the two magnitudes compared, that which is taken as the standard of comparison is called the *antecedent term of the ratio,* or simply, the *antecedent;* and that which is compared with it, the *consequent.* The leading idea of ratio is, the

number of times that the antecedent is contained in the consequent; and hence the doctrine of ratio becomes, essentially, a branch of arithmetic *.

The manner of writing a ratio is $a : b$, where a is the antecedent and b the consequent. The reading it is, a is to b; and the expression of the fundamental idea is $\frac{b}{a}$, written as a fraction, the numerator and denominator of which are the consequent and antecedent respectively.

77. *Proportion* is the equality of two ratios, expressed as fractions. Thus, if $\frac{b}{a} = \frac{d}{c}$, the magnitudes a, b, c, d, are said to be *proportionals*, or to be *in proportion*. In geometrical investigations it is, however, more usual to write them $a : b :: c : d$, the verbal interpretation of which is either

a is to b as c is to d, or
a has the same ratio to b that c has to d.

78. When there is any number of magnitudes of the same kind, the ratio of the first to the last of them is said to be compounded of the ratios of the first to the second, the second to the third, the third to the fourth, and so on to the last. This is expressed by the term *compound ratio*.

79. When all these ratios, viz. that of the first term to the second, the second to the third, and so on, are all equal, the terms are said to form *a geometrical progression*, and are said to be *continued proportionals*.

80 When the ratios are equal, and there are only three terms, (or two ratios,) the third is called a *third proportional to the first and second;* and the first is said to have to the third the *duplicate ratio* of that which the first has to the second. The middle term is called a *mean proportional* between the first and third.

In like manner, when there are three equal ratios, the first term is said to have to the fourth, the *triplicate ratio* of the first to the second, and so on, however many equal ratios there may be.

There are other technical terms employed to signify certain modifications

---

* The method of treating the doctrine of ratio by the Greek geometers was precisely similar in all its essential characters to their method of treating theoretical arithmetic. The modern method of discussing the properties of numbers has superseded the Greek one; but in treating the doctrine of ratio, the original mode is still adhered to by the great majority of geometrical writers, on account of its supposed superiority of logical conclusiveness. The great beauty of that method of investigating the properties of ratios, no one doubts; but its superior conclusiveness may be very fairly questioned, and its great complexity renders it a serious obstacle to the progress of geometrical study.

The great logical difficulty that has been felt in treating ratio directly and formally as a branch of arithmetic, has arisen from the possible incommensurability of the two terms of the ratio. Now if it were essential that the specific ratio itself should be assigned between the two terms, there would be some force in this objection; but as in all our investigations, and in all the uses we make of the doctrine in theoretical researches, resolve themselves into investigations respecting the equality or inequality of two or more ratios, as the result of given conditions, it is obviously sufficient that we should be able to determine the essential equality or inequality of those several ratios, without discussing the actual values of the fractional expressions themselves. Such ratios themselves may be unknown, indeterminable, or irrational; and yet their equality or inequality may be determined as completely by arithmetical considerations, as by the method of the Greeks. In fact, all the reasonings in which ratio is employed are conducted altogether independently of the *actual value* of the fraction $\frac{a}{b}$, and which may, therefore, with perfectly logical accuracy, be denoted by any symbol whatever, as m, or n, $f(r)$, or any other.

under which magnitudes originally proportional will still continue so. These cases being enunciated and proved in the following series, the several terms or phrases by which they are designated, are annexed to the propositions themselves.

81. A line is said to be *divided in extreme and mean ratio*, when the whole line is to the greater segment as the greater segment is to the less; or conversely to be *extended in extreme and mean ratio*, when the extended part is to the original line as the original line is to the whole line composed of the original one and the extended part.

82. The *altitude* of a triangle or a parallelogram is the perpendicular distance (or simply the distance, *def.* 50) of the vertex of the triangle, or the opposite side of the parallelogram from the base.

83. Two pairs of magnitudes are said to be *reciprocally proportional*, when the first of the first pair is to the first of the second pair, as the latter of the second pair is to the latter of the first pair. Thus, if a, b, c, d, taken in order were the two pairs, they are reciprocally proportional when $a : c :: d : b$.

84. A line is said to be divided in *harmonical ratio*, (or simply divided *harmonically*,) when it is divided and extended in the same ratio.

85. A *transversal* is any straight line or circle which is drawn to cut a system of straight lines.

86. When a straight line is divided harmonically, and lines are drawn from the points of division to any fifth point, the four lines so drawn are called an *harmonical fasceau.*

A convenient mode of writing this is as follows:

Let A, B, C, D, be the four points of the harmonical line, and E the point of the *fasceau*; then $E\{ABCD\}$ denotes lines drawn as in the definition.

THEOREM LXVI.

Equimultiples of two magnitudes have the same ratios as the magnitudes themselves.

Let a, b, be the two magnitudes, and ma, mb, their equimultiples. Then $a : b :: ma : mb$. For the ratios $\frac{b}{a}$ and $\frac{mb}{ma}$ are equal, whatever be the value of the multiple m, whether integer, fractional, or irrational.

Cor. Hence any equisubmultiples of two magnitudes have the same ratio as the magnitudes themselves.

THEOREM LXVII.

If four magnitudes of the same kind be proportional, then the antecedents will have the same ratio as the consequents.

[This is called *alternation* or *permutation* of the terms.]

Let $a : b :: c : d$; then $a : c :: b : d$. Then since (*def.* 77) $\frac{b}{a} = \frac{d}{c}$ gives $\frac{c}{a} = \frac{d}{b}$, and this fulfils the definition of proportional terms, or $a : c :: b : d$.

THEOREM LXVIII.

If four quantities taken in order be proportionals, then will the first consequent be to the first antecedent as the second consequent is to the second antecedent.

[This is called *proportion by inversion.*]

LET $a : b :: c : d$, then also we shall have $b : a :: d : c$.

For since $\frac{b}{a} = \frac{d}{c}$, we have $\frac{a}{b} = \frac{c}{d}$, or again, by the definition, $b : a :: d : c$.

THEOREM LXIX.

If four magnitudes be proportional, then the sum or difference of the first and second will be to the first or second as the difference of the third and fourth is to the third or fourth.

[This is termed proportion by *composition* or *division*, according as the sums or differences are used].

LET $a : b :: c : d$; then we shall have to prove that $a \pm b : a :: c \pm d : c$, and that $a \pm b : b :: c \pm d : d$.

Now, since $\frac{b}{a} = \frac{d}{c}$, we have $\frac{a}{b} = \frac{c}{d}$, and hence also $1 \pm \frac{b}{a} = 1 \pm \frac{d}{c}$, and $\frac{a}{b} \pm 1 = \frac{c}{d} \pm 1$. Whence also, $\frac{a \pm b}{a} = \frac{c \pm d}{c}$, and $\frac{a \pm b}{b} = \frac{c \pm d}{d}$; that is, again, after inversion, $a \pm b : a :: c \pm d : c$, and $a \pm b : b :: c \pm d : d$.

Cor. 1. Hence also, $a + b : a - b :: c + d : c - d$. For by the preceding $\frac{a+b}{a} = \frac{c+d}{c}$, and $\frac{a-b}{a} = \frac{c-d}{c}$; therefore we get $\frac{a-b}{a+b} = \frac{c-d}{c+d}$, or $a + b : a - b :: c + d : c - d$.

Cor. 2. Also $a : c :: a \pm b : c \pm d$, and $c : d :: a \pm c : b \pm d$.

THEOREM LXX.

If, of four proportional magnitudes there be taken any equimultiples whatever of the two antecedents, and any whatever of the two consequents, these multiples will be proportionals.

LET $a : b :: c : d$, then also $ma : nb :: mc : nd$. For $\frac{b}{a} = \frac{d}{c}$, and hence $\frac{nb}{ma} = \frac{nd}{mc}$: or, which is the same thing, $ma : nb :: mc : nd$.

THEOREM LXXI.

If there be four proportional magnitudes, and the two consequents be either augmented or diminished by magnitudes which have the same ratio as the two antecedents, the sums or differences form with the two antecedents a set of proportionals.

LET $a : b :: c : d$, and $e : f :: a : c$; then will $a : b \pm e :: c : d \pm f$.

For, from the two given proportions we have $\frac{c}{a} = \frac{d}{b}$ and $\frac{c}{a} = \frac{f}{e}$, hence b . e :: $d : f$, and $b \pm e : d \pm f :: b : d :: a : c$.

Cor. The variation of this theorem is obvious. viz.: $a \pm e : c \pm f :: b : d$.

THEOREM LXXII.

If any number of magnitudes be proportional, then any one of the antecedents is to its consequent as all the antecedents taken together are to all the consequents taken together.

Let $a : b :: c : d :: e : f :: g : h$ ...... Then $a : b :: a + c + e + g$ .... $: b + d + f + h$ ....

For $\frac{b}{a} = \frac{d}{c} = \frac{b+d}{a+c} = \frac{f}{e} = \frac{b+d+f}{a+c+e} = \frac{h}{g} = \frac{b+d+f+h}{a+c+e+g}$, and so on to any extent. Hence the conclusion follows.

THEOREM LXXIII.

If a whole magnitude be to a whole as a part taken from the first is to a part taken from the other: then the remainder will be to the remainder as the whole to the whole.

Let $a : b :: \frac{m}{n} a : \frac{m}{n} b$; then $a : b :: a - \frac{m}{n} a : b - \frac{m}{n} b$.

For $\frac{b}{a} = \dfrac{b - \frac{m}{n} b}{a - \frac{m}{n} a}$, or in another form as stated in the theorem.

THEOREM LXXIV.

If there be several pairs of ratios which are equal each to each, then the ratio compounded of all the first ratios will be equal to the ratio compounded of all the others.

Let
$$\begin{array}{l} a : b :: c : d \\ a_1 : b_1 :: c_1 : d_1 \\ a_2 : b_2 :: c_2 : d_2 \\ \cdots\cdots\cdots\cdots \\ a_m : b_m :: c_m : d_m \end{array}$$

Then will the ratio compounded of $a : b$, $a_1 : b_1$, $a_2 : b_2$, ... $a_m : b_m$, be equal to that compounded of $c : d$, $c_1 : d_1$, $c_2 : d_2$, ... $c_m : d_m$.

For $\frac{b}{a} = \frac{d}{c}$, $\frac{b_1}{a_1} = \frac{d_1}{c_1}$, $\frac{b_2}{a_2} = \frac{d_2}{c_2}$, ... $\frac{b_m}{a_m} = \frac{d_m}{c_m}$. Whence it follows that $\frac{b}{a} \cdot \frac{b_1}{a_1} \cdot \frac{b_2}{a_2} \cdots \frac{b_m}{a_m} = \frac{d}{c} \cdot \frac{d_1}{c_1} \cdot \frac{d_2}{c_2} \cdots \frac{d_m}{c_m}$.

Cor. 1. If there be magnitudes common to the numerator and denominator of either multiplied fraction, they may be cancelled, on the principle of the common measure.

Cor. 2. If the magnitudes be numerically expressed, we shall have, as at p. 166,

$$aa_1a_2 \ldots\ldots a_m : bb_1b_2 \ldots b_m :: cc_1c_2 \ldots\ldots c_m : dd_1d_2 \ldots d_m.$$

Cor. 3. If all these ratios are equal, and the magnitudes expressed numerically, then we have $\frac{b^m}{a^m} = \frac{d^m}{c^m}$, and $\frac{b^{\frac{m}{n}}}{a^{\frac{m}{n}}} = \frac{d^{\frac{m}{n}}}{c^{\frac{m}{n}}}$; and hence, $a^m : b^m :: c^m : d^m$, and $a^{\frac{m}{n}} : b^{\frac{m}{n}} :: c^{\frac{m}{n}} : d^{\frac{m}{n}}$.

THEOREM LXXV.

Of four proportional magnitudes, if the first be greater than the second, the third is greater than the fourth; if equal, equal; and if less, less.

Let $a : b :: c : d$; then if a be greater than b, c is greater than d; if equal, equal, and if less, less.

For $\frac{b}{a} = \frac{d}{c}$. Then if a be greater than b, the fraction $\frac{b}{a}$ is less than unity, and hence $\frac{d}{c}$ is also less than unity, or c is greater than d. If a be equal to b, the fraction $\frac{b}{a}$ is equal to unity, and hence $\frac{d}{c}$ is also unity, or $c = d$. In like manner, if a be less than b, c is less than d.

Cor. Since (*th.* 69) we have $a : c :: b : d$, the same reasoning will lead to the conclusion, that if four magnitudes of the same kind be proportionals. then the second will be less, equal to, or greater than the fourth, according as the first is less, equal to, or greater than the third.

THEOREM LXXVI.

If any equimultiples whatever of the first and third of four proportionals *magnitudes be taken, and any whatever of the second and fourth; then, according as the multiple of the first be greater than, equal to, or less than that of the second, that of the third will be greater than, equal to, or less than that of the fourth.*

Let $a : b :: c : d$; then it is to be shown that

(1) if ma be greater than nb, mc is also greater than nd;

(2) if ma be equal to nb, mc is also equal to nd;

(3) if ma be less than nb, mc is also less than nd.

For, since $a : b :: c : d$, $ma : nc :: mb : nd$ (*th.* 70); and since these last are proportionals, according as ma is greater than, equal to, or less than nc, mb is greater than, equal to, or less than nd (*th.* 75).

THEOREM LXXVII.

If there be four magnitudes such that when any equimultiples whatever are taken of the first and third, and any whatever of the second and fourth, and if when the multiple of the first is greater than, equal to, and less than that of the second, that of the third is greater than, equal to, and less, respectively, than that of the fourth: these four magnitudes will be proportional.

Let a, b, c, d, be four magnitudes, and m, n, any numbers whatever; and when ma is greater than, equal to, and less than nb, let mc be greater than, equal to, and less than nd: then we have $a : b :: c : d$.

For, if possible, let the fourth magnitude not be a fourth proportional to a, b, c, and let the fourth proportional to them be $d_{,}$. Then if $d_{,} = d + d'$, we have

$a : b :: c : d + d'$. Now in this case we have $ma : nb :: mc : nd + nd'$. Hence,

if ma be greater than nb, mc is greater than $nd + nd'$
........equal to .... equal to ....
........less than .... less than............

But by hypothesis,

if ma be greater than nb, mc is greater than nd
...... equal to .... equal to ......
...... less than .... less than......

Now the second of both of these sets of conditions can only be fulfilled by $d' = 0$; and of the other two of each set, the first is not necessarily fulfilled by any other value of d'. Whence that the three conditions may be fulfilled, we must have d a fourth proportional to a, b, c, d; that is, under the given circumstances, $a : b :: c : d$.

THEOREM LXXVIII.

If any number of quantities be continued proportionals, then the ratio of the first to the last is that power of the ratio which expresses the number of ratios compounded.

Let there be n equal ratios $a : b, b : c, c : d, \ldots . p : q$ compounded: then $\frac{q}{a} = \left(\frac{b}{a}\right)^n$

For $\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = \ldots = \frac{q}{p}$. Hence $\frac{b}{a} \cdot \frac{c}{b} \cdot \frac{d}{c} \ldots . \frac{q}{p} = \frac{b}{a} \cdot \frac{b}{a} \ldots$ (n terms)

or, which is the same thing, $\frac{q}{a} = \left(\frac{b}{a}\right)^n$.

Cor. Hence the duplicate ratio is the square of the simple ratio, the triplicate ratio is its cube, and so on.

THEOREMS DEPENDING ON RATIOS.

THEOREM LXXIX.

Parallelograms, or triangles, having equal altitudes, are to one another as their bases; those having equal bases are to one another as their altitudes; and those having neither equal bases nor altitudes, are to one in a ratio compounded of the ratios of their bases and the ratio of their altitudes.

First. (1). Let BADC, GDEF be any two parallelograms having the same altitude (and, therefore, when their bases AD, DE, are in the same straight line AE, their opposite sides BC, GF, are in the same straight line BF parallel to AE): then they are to each other as their bases AD, DE.

In AD produced, take any number of parts AL, LS, each equal to AD; and any number EH, HK, KV, each equal to DE; and draw LM, ST, EN, HP, KQ, VW, all parallel to AB or DC, and meeting BF, produced, as in the figure.

Then each of the parallelograms TL, MA, is equal to BD; and there are as

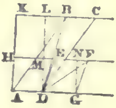
many of them as there were taken lines AL, LS, equal to AD. Hence, whatever multiple the base SD is of the base DA, the same multiple is the parallelogram SC of the parallelogram AC. In like manner, whatever multiple the base DV is of the base DE, the same multiple is the parallelogram DW of the parallelogram DN, or (*th.* 25) of the parallelogram DF.

Again, if SD, the multiple of AD, be greater than DV, the multiple of DE, the multiple SC of AC will be greater than the multiple DW of DN or DF; if equal, equal; if less, less. Hence, (*th.* 77,) parallelogram AC : parallelogram DN : : base AD : base DE.

(2.) Let ARD, DGE, be two triangles of equal altitudes, they will be to each other as their bases, AD, DE.

For, being of equal altitudes, they are between the same parallels; and as each of the triangles ARD, DGE, is the half of the parallelograms AC, DF, having the same base and altitude, they are to one another as those parallelograms. But the parallelograms have been proved to have the same ratio as their bases; and hence triangle ARD : triangle DGE :: base AD : base DE.

Secondly. (1). Let the parallelograms AC, DF, have equal bases AD, DG: they will be to each other as their altitudes.

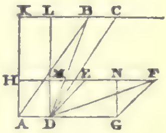


For, draw AK, DL, GN, perpendicular to AG, and produce BC, EF, to meet them, as in the figure. Then the lines BC, EF, being parallel to AG, are parallel to each other (*th.* 15); and the parallelograms AL, DN, are rectangles (*th.* 22, *cor.* 1), and equal to AC, DF, respectively (*th.* 25); and the rectangle AM is equal to the rectangle DN. But the rectangles AM, AL, are to one another as AH, AK, by the former part of the proposition; that is, as the altitudes of the parallelograms AC, DF. Whence also we have parallelogram AC : parallelogram DF : : altitude AH : altitude AK.

(2). The triangles ABD, DFG, being the halves of the parallelograms AC, DF, are to one another in the same ratio; that is,

triangle ABD : triangle DFG : : base AD : base DG.

Thirdly. (1). Let ABCD, DEFG, be any two parallelograms, having neither their bases nor altitudes equal, and make the same construction as in the last case: they will be to each other in a ratio compounded of the ratio of their bases AD, DG, and the ratio of their altitudes AK, AH.



For, by the first and second cases respectively, we have

$$\frac{\text{parellelogram AM}}{\text{parallelogram DN}} = \frac{AD}{DG}, \text{ and } \frac{\text{parallelogram AL}}{\text{parallelogram AM}} = \frac{AK}{AH}: \text{ hence also}$$

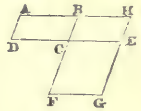
$$\frac{AD}{DG} \cdot \frac{AK}{AH} = \frac{\text{parallelogram AM}}{\text{parallelogram DN}} \cdot \frac{\text{parallelogram AL}}{\text{parallelogram AM}} = \frac{\text{parallelogram AC}}{\text{parallelogram DF}}, \text{ or}$$

the parallelograms AC, DF, have the ratio which is compounded of the ratio of their bases AD, DG, and the ratio of their altitudes AK, AH.

(2). Since the triangles ABD, DFG, have the same bases and altitudes as the parallelograms AC, DF, they have the same ratios as the parallelograms themselves, and hence the proposition is also true respecting triangles.

THEOREM LXXX.

Parallelograms, or triangles, which have one angle of the one equal to one angle of the other, have to one another a ratio compounded of the ratios of the sides about the equal angles.



(1). Let ABCD, FCEG, be two parallelograms, having the angles DCB, ECF equal to one another: then they shall be to one another in a ratio compounded of the ratios of DC, CE, and BC, CF.

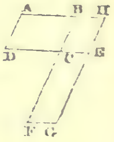
For (*th.* 79) $\frac{\text{parallelogram AC}}{\text{parallelogram CH}} = \frac{\text{DC}}{\text{CE}}$, and $\frac{\text{parallelogram CH}}{\text{parallelogram CG}} = \frac{\text{BC}}{\text{CF}}$; hence $\frac{\text{parallelogram AC}}{\text{parallelogram CH}} \cdot \frac{\text{parallelogram CH}}{\text{parallelogram CG}} = \frac{\text{parallelogram AC}}{\text{parallelogram CG}} = \frac{\text{DC}}{\text{CE}} \cdot \frac{\text{CB}}{\text{CF}}$.

(2). Let BCD, FCE, be triangles having the angles at C equal; they shall be to one another in a ratio compounded of the ratio of the sides DC, CE, and BC, CF.

For the triangles BCD, FCE, being the halves of the parallelograms AC, CG, they have the same ratio as the parallelograms: that is, by the last case, the ratio compounded of the ratios of the sides.

THEOREM LXXXI.

In parallelograms, or triangles, having one angle of the one equal to one angle of the other, if the sides about the equal angles are reciprocally proportional, the parallelograms or triangles are equal; and if they be equal, the sides about the equal angles are reciprocally proportional.



First, let the parallelograms AC, CG, have their angles at C equal, and the sides about C reciprocally proportional, (that is, DC : CE :: CF : CB) then they will be equal.

For by the last proposition we have $\frac{\text{parallelogram AC}}{\text{parallelogram CG}} = \frac{\text{DC}}{\text{CE}} \cdot \frac{\text{CB}}{\text{CF}}$; and since also $\frac{\text{DC}}{\text{CE}} = \frac{\text{CF}}{\text{CB}}$, we have $\frac{\text{DC}}{\text{CE}} \cdot \frac{\text{CB}}{\text{CF}} = 1$, and therefore $\frac{\text{parallelogram AC}}{\text{parallelogram CG}} = 1$, or parallelogram AC = parallelogram CG.

In like manner, under the same circumstances, the triangles BCD, FCE, being the halves of the equal parallelograms, are also equal.

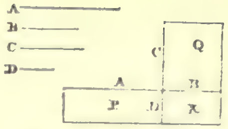
Secondly. Let the parallelograms AC, CG, be equal, and have the angles at C equal: then the sides shall be reciprocally proportional, or DC : CE :: CF : CB.

For since $\frac{\text{parallelogram AC}}{\text{parallelogram CG}} = \frac{\text{DC}}{\text{CE}} \cdot \frac{\text{CB}}{\text{CF}}$, and the parallelograms are equal, we have $\frac{\text{DC}}{\text{CE}} \cdot \frac{\text{CB}}{\text{CF}} = 1$, or $\frac{\text{DC}}{\text{CE}} = \frac{\text{CF}}{\text{CB}}$; or again, finally, DC : CE :: CF : CB.

In the same way, it may be proved for the triangles BCD, FCE.

Cor. 1. If four straight lines be proportional, the rectangle of their extremes is equal to the rectangle of the means: and if the rectangle of the extremes be equal to the rectangle of the means, the four straight lines are proportional.

Let the four straight lines A, B, C, D, be proportionals, that is, A : B :: C : D; then shall the rectangle of A and D be equal to the rectangle of B and C.



For place A, B, C, D, meeting in a point, and forming four right angles at their point of intersection, and draw lines parallel to them to complete the figure; where P is the rectangle of A, and D, Q, that of B and C, and R that of D and B. Then the figures P, Q, R, are rectangles, and the theorem is a case of the proposition, in which the alleged properties have been generally proved.

Cor. 2. If three straight lines be proportional, the rectangle of the extremes is equal to the square of the mean; and if the rectangle of the extremes be equal to the square of the mean, the three straight lines are proportional.

For in this case B = C, and the rectangle of B and C becomes the square of B or C. Whence by the last *cor.* the truth follows.

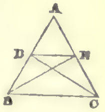
Scholium.

Since it appears, by the rules of proportion in arithmetic and algebra, that when four quantities are proportional, the product of the extremes is equal to the product of the two means; and by this theorem, that the rectangle of the extremes is equal to the rectangle of the two means; it follows, that the area or space of a rectangle is represented or expressed by the product of its length and breadth multiplied together: and, in general, the area of a rectangle in geometry is represented by the product of the measures of its length and breadth, or base and height; and a square is similarly represented by the measure of the side multiplied by itself. Hence, what is shown of such products, is to be also understood of the squares and rectangles.

THEOREM LXXXII.

If a line be drawn in a triangle parallel to one of its sides, it will cut the other two sides proportionally.

Let DE be parallel to the side BC of the triangle ABC; then will AD : DB :: AE : EC.



For, draw BE and CD. Then the triangles DBE, DCE, are equal to each other, because they have the same base DE, and are between the same parallels DE, BC (*th.* 25). But the two triangles ADE, BDE, on the bases AD, DB, have the same altitude; and the two triangles ADE, CDE, on the bases AE, EC, have also the same altitude; and because triangles of the same altitude are to each other as their bases, therefore

$$\triangle \text{ADE} : \text{BDE} :: \text{AD} : \text{DB}, \triangle \text{ADE} : \text{CDE} :: \text{AE} : \text{EC}.$$

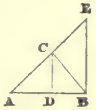
But BDE = CDE; and equals must have to equals the same ratio; therefore AD : DB :: AE : EC. In a similar manner, the theorem is proved when the sides of the triangle are cut in prolongation beyond either the vertex or the base.

Cor. Hence, also, the whole lines AB, AC, are proportional to their corresponding proportional segments (*cor. th.* 66),

viz. AB : AC :: AD : AE,
and AB : AC :: BD : CE.

THEOREM LXXXIII.

A line which bisects any angle of a triangle, divides the opposite side into two segments, which are proportional to the two other adjacent sides.



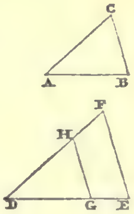
Let the angle ACB, of the triangle ABC, be bisected by the line CD, making the angle ACD equal to the angle DCB: then will the segment AD be to the segment BD, as the side AC is to the side CB. Or, AD : DB :: AC : CB.

For, let BE be parallel to CD, meeting AC produced at E. Then, because the line BC cuts the two parallels CD, BE, it makes the angle CBE equal to the alternate angle DCB (*th.* 12), and therefore also equal to the angle ACD, which is equal to DCB by the supposition. Again, because the line AE cuts the two parallels DC, BE, it makes the angle E equal to the angle ACD on the same side of it (*th.* 14). Hence, in the triangle BCE, the angles B and E, being each equal to the angle ACD, are equal to each other, and consequently their opposite sides CB, CE, are also equal (*th.* 3).

But now, in the triangle ABE, the line CD, being drawn parallel to the side BE, cuts the two other sides AB, AE, proportionally (*th.* 82), making AD to DB, as is AC to CE or to its equal CB.

THEOREM LXXXIV.

Equiangular triangles are similar, or have their like sides proportional.



Let ABC, DEF, be two equiangular triangles, having the angle A equal to the angle D, the angle B to the angle E, and consequently the angle C to the angle F; then will AB : AC :: DE : DF.

For, make DG = AB, and DH = AC, and join GH. Then the two triangles ABC, DGH, having the two sides AB, AC, equal to the two DG, DH, and the contained angles A and D also equal, are identical, or equal in all respects (*th.* 1), namely, the angles B and C are equal to the angles G and H. But the angles B and C are equal to the angles E and F by the hypothesis; therefore also the angles G and H are equal to the angles E and F (*ax.* 1), and consequently the line GH is parallel to the side EF (*cor.* 1, *th.* 14).

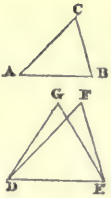
Hence then, in the triangle DEF, the line GH, being parallel to the side EF, divides the two other sides proportionally, making DG : DH :: DE : DF (*cor. th.* 82). But DG and DH are equal to AB and AC; therefore also AB : AC :: DE : DF.

THEOREM LXXXV.

Triangles which have their sides proportional, are equiangular.

In the two triangles ABC, DEF, if AB : DE :: AC : DF :: BC : EF; the two triangles will have their corresponding angles equal.

For, if the triangle ABC be not equiangular with the triangle DEF, suppose some other triangle, as DEG, to be equiangular with ABC. But this is impossible: for if the two triangles ABC, DEG, were equiangular, their sides would be proportional (*th.* 84). So that, AB being to DE as AC to DG, and AB to DE as BC to EG, it follows that DG and EG, being fourth proportionals to the same three quantities, as well as the two DF, EF, the former, DG, EG, would be equal to the latter, DF, EF. Thus, then, the two triangles DEF, DEG, having their three sides equal, would be identical (*th.* 5); which is absurd, since their angles are unequal.

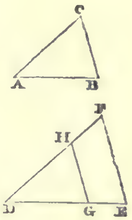


THEOREM LXXXVI.

Triangles, which have an angle in the one equal to an angle in the other, and the sides about these angles proportional, are equiangular.

Let ABC, DEF, be two triangles, having the angle A equal to the angle D, and the sides AB, AC, proportional to the sides DE, DF: then will the triangle ABC be equiangular with the triangle DEF.

For, make DG equal to AB, and DH to AC, and join GH. Then, the two triangles ABC, DGH, having two sides equal, and the contained angles A and D equal, are identical and equiangular (*th.* 1), having the angles G and H equal to the angles B and C. But, since the sides DG, DH, are proportional to the sides DE, DF, the line GH is parallel to EF (*th.* 82); hence the angles E and F are equal to the angles G and H (*th.* 14), and consequently to their equals B and C.

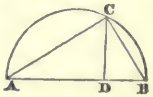


THEOREM LXXXVII.

In a right-angled triangle, a perpendicular from the right angle, is a mean proportional between the segments of the hypothenuse; and each of the sides, about the right angle, is a mean proportional between the hypothenuse and the adjacent segment.

Let ABC be a right-angled triangle, and CD a perpendicular from the right angle G to the hypothenuse AB; then will

CD be a mean proportional between AD and DB;
AC a mean proportional between AB and AD;
BC a mean proportional between AB and BD.



Or, AD : CD :: CD : DB; and AB : BC :: BC : BD; and AB : AC :: AC : AD.

For, the two triangles ABC, ADC, having the right angles at C and D equal, and the angle A common, have their third angles equal, and are equiangular (*cor.* 1, *th.* 17). In like manner, the two triangles ABC, BDC, having the right angles at C and D equal, and the angle B common, have their third angles equal, and are equiangular.

Hence then, all the three triangles, ABC, ADC, BDC, being equiangular, will have their like sides proportional (*th.* 84); viz. AD : CD :: CD : DB; AB : AC :: AC : AD; and AB : BC :: BC : BD.

Cor. 1. Because the angle in a semicircle is a right angle (*th.* 52); it follows, that if, from any point C in the periphery of the semicircle, a perpendicular be drawn to the diameter AB; and the two chords CA, CB, be drawn to the extremities of the diameter; then are AC, BC, CD, the mean proportionals as in this theorem, or (*th.* 77), $CD^2 = AD \cdot DB$; $AC^2 = AB \cdot AD$; and $BC^2 = AB \cdot BD$.

Cor. 2. Hence $AC^2 : BC^2 :: AD : BD$.

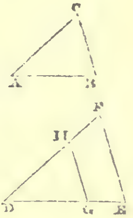
Cor. 3. Hence we have another demonstration of *th.* 34.

For since $AC^2 = AB \cdot AD$, and $BC^2 = AB \cdot BD$.

By addition $AC^2 + BC^2 = AB\,(AD + BD) = AB^2$.

THEOREM LXXXVIII.

Equiangular or similar triangles, are to each other as the squares of their like sides.



Let ABC, DEF, be two equiangular triangles, AB and DE being two like sides: then will the triangle ABC be to the triangle DEF, as the square of AB is to the square of DE, or as AB^2 to DE^2.

For, the triangles being similar, they have their like sides proportional (*th.* 84), and are to each other as the rectangles of the like pairs of their sides (*cor.* 4, *th.* 81);

theref. $AB : DE :: AC : DF$ (*th.* 84),

and $AB : DE :: AB : DE$ of equality:

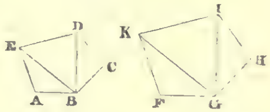
theref. $AB^2 :: DE^2 :: AB \cdot AC : DE \cdot DF$ (*th.* 75).

But $\triangle ABC : \triangle DEF :: AB \cdot AC : DE \cdot DF$ (*cor.* 4, *th.* 81).

theref. $\triangle ABC : \triangle DEF :: AB^2 : DE^2$.

THEOREM LXXXIX.

All similar figures are to each other, as the squares of their like sides.



Let ABCDE, FGHIK, be any two similar figures, the like sides being AB, FG, and BC, GH, and so on in the same order: then will the figure ABCDE be to the figure FGHIK, as the square of AB to the square of FG, or as AB^2 to FG^2.

For, draw BE, BD, GK, GI, dividing the figures into an equal number of triangles, by lines from two equal angles B and G.

The two figures being similar, (*hypoth.*) they are equiangular, and have their like sides proportional (*def.* 67).

Then, since the angle A is = the angle F, and the sides AB, AE, proportional to the sides FG, FK, the triangles ABE, FGK, are equiangular (*th.* 86). In like manner, the two triangles BCD, GHI, having the angle C = the angle H, and the sides BC, CD, proportional to the sides GH, HI, are also equiangular. Also, if from the equal angles AED, FKI, there be taken the equal angles AEB, FKG, there will remain the equals BED, GKI; and if from the equal angles CDE, HIK, be taken away the equals CDB, HIG, there will remain the equals BDE, GIK; so that the two triangles BDE, GIK, having two angles equal, are

also equiangular. Hence each triangle of the one figure is equiangular with each corresponding triangle of the other.

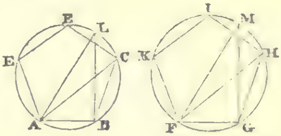
But equiangular triangles are similar, and are proportional to the squares of their like sides (*th.* 88).

$$\text{Therefore } \triangle ABE : \triangle FGK :: AB^2 : FG^2,\ \triangle BCD : \triangle GHI :: BC^2 : GH^2,$$
$$\triangle BDE : \triangle GIK :: DE^2 : IK^2.$$

But as the two polygons are similar, their like sides are proportional, and consequently their squares also proportional; so that all the ratios AB^2 to FG^2, and BC^2 to GH^2, and DE^2 to IK^2, are equal among themselves, and consequently the corresponding triangles also, ABE to FGH, and BCD to GHI, and BDE to GIK, have all the same ratio, viz. that of AB^2 to FG^2: and hence all the antecedents, or the figure ABCDE, have to all the consequents, or the figure FGHIK, still the same ratio, viz. that of AB^2 to FG^2 (*th.* 72).

THEOREM XC.

Similar figures inscribed in circles, have their like sides, and also their whole perimeters, in the same ratio as the diameters of the circles in which they are inscribed.



Let ABCDE, FGHIK, be two similar figures, inscribed in the circles whose diameters are AL and FM; then will each side AB, BC, ... of the one figure be to the like side GF, GH, ... of the other figure, or the whole perimeter AB + BC + ... of the one figure, to the whole perimeter FG + GH + ... of the other figure, as the diameter AL to the diameter FM.

For, draw the two corresponding diagonals AC, FH, as also the lines BL, GM. Then, since the polygons are similar, they are equiangular, and their like sides have the same ratio (*def.* 67); therefore the two triangles, ABC, FGH, have the angle B = the angle G, and the sides AB, BC, proportional to the two sides FG, GH; consequently these two triangles are equiangular (*th.* 86), and have the angle ACB = FHG. But the angle ACB = ALB, standing on the same arc AB; and the angle FHG = FMG, standing on the same arc FG; therefore the angle ALB = FMG (*ax.* 1). And since the angle ABL = FGM, being both right angles, because in a semicircle; therefore the two triangles ABL, FGM, having two angles equal, are equiangular; and consequently their like sides are proportional (*th.* 84); hence AB : FG :: the diameter AL : the diameter FM.

In like manner, each side BC, CD, ... has to each side GH, HI, ... the same ratio of AL to FM: and consequently the sums of them are still in the same ratio, viz. AB + BC + CD .... is to FG + GH + HI .... is to diameter AL as diameter FM (*th.* 72).

THEOREM XCI.

Similar figures inscribed in circles, are to each other as the squares of the diameters of those circles.

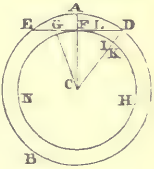
Let ABCDE, FGHIK, be two similar figures, inscribed in the circles whose diameters are AL and FM; then the surface of the polygon ABCDE will be to the surface of the polygon FGHIK, as AL^2 to FM^2. [*See fig. th.* 90.]

For, the figures being similar, are to each other as the squares of their like sides, AB^2 to FG^2 (*th.* 88). But, by the last theorem, the sides AB, FG, are as the diameters AL, FM; and therefore the squares of the sides AB^2 to FG^2, as the squares of the diameters AL^2 to FM^2 (*th.* 74). Consequently the polygons ABCDE, FGHIK, are also to each other as the squares of the diameters AL^2 to FM^2 (*ax.* 1).

THEOREM XCII.

The area of any circle is equal to the rectangle of half its circumference and half its diameter.

The area of any circle ABD is equal to the rectangle contained by the radius, and a straight line equal to half the circumference.

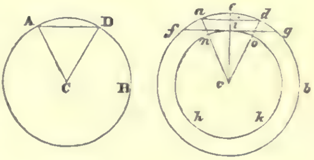


If not, let the rectangle be less than the circle ABD, or equal to the circle FNH; and imagine ED drawn to touch the interior circle in F, and meet the circumference ABD in E and D. Join CD, cutting the arc of the interior circle in K. Let FH be a quadrantal arc of the inner circle, and from it take its half, from the remainder its half, and so on, until an arc FI is obtained, less than FK. Join CI, produce it to cut ED in L, and make FG = FL; so shall LG be the side of a regular polygon circumscribing the circle FNH. It is manifest that this polygon is less than the circle ABD, because it is contained within it. Because the triangle GCL is half the rectangle of base GL and altitude CF, the whole polygon of which GCL is a constituent triangle, is equal to half the rectangle whose base is the perimeter of that polygon and altitude CF. But that perimeter is less than the circumference ABD, because each portion of it, such as GL, is less than the corresponding arch of circle having radius CL, and therefore, *à fortiori*, less than the corresponding arch of circle with radius CA. Also CE is less than CA. Therefore the polygon of which one side is GL, is less than the rectangle whose base is half the circumference ABD and altitude CA; that is, (*hyp.*) less than the circle FNH, which it contains; which is absurd. Therefore, the rectangle under the radius and half the circumference is not *less* than the circle ABD. And by a similar process it may be shown that it is not *greater*. Consequently, it is equal to that rectangle.

THEOREM XCIII.

The circumferences of circles are to each other as their radii.

The circumferences of two circles ABD, *abd*, are as their radii.



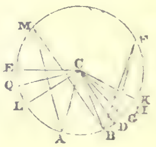
If possible, let the radius AC, be to the radius *ac*, as the circumference ABD to a circumference *ihk* less than *abd*. Draw the radius *cie*, and the straight line *fig* a chord to the circle *abd*, and a tangent to the circle *ihk* in *i*. From *eb*, a quarter of the circumference of *abd*, take away its half, and then the half of the remainder, and so on, until there be obtained an arc *ed* less than *eg*; and from *d* draw *ad* parallel to *fg*, it will be the side of a regular polygon inscribed in the circle *abd*, yet evidently greater than the circle *ihk*, because each of its constituent triangles, as *acd*, con-

tains the corresponding circular sector *cno*. Let AD be the side of a similar polygon inscribed in the circle ADB, and draw the radii AC, CD, *ac*, *cd*. The similar triangles ACD, *acd*, give AC : *ac* :: AD : *ad*, and :: perim. of polygon in ABD : perim. of polygon in *abd*. But, by the preceding theorem, AC : *ac* :: circumf. ABD : circumf. *abd*. The perimeters of the polygons are, therefore, as the circumferences of the circles. But this is impossible; because (*hyp.*) the perimeter of polygon in ABD is *less* than the circumference; while, on the contrary, the perimeter of polygon in *adb* is *greater* than the circumference *ihk*. Consequently, AC is not to *ac*, as circumference ADB, to a circumference *less* than *adb*. And by a similar process it may be shown, that *ac* is not to AC, as the circumference *abd*, to a circumference *less* than ABD. Therefore AC : *ac* :: circumference ABD : circumference *abd*.

Cor. If R, r be the radii, D, d the diameters, and C, c the circumferences, we have, by this theorem, C : c :: R : r; or, if $C = \pi R$, $c = \pi r$; and, by the former, area (A) : area (a) :: $\frac{1}{2}$RC : $\frac{1}{2}rc$: we have A : a :: $\frac{1}{2}\pi R^2 : \frac{1}{2}\pi r^2 :: R^2 : r^2 :: D^2 : d^2 :: C^2 : c^2$.

THEOREM XCIV.

Angles at the centre of a circle, angles at the circumference of a circle, and sectors of circles, have all the same ratio as the arcs by which they are subtended.



Let AB, BD be two arcs of a circle subtending the angles ACB, BCD at the centre, the angles AMB, BFD at the circumference, and the sectors: then,

(1) angle ACB : angle BCD :: arc AB : arc BD,
(2) angle AMB : angle BFD :: arc AB : arc BD,
(3) sector ACB : sector BCD :: arc AB : arc BD,

First. Take any number of arcs AL, LE, each equal to AB, and any number DG, GH, HK, each equal to BD, and draw CL, CE, CG, CH, CK.

Then since BA, AL, LE, are all equal, the angles ACB, LCA, ECL, are all equal (*ax.* 11), and hence whatever multiple the arc EB is of the arc AB, the same multiple is the angle ECB of the angle ACB. In like manner, whatever multiple the arc BK is of the arc BD, the same multiple is the angle KCB of the angle DCB.

Again, if the arc BE be greater than the arc BK, the angle ECB is greater than the angle BCK; if equal, equal; if less, less: and these are equimultiples of AB, and ACB the first and third, and of BD, BCD the second and fourth. Hence (*th.* 77) it follows that

angle ACB : angle BCD :: arc AB : arc BD.

Secondly. The angles AMB, BFD, at the circumference being the halves of the angles ACB, BCD, respectively, at the centres, have the same ratios; that is,

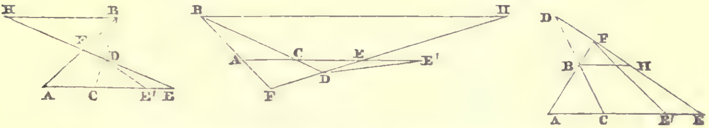
angle AMB : angle BFD :: angle ACB : angle BCD :: arc AB : arc BD.

Thirdly. The sectors ECL, LCA, ACB, are equal, and also the sectors BCD, DCG, GCH, HCK, are equal. Conceive the sector ACB to be placed upon ECL, so that CB shall coincide with CL; then the angles ACB, ECL, being equal, as before, the side AC will coincide with CE. Then the arc AB will coincide with the arc LE. For if not, let it take some other position, as EPL, and draw CPQ, cutting the arcs in P and Q; then since PC and QC are radii of the same circle they are equal: whence it is impossible that the arcs AB, EL should not coincide. The sector ECL is therefore equal to the sector ACB. In

like manner, the sector LCA is equal to ACB; and the sectors DCG, GCH, HCK, are each equal to the sector BCD. Hence it may be proved, as in the first case, that the sectors are to one another as the arcs on which they stand.

THEOREM XCV.

If the three sides of a triangle be cut by any straight line, any one side will be divided in a ratio compounded of the ratios of the segments of the other two.



Let ABC be a triangle cut by any line, straight or transversal, in D, E, F: then

AE : EC :: AF . BD : FB . DC	BF : FA :: CE . BD : AE . CD
CD : DB :: AF . EC : FB . AE	AF . BD . CE = FB . DC . AE.

For, draw BH parallel to AC meeting EDF in H. Then,

by sim. trian. HBF, AFE, AE : BH :: AF : FB, and

by sim. trian. HBD, EDC, BH : EC :: BD : DC;

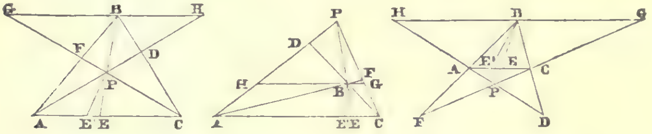
hence, by composition, cancelling BH from the first and second terms,

AE : EC :: AF . BD : FB . DC.

The two next are obtained in a similar manner; and the last by *th.* 81, p. 230.

THEOREM XCVI.

If three straight lines be drawn from the angles of a triangle through any point, to meet the opposite sides, the segments of any one side will be divided in a ratio compounded of the ratios of the segments of the other two.



Let P be any point, through which lines AP, BP, CP, from the angles of the triangle ABC are drawn to meet the sides in D, E, F, respectively: then	AE : EC :: AF . BD : FB . DC CD : DB :: AF . EC : FB . AE BF : FA :: CE . BD : AE . CD AF . BD . CE = FB . CD . EA.

For, through B draw HG parallel to AC meeting AP, CP, in H and G. Then

by parallels AC, HG, cut by HP, BP, GP, AE : EC :: BH : BG,

sim. trian. FAC, FBG, AC : BG :: FA : FB, and

sim. trian. DCA, DBH, HB : AC :: BD : DC;

hence, by composition of ratios, we get

AE : EC :: AF . BD : FB . DC.

Similarly the two next may be obtained; and the last as the corresponding one of the last theorem.

Scholium.

Though the sides of the triangles in these two propositions (*th.* 95, 96) have the same relation as to ratio amongst their segments, yet there is an essential difference as to the number of intersections made in the sides produced and unproduced. In the latter theorem, there are always *two* or *none* produced, but never one singly or all three; that is, an *even* number: in the former, always *one* or *three*, but never two or none; that is, an odd number. This distinction will appear to be of importance in the next theorem, which is the converse of the two last.

THEOREM XCVII.

If the sides of a triangle be divided so that the segments of one side have the ratio compounded of the ratios of the segments of the other two sides, then:

1. *If two or more of these sides be so divided in prolongation, lines drawn from the points of section to the opposite angles, will all pass through the same point.*
2. *If one or three of the sides be divided in prolongation, the three points of section will be in one straight line.*

First. (*figs. th.* 96.) Let AE : EC :: AF . BD : FB . DC, two sides or more being produced, then AD, BE, CF, pass through one point.

For if not, let CF, AD, intersect in P, and draw BP to meet AC in E′. Then (*th.* 96) AE′ : E′C :: AF . BD : FB . DC. Whence AE′ : E′C :: AE : EC, and AE′ $\pm$ E′C : AE′ :: AE $\pm$ EC : AE. But AE′ $\pm$ E′C $=$ AE $\pm$ EC; hence AE′ $=$ AE the less to the greater, which is impossible. The three lines AD, BE, CF, therefore pass through the same point P.

Second. (*figs. th* 95.) Let AE : EC :: AF . BD : FB . DC, one side or three being produced; then D, E, F, are in one straight line.

For if not, let FD meet AC in E′. Then, (*th.* 95,) we have AE′ : E′C :: AF . BD : FB . DC, and hence AE′ : E′C :: AE : EC, and AE′ $\mp$ E′C : AE′ :: AE $\mp$ EC : AE; but AE′ $\mp$ ′EC $=$ AE $\mp$ EC, and hence AE′ $=$ AE, the less to the greater, which is impossible. Whence D, E, F, are in one line.

Scholium.

This proposition furnishes a ready method of proving a great number of elegant theorems. For instance the following:

(1). The three perpendiculars from the angles of a triangle to the opposite sides, meet in one point.

(2). The lines which bisect the angles of a triangle, either all internally, or one internally and the other two externally, meet in one point.

(3). The lines which bisect the sides of a triangle meet in one point.

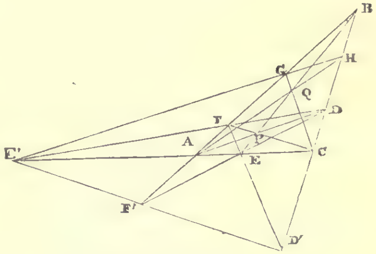
(4). Lines drawn from the angles of a triangle to the points of contact of its inscribed circle, meet in one point.

(5). Lines drawn from the angles of a triangle to the points of contact of circles each touching one side exteriorly and the other two produced, meet in one point.

(6). If the exterior angles of a triangle be bisected by lines which are produced to cut the opposite sides, the bisecting lines intersect those sides in three points which lie in one straight line.

THEOREM XCVIII.

If through any point lines be drawn from the angles of a triangle to cut the opposite sides, and the points of section be joined two and two by lines which are produced to cut the remaining sides of the triangle; then the sides of the triangle will all be divided in harmonical proportion, each in the two points in which it is cut by the lines described, and the last-mentioned three points of section will be in one straight line.



Let ABC be a triangle, from the angles A, B, C, of which lines are drawn through any point P to meet the opposite sides in D, E, F, respectively; and let EF be drawn to meet BC in D′, FD to meet AC in E′, and DE to meet AB in F′: then

First, BD : DC :: BD′ : D′C,
CE : EA :: CE′ : E′A,
and FA : BF :: F′A : BF′.

For, (*th.* 96,) AE : EC :: AF . BD : FB . CD,
and, (*th.* 95,) AE′ : E′C :: AF . BD : FB . CD.

Whence AE : EC :: AE′ : E′C; and similarly of the other two proportions. The lines are therefore divided harmonically (*def.* 84).

Second. The points, D′, E′, F′, are in one straight line. For, compounding the three ratios, we have

BD . CE . FA : DC . EA . FB :: BD′ . CE′ . F′A : D′C . E′A . F′B;

and since the first term is equal to the second, the third term is equal to the fourth, and CE′ : E′A :: F′B . CD′ : F′A . BD′. Hence (*th.* 97) the points D′, E′, F′, are in one straight line.

Cor. 1. If from one angle, as B, of a triangle ABC, a line BE be drawn to cut the opposite side in E, and from the other angles any number of pairs of lines be drawn to meet in BE and cut the opposite sides, as AD, CF, meeting AE in P, AH, CG, meeting it in Q, and so on; then DF, HG, and so on, will all meet AC in the same point E′. For they all divide, with E, the side AC harmonically in their points of intersection. These points, must, therefore, coincide.

Cor. 2. If from any point E′ in one side of a triangle lines be drawn to cut the other two, as E′GH, E′FD, and so on; and lines be drawn from these points to the angles opposite, they will two and two cut each other in points, all of which lie in one straight line.

For (*th.* 97) BQ, BP, and so on, all cut the side AC in points such that with E′ divide it harmonically. Hence these points must coincide, and therefore BQ, BP, ... must also coincide, or the points P, Q, ... all lie in one straight line *.

---

* Several simple and yet very important properties of harmonical lines are thrown together in this note, chiefly to avoid the necessity of formal enunciations, which, if given in words, would occupy considerable space.

1. Let AC : CB :: AD : DB be the general division of the line. Bisect AB in M and CD in M′. Then

2. CA : AD :: CB : BD, or the line CD is harmonically divided in A and B; and the same points of division result from supposing either AB or CD to be the given line.

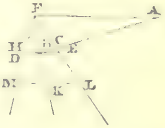
3. From

THEOREM XCIX.

If a straight line be divided harmonically, and from the four points of section, straight lines be drawn through any point in the same plane: then

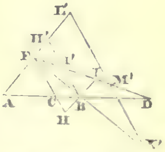
1. *Any straight line drawn parallel to one of those four lines will be bisected by the other three.*
2. *Any line oblique to them all will be harmonically divided by them at the points of intersection.*

First. Let AC : CB :: AD : DB, and the lines be drawn from A, B, C, D, to meet at F: then, if a line MKL be drawn parallel to AF, one of the *extreme* lines of the fasceau, it will be bisected in K.



Through B draw HBE parallel to ML or FA. Then, by parallels AD : DB :: AF : BH, and AC : CB :: AF : BE; but by hypothesis, AD : DB :: AC : CB, and therefore AF : BE :: AF : BH, or BE = BH. Again, by parallels, ML, HE, HB : BE :: MK : KL, or MK = KL.

Next, let the line $L'K'M'$ be parallel to one of the intermediate lines FC, it will be bisected in M'.



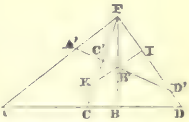
Through B draw $BI'H'$ parallel to $L'K'$ or FC, cutting the extreme lines of the fasceau in I', H'; and draw HI through B parallel to AF.

Then, because BH' is parallel to FH, and BH to FH', we have $FH' = BH$: but by the previous case, $BH = BI$; and hence FH' is equal and parallel to BI, and the diagonal $H'B$ is bisected in I' by the diagonal FI, or extreme sector FD. Also, by parallels, $HI' : I'B :: L'M' : M'K'$, and $HI' = I'B$; hence $L'M' = M'K'$.

Cor. Conversely, if from any point two lines be drawn to the extremities of a given line, a third to bisect that line, and a fourth parallel to it, they will form an harmonical fasceau.

Second. If any line cut an harmonical fasceau F{ABCD} it will be divided harmonically in the points A', B', C', D', of intersection.

For, through B' draw KI parallel to AF: then by the former part of the proposition, KI is bisected in B'; and by parallels



$A'F : B'I :: A'D' : B'D'$, and
$A'F . B'K : A'C' : C'B'$: whence
$A'D' : D'B' :: A'C' : C'B'$, and $A'B'$ is harmonically divided in C', D'.

---

3. From (1) we have AC — CB : AC + CB :: AD — DB : AD + DB; that is, 2MC : 2MB :: 2MB : 2MD, or MC : MB :: MB : MD; or MB is a mean proportional between MC and MD.

4. In like manner, $M'B : M'C :: M'C : M'A$, or $M'C$ is a mean proportional between $M'A$ and $M'B$.

5. From (3) we have MD : MD $\pm$ MD :: MB : MB $\pm$ MC, or which is the same thing, MD : DB :: MB : BC.

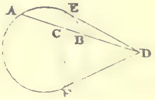
6. By *th.* 33, $AD . DB = MD^2 - MB^2$, and $AC . CB = MB^2 - MC^2$; hence the difference gives us at once

$$AD . DB - AC . CB = MD^2 - 2MB^2 + MC^2, \text{ or by (3)}$$
$$= MD^2 - 2MC . MD + MC^2, \text{ or } th. 32,$$
$$= (MD - MC)^2 = CD^2.$$

THEOREM C.

If two tangents from one point to a circle, and the chord joining the points of contact, be drawn; then any line drawn from the intersection of the tangents which cuts the circle will be harmonically divided at its intersection with the circle and its chord.

Let the tangents DE, DF, drawn from D, touch the circle in E and F, and let EF be joined: then any line DA cutting the circle in A and B and the chord in C will be harmonically divided, such that AC : CB :: AD : DB.

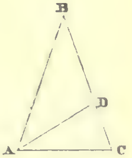


For $DC^2 + EC \cdot CF = DE^2$ (*th.* 39) $= AD \cdot DB$ (*th.* 61, *cor.* 1). Whence $DC^2 = AD \cdot DB - EC \cdot CF = AD \cdot DB - AC \cdot CB$. Hence (*conv. of note* 6, *p.* 341) the line AB is harmonically divided in C and D.

THEOREM CI.

If each of the angles at the base of an isosceles triangle be double of the vertical angle, the base is the greater segment of the side divided in extreme and mean ratio; and if the base of an isosceles triangle be the greater segment of the side divided in extreme and mean ratio, each of the angles at the base is double of the vertical angle.

1. Let the angles BAC, BCA, in the isosceles triangle ABC be each double of the third, ABC; then AC will be the greater segment of AB or BC divided in extreme and mean ratio.



For, draw AD bisecting the angle BAC; and since BAC is double of ABC, its half, BAD, is equal to ABD or DAC: but ADC is equal to BAD and ABD; that is, to BAC or BCA, and likewise AD, DB, are equal, since the angles BAD, ABD, are equal. Hence the triangle DAC is similar to ABC, and AC, AD, equal: and therefore AB : AC :: AC : CD; or since AD, AC, BD, are all equal, CB : BD :: BD : DC.

2. Let the base AC of the isosceles triangle ABC be the greater segment of the side BC divided in extreme and mean ratio; then each of the angles BAC, BCA, will be double of ABC.

For make BD equal to AC, and join AD; and since CB : BD :: BD : DC, we have AB : AC :: AC : CD, and the angles BAC, ACD, equal, and therefore the triangles BAC, ACD, similar. Whence, since the sides AB, BC, are equal, the sides AC, AD, are also equal, and the angles ACD, ADC, also equal. Now by construction, BD is equal to AC; and therefore, also, to AD, or ADB is isosceles; and the angle ADC being equal to ABD and BAD, is double of one of them ABD: but ADC is equal to the angle DCA, and therefore to each of the angles BAC, BCA, of the triangle ABC; and hence each of these angles is double of the angle ABC at the vertex.

MISCELLANEOUS EXERCISES IN PLANE GEOMETRY*.

1. From two given points on the same side of a line given in position, to draw two lines which shall meet in that line, and make equal angles with it.

2. If two circles cut each other, and from either point of intersection diameters be drawn; the extremities of these diameters and the other point of intersection shall be in the same straight line.

3. If a line touching two circles cut another line joining their centres, the segments of the latter will be to each other as the diameters of the circles.

4. If a straight line touch the interior of two concentric circles, and be placed in the outer, it will be bisected at the point of contact.

5. If from the extremities of the diameter of a semicircle perpendiculars be let fall on any line cutting the semicircle, the parts intercepted between those perpendiculars and the circumference are equal.

6. If on each side of any point in a circle any number of equal arcs be taken, and the extremities of each pair joined; the sum of the chords so drawn will be equal to the last chord produced to meet a line drawn from the given point through the extremity of the first arc.

7. If two circles touch each other externally or internally, any straight line drawn through the point of contact will cut off similar segments.

8. If two circles touch each other externally or internally, two straight lines drawn through the point of contact will intercept arcs, the chords of which are parallel.

9. If two circles touch each other, and also touch a straight line; the part of the line between the points of contact is a mean proportional between the diameters of the circles.

10. If a common tangent be drawn to any number of circles which touch each other internally, and from any point in this tangent as a centre a circle be described cutting the others, and from this centre lines be drawn through the intersections of the circles respectively; the segments of them within each circle will be equal.

11. If the radius of a circle be a mean proportional to two distances from the centre in the same straight line, the lines drawn from their extremities to any point in the circumference will have the same ratio that the distances of these points from the circumference have.

12. In a circle to place a straight line of a given length, so that perpendiculars drawn to it from two given points in the circumference may have a given ratio.

13. If any two chords be drawn in a circle, to intersect at right angles, then will the squares upon the four segments of those chords be together equal to the square upon the diameter of the circle.

14. If the tangents drawn to every two of three unequal circles be produced till they meet, the points of intersection will be in a straight line.

---

* It is not expected that the student should go through *all* these exercises in his first study of geometry; but that the tutor should select from them fewer or more according to the capacity and talent of his pupil; requiring demonstrations of the theorems, and both constructions and demonstrations of the problems, thus selected.

15. If the points of bisection of the sides of a given triangle be joined, the triangle so formed will be one-fourth of the given triangle.

16. The three straight lines which bisect the three angles of a triangle meet in the same point.

17. If from the angles of a triangle, lines, each equal to a given line, be drawn to the opposite sides (produced if necessary); and from any point within, lines be drawn parallel to these, and meeting the sides of the triangle; these lines will together be equal to the given line.

18. The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of two opposite sides, are together half of the parallelogram.

19. If in the sides of a square, at equal distances from the four angles, four other points be taken, one in each side; the figure contained by the straight lines which join them shall also be a square.

20. Determine the figure formed by joining the points of bisection of the sides of a trapezium, and its ratio to the trapezium.

21. Determine the figure formed by joining the points where the diagonals of the trapezium cut the parallelogram (in the last problem), and its ratio to the trapezium.

22. If the sides of any pentagon be produced to meet, the angles formed by these lines are together equal to two right angles.

23. If the sides of any hexagon be produced to meet, the angles formed by these lines are together equal to four right angles.

24. If squares be described on the three sides of a right-angled triangle, and the extremities of the adjacent sides be joined; the triangles so formed are equal to the given triangle and to each other.

25. If from the angular points of the squares described upon the sides of a right-angled triangle, perpendiculars be let fall upon the hypothenuse produced, they will cut off equal segments; and the perpendiculars will together be equal to the hypothenuse.

26. If squares be described on the hypothenuse and sides of a right-angled triangle, and the extremities of the sides of the former and the adjacent sides of the others be joined; the sum of the squares of the lines joining them will be equal to five times the square of the hypothenuse.

27. If through any point within a triangle lines be drawn from the angles to cut the opposite sides, the segments of any one side will be to each other in the ratio compounded of the ratios of the segments of the other sides.

28. If a line be drawn from the vertex to any point in the base of a triangle, the sum of the two solids under the squares of the two sides and the alternate segments of the base will be equal to the solid under the whole base and its two segments, together with the solid under the same base and the square of the dividing line.

29. Determine a point in a line given in position, to which lines drawn from two given points may have the greatest difference possible.

30. Divide a given triangle into any number of parts, having a given ratio to each other, by lines drawn parallel to one of the sides of the triangle.

31. Through a given point between two straight lines containing a given angle, to draw a line which shall cut off a triangle equal to a given figure.

32. Divide a circle into any number of concentric equal annuli.

33. Divide it into annuli which shall have a given ratio.

34. In any quadrilateral figure circumscribing a circle, the opposite sides are equal to half the perimeter.

35. Inscribe a square in a given right-angled isosceles triangle.

36. Inscribe a square in a given quadrant of a circle.

37. Inscribe a square in a given semicircle.

38. Inscribe a square in a given segment of a circle.

39. Having given the distance of the centres of two equal circles which cut each other, inscribe a square in the space included between the two circumferences.

40. In a given segment of a circle inscribe a rectangular parallelogram whose sides shall have a given ratio.

41. In a given triangle inscribe a triangle similar to a given triangle.

42. In a given equilateral and equiangular pentagon inscribe a square.

43. In a given triangle inscribe a rhombus, one of whose angles shall be in a given point in the side of the triangle.

44. Inscribe a circle in a given quadrant.

45. If on the diameter of a semicircle two equal circles be described, and in the curvilinear space included by the three circumferences a circle be inscribed; its diameter will be to that of the equal circles in the proportion of two to three.

46. If through the middle point of any chord of a circle two chords be drawn, the lines joining their extremities will intersect the first chord at equal distances from the middle point.

47. If in a right-angled triangle a perpendicular be drawn from the right angle to the hypothenuse, and circles inscribed within the triangles on each side of it, their diameters will be to each other as the subtending sides of the right-angled triangle.

48. If in a right-angled triangle a perpendicular be drawn from the right angle to the hypothenuse, and circles inscribed within the triangles on each side of it, they will be to each other as the segments of the hypothenuses made by the perpendicular.

49. In any triangle, if perpendiculars be drawn from the angles to the opposite sides, they will all meet in a point.

50. Three equal circles touch each other; compare the area of the triangle formed by joining their centres with the area of the triangle formed by joining the points of contact.

51. If a four-sided rectilinear figure be described about a circle, the angles subtended at the centre of the circle, by any two opposite sides of the figure, are together equal to two right angles.

52. If two given straight lines touch a circle, and if any number of other tangents be drawn, all on the same side of the centre, and all terminated by the two given tangents, the angles which they subtend at the centre of the circle shall be equal to one another.

53. If two circles cut each other, and from any point in the prolongation of the straight line which joins their intersections, two tangents be drawn, one to each circle, they shall be equal to each other.

54. To cut off from a given parallelogram a similar parallelogram which shall be any given part of it.

55. If there be any right-lined hexagonal figure, and two contiguous sides be in succession equal and parallel to two other contiguous and opposite sides, each to each; then, first, the two remaining sides will be respectively equal and parallel; secondly, the opposite angles (viz. the first and fourth, second and fifth, third and sixth,) will be equal to one another; thirdly, any diagonal joining two of those opposite angles, will divide the figure into two equal parts.

56. In any pentagonal right-lined figure, thrice the sum of the squares of the

sides will be equal to the sum of the squares of the diagonals, together with four times the sum of the squares of the five right lines joining, in order, the middle points of those diagonals.

57. If there be any rectilinear figure having an even number of sides in a circle, the sum of all the angles of those angles of the figure, beginning at any one, which succeed one another, according to the odd numbers, will be equal to the sum of all the angles which succeed one another according to the even numbers.

58. If each side of any rectilinear figure, whose sides are even in number, touch a circle, the sum of the first, third, fifth, &c., beginning at any one side, and proceeding in order according to the odd numbers, will be equal to the sum of the remaining second, fourth, sixth, &c., sides, proceeding according to the even numbers.

59. If two circles intersect one another, and any right line be drawn cutting the circles, it will be proportionally divided by the circumferences of the circles.

60. Given the perpendicular drawn from the vertical angle to the base, and the difference between each side and the adjacent segment of the base made by the perpendicular; to construct the triangle.

61. Given the vertical angle, the perpendicular drawn from it to the base, and the ratio of the segments of the base made by it; to construct the triangle.

62. Given the vertical angle, the difference of the two sides containing it, and the difference of the segments of the base made by a perpendicular from the vertex; to construct the triangle.

63. Given the lengths of three lines drawn from the angles to the points of bisection of the opposite sides; to construct the triangle.

64. The sum of the descending infinite series $a + b + \frac{b^2}{a} + \frac{b^3}{a^2} + \ldots$ is well known to be expressed by $\frac{a^2}{a - b}$, or a third proportional to $a - b$ and a. Demonstrate this upon geometrical principles.

65. Every equilateral polygon circumscribed about a circle, or inscribed in a circle, is equiangular; and every equiangular polygon so circumscribed or inscribed is equilateral.

66. If from the points of contact of a regular circumscribed polygon, lines be drawn from each point of contact to its adjacent ones, the polygon so described will be regular; and if to the circle at the angular points of a regular inscribed polygon, tangents be drawn, these, by their successive adjacent intersections, will form a regular circumscribed polygon.

67. Every regular polygon is capable of inscription and circumscription by circles.

68. If in a regular inscribed polygon of an *odd* number of sides, parallels to each side be drawn through the angles opposite to those sides respectively, they will form by their intersections a regular circumscribed polygon.

69. If in a regular inscribed polygon of an even number of sides, lines be drawn parallel to those which join every two adjacent sides through the angle, most distant from these lines, the lines so drawn will be tangents to the circle, and their assemblage will constitute a regular circumscribed polygon.

70. If straight lines be drawn through any point to cut a circle, and the fourth harmonical points in each of them, (the given point and the intersections with the circle being the other three,) be found: all these fourth points will be in one straight line.

71. If two lines be drawn from any point without a circle to intersect it, and lines be drawn to the alternate points of intersection, these will always intersect in the chord which joins the points of contact of the tangents drawn to the circle from the point without it.

72. If the radius of a circle be divided in extreme and mean ratio, the greater segment is the side of the regular decagon inscribed in that circle; and the sum of the squares of the radius and its greater segment is equal to the square of the side of the inscribed regular pentagon.

73. If a tangent be drawn to a circle equal to its diameter, and from the extremity of the tangent a line be drawn through the centre, and from the points of intersection of this line with the circle, lines be drawn to the point of contact: the greater of these will be the radius of a circle in which the less will be the side of the inscribed decagon, and in which the tangent will be the side of the inscribed pentagon.

OF PLANES AND SOLIDS.

The figures which we have hitherto considered are such as lie entirely on one plane: in those which follow, the intersections of different planes with one another, or with given straight lines, the volumes of space enclosed with certain combinations of planes, and other topics of the same kind, are the objects of research. The conception of the figures and of their properties is greatly facilitated by the use of models, and no student should proceed without them; though, of course, no great regard to extreme precision is requisite in their construction.

DEFINITIONS.*

1. (88.) The *common section of two planes* is the line in which they meet or cut each other.

2. (89.) A *line is perpendicular to a plane*, when it is perpendicular to every line in that plane which meets it; and the point in which the perpendicular meets the plane is called the foot of the perpendicular.

3. (90.) *One plane is perpendicular to another plane*, when every line of the one, which is perpendicular to the line of their common section, is perpendicular to the other.

4. (91.) The *inclination of one plane to another*, or the angle they form between them, is the angle contained by two lines, drawn from any point in the common section and at right angles to it, one of these lines in each plane. This is often called a *dihedral* angle.

5. (91*a*.) If from a point in a line which meets a plane, a perpendicular be drawn to the plane, and the points of intersection of these two lines with the plane be joined, the angle formed by the line in the plane and the line which meets the plane is called the *inclination of the line to the plane.*

6. (92.) *Parallel planes* are such as being produced ever so far in every direction will never meet.

7. (93.) A *solid angle* is that which is made by three or more plane angles meeting each other in the same point.

---

* A modified arrangement of the definitions and propositions of this subject has rendered it necessary to commence both as with a new subject. The numbers of the last edition, however, for obvious reasons, being desirable to be retained, they are here marked in parentheses.

8. (94.) *Similar solids,* contained by plane figures, are such as have all their solid angles equal, each to each, and are bounded by the same number of similar planes, and placed in the same consecutive order.

9. (95.) A *prism* is a solid whose ends are parallel, equal, and like plane figures; and its sides, connecting those ends, are parallelograms.

10. (96). A *prism* takes particular names according to the figure of its base or ends, whether *triangular* prism, *square* prism, *rectangular* prism, *pentagonal* prism, *hexagonal* prism, and so on.

11. (97.) A *right prism* is that which has the planes of the sides perpendicular to the planes of the ends or base.

12. (98.) A *parallelopiped*, or *parallelopipedon,* is a prism bounded by six parallelograms, every opposite two of which are equal and parallel.

13. (99.) A *rectangular parallelopipedon* is that whose bounding planes are all rectangles.

14. (100.) A *cube* is a square prism, being bounded by six equal square sides or faces.

15. (101.) A *cylinder* is a round prism, having circles for its ends. It is conceived to be formed by the rotation of a right line about the circumferences of two equal and parallel circles, always parallel to the axis.

16. (102.) The *axis of a cylinder* is the right line joining the centres of the two parallel circles about which the figure is described.

When the axis of the cylinder is at right angles to the planes of the parallel ends, the cylinder is called a *right,* and when oblique to them an *oblique* cylinder.

17. (103.) A *pyramid* is a solid, whose base is any right lined plane figure, and its sides triangles, having all their vertices meeting together in a point without the plane of the base, called the vertex of the pyramid.

18. (104.) A pyramid, like the prism, takes its particular name from the figure of the base; as a *triangular, quadrangular, etc. pyramid.*

19. (105.) A *cone* is a round pyramid, having a circular base. It is conceived to be generated by the rotation of a right line about the circumference of a circle, one end of which is fixed at a point without the plane of that circle.

20. (106.) The *axis of a cone* is the right line joining the vertex, or fixed point, and the centre of the circle about which the figure is described.

When the axis of the cone is at right angles to its base, the cone is said to be a *right,* and when oblique to the base, an *oblique cone.*

21. (107.) *Similar cones* and *similar cylinders* are such as have their altitudes, the diameters of their bases, and their axes, proportional.

22. (108.) A *sphere* is a solid bounded by one curve surface, which is every where equally distant from a certain point called the *centre.* It is sometimes conceived to be generated by the rotation of a semicircle about its diameter, which remains fixed.

23. (109.) The *axis of a sphere* is the right line about which the semicircle revolves, and the centre is the same as that of the revolving semicircle.

24. (110.) The *diameter of a sphere* is any right line passing through the centre, and terminated both ways by the surface.

25. (111.) The *altitude of a solid* is the perpendicular drawn from the vertex to the opposite face, considered as its base.

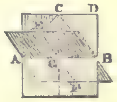
26. (112.) By the *distance of a point from a plane* is meant the *shortest line* that can be drawn from that point to meet the plane. It is subsequently shown that this is the *perpendicular* (*th.* 3, *cor.* 1).

THEOREMS.

SECT. I.—OF LINES AND PLANES.

THEOREM I. (96.)

Two straight lines which meet each other; the three sides of a triangle; any three points in space; or two parallel lines;—are in the same plane, and being given determine its position.



First. LET AB, AC, be two straight lines which intersect each other in A. A plane may be made to pass through AB in any direction, and hence it may be turned about AB till it also passes through C. Then the line AC which has two of its points, A and C, in this plane, lies wholly in the plane, and the plane itself is fixed in its position.

Second. A triangle ABC, or any three points in space not in the same right line, determine the position of a plane.

Third. Also two parallels, AB, CD, determine the position of the plane in which they are situated. For the plane may be turned about one of them to touch a point of the other, and the second line being in the same plane as the first, and passing through a point in it, the plane must be that just determined.

THEOREM II. (97.)

The common section of two planes is a right line.

(Same figure.)

LET ACBDA, AEBFA, be two planes cutting each other, and A, B, two points in which the two planes meet; drawing the line AB, this line will be the common intersection of the two planes.

For, because the right line AB touches each of the planes in the points A and B, it touches them in all other points (*def.* 20): this line is therefore common to the two planes. That is, the common intersection of the two planes is a right line.

THEOREM III. (98.)

If a straight line be perpendicular to any two other straight lines in their point of intersection, it shall also be at right angles to the plane which passes through them, that is, to the plane in which they are.

LET PC, PB, be two lines intersecting in P, and AP be another line passing through P at right angles to PB and PC, then AP will be perpendicular to any line PQ in the plane BPC.

For, in PQ take any point Q, and draw QR parallel to PC, meeting PB in R; and take RB equal to RP, and draw BQ to meet PC in C.

Join AC, AQ, AB. Then, since the line QR is drawn

parallel to the side PC of the triangle PCB, we have (*th.* 82) PR : RB : : CQ : QB, and PR = RB; hence BC is bisected in Q.

Then, (*th.* 38,) $PC^2 + PB^2 = 2PQ^2 + 2CQ^2$, and $AC^2 + AB^2 = 2AQ^2 + 2CQ^2$; and taking the first equation from the second, we shall have (*th.* 34)

$$AP^2 + AP^2 = 2AQ^2 - 2PQ^2, \text{ or } AQ^2 = AP^2 + PQ^2.$$

Whence (1 *cor. th.* 34) APQ is right angled at P, or AP is perpendicular to PQ. And the same may be proved for any other line drawn in the plane MN through the point P. The line AP is, therefore, perpendicular to every straight line in the plane MN passing through P; and hence to the plane in which those lines are (*def.* 2).

Cor. 1. The perpendicular AP is the shortest line that can be drawn from A to the plane. See *def.* 26.

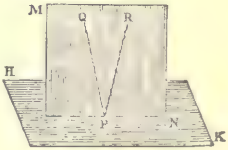
Cor. 2. Oblique lines which meet the plane at the same distance from the foot of the perpendicular and proceed from the same point in the perpendicular, are equal to one another: and that which meets the plane at a less distance from P is less than that which meets it at a more remote distance.

THEOREM IV. (99.)

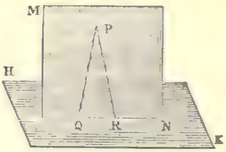
There can only be one line perpendicular to a given plane, and passing through a given point, whether that point be in the plane or without it.

For suppose there can be two.

First. Let P be in the plane HK, and the two perpendiculars be PQ, PR. Through QPR let a plane pass, cutting HK in PN. Then the angles QPN, RPN will both be right angles (*def.* 89), and hence equal to one another (*ax.* 10): that is, a part equal to the whole, which is absurd. Hence PR is perpendicular to the plane HK.



Second. Let P be without the plane HK, and let PQ, PR be the two perpendiculars admitted for the moment to be drawn from P to HK. Let the plane MN passing PQ, and PR, cut the plane HK in QR. Then the angles PQR, PRQ are both right angles, which is impossible (*th.* 17). Hence, there cannot be two perpendiculars drawn to the same plane from a point without it.

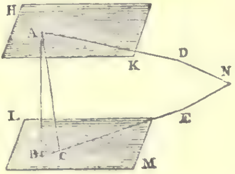


THEOREM V. (100.)

If a straight line be perpendicular to one of two parallel planes, it will be perpendicular to the other.

Let HK, LM, be two parallel planes, and AB be perpendicular to HK, it shall also be perpendicular to LM.

For if not, from A draw AC perpendicular to LM, meeting it in C, and through ABC draw a plane cutting the planes HK, LM, in AD and BCE. Then, since AC is perpendicular to LM, the angle ACB is a right angle, and hence ABC is



less than a right angle. Hence, since BAD is a right angle, the two angles DAB and ABC are less than two right angles, and hence the lines AD, BE, in the same plane will meet if sufficiently produced. But AD is in the plane HK, and BC in LM, hence HK also meets LM: which is impossible, since by hypothesis they are parallel.

THEOREM VI. (101.)

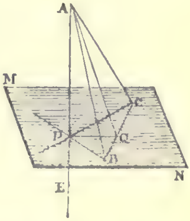
[*See figure to theorem* 5].

If two planes be perpendicular to the same straight line, they are parallel to one another.

Let the planes HK and LM be perpendicular to the line AB, they will be parallel to one another. For if they be not parallel they must meet. Let N be a point in their common intersection and join NA, NB. Then since AB is perpendicular to the plane HK, it is perpendicular to NA drawn through A in that plane, and NAB is a right angle. In like manner, NBA is a right angle. But NAB being a triangle, the two angles NAB, NBA, are together less than two right angles. Hence the perpendiculars from A, B, in the planes HK and LM, do not meet at N. In the same manner it can be proved that they do not meet at any other point; and hence that the planes HK, LM, have not any point common, and are therefore parallel.

THEOREM VII. (102.)

If from the foot of the perpendicular to any plane a line be drawn at right angles to a line in that plane, any line drawn from the point of intersection to a point in the line which is perpendicular to the plane will also be perpendicular to the line which lies in the plane.



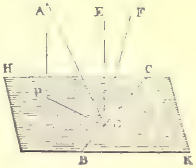
Let AP be perpendicular to the plane MN, and BC be a line situated in that plane: if from P, the foot of the perpendicular, the line PQ be drawn perpendicular to BC, then QA drawn to any point A, in AP, will also be perpendicular to BC.

Take BQ = QC, and join PB, PC, AB, AC. Then since BQ = QC, and PQB = PQC, and PQ common, we have also PB = PC. Again, since PB = PC, APB = APC (*def.* 89), and AP common, we have also AB = AC. Then AQ, QB, being equal to AQ, QC, each to each, and AB equal to AC, the angle AQB = AQC, and hence they are right angles.

Cor. BC is perpendicular to the plane APQ, for BC is perpendicular to AQ and PQ, which determine that plane (*th.* 1).

THEOREM VIII. (103.)

If a plane be perpendicular to one of two parallel lines, it will be perpendicular to the other; and if two straight lines are perpendicular to the same plane, they are parallel to one another.



First. Let AP be perpendicular to the plane HK, and DE be parallel to AP; DE will also be perpendicular to HK.

Let the plane APDE which includes the parallels AP and DE, intersect the plane HK in PD; and in the plane HK let the line BC be drawn perpendicular to PD, and join AD.

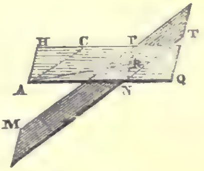
By *cor. th.* 102, BC is perpendicular to the plane APDE, and therefore the angle BDE is a right angle. Also, since AP, DE, are parallels, and APD is a right angle (*def.* 89), the angle EDP is a right angle. Hence ED is perpendicular to the two straight lines DP, DB, and hence is perpendicular to the plane HK in which they lie.

Second. Let AP, DE be perpendicular to the plane HK, they shall be parallel to one another.

For if this be denied, let some other line DF through D be supposed parallel to AP. Then DF will (by the former part of the proposition) be perpendicular to HK. But, by hypothesis, DE, also passing through D, is perpendicular to HK: and hence through the same point D there can be two perpendiculars drawn to the plane, HK, which, (*th.* 99,) is impossible. Hence AP, DE, are parallel.

THEOREM IX. (104.)

If each of two lines which intersect one another be parallel to a plane, the plane in which these lines are situated, will also be parallel to it.



Let the two straight lines AR, AC, be each of them parallel to the plane MN; then the plane HQ, in which AR, AC, are situated, will be parallel to MN.

For if HQ be not parallel to MN they will meet, and their intersection will be a straight line (*th.* 2). Let NP be their intersection. Then, since NP is in the same plane with AR and AC, which meet in A, it cannot be parallel to both of them, and therefore will cut one at least, as AR, in the point R. Now the point R is situated also in the plane MN, and hence AR meets MN: but AR is parallel (*hyp.*) to the plane MN, which is impossible. Whence the plane in which AR, AC, are situated, cannot meet the plane MN to which AR, AC, are parallel; that is, HQ is parallel to MN.

Cor. Hence, through any given line which is parallel to a plane, a second plane may always be made to pass, that shall be parallel to that plane. For through any point in the line another line parallel to the given plane may be drawn; and the plane of these two lines will be parallel to the other plane.

THEOREM X. (105.)

[*See figure to theorem* 1.]

If one plane meet another plane, it will make angles with that other plane, which are together equal to two right angles.

Let the plane ACBD meet the plane AEBF; these planes make with each other two angles whose sum is equal to two right angles.

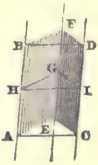
For, through any point G, in the common section AB, draw CG, EF, perpendicular to AB. Then, the line CG makes with EF two angles together equal to two right angles. But these two angles are (*def.* 91) the angles of inclination of the two planes. Therefore, the two planes make angles with each other, which are together equal to two right angles.

Scholium.

In like manner it may be demonstrated, that planes which intersect have their vertical or opposite angles equal; also, that parallel planes cut by a third make their alternate angles equal; and so on, as in parallel lines; but with obvious limitations. See pp. 301, 302.

THEOREM XI. (106.)

If two lines be parallel to a third line, though not in the same plane with it; they will be parallel to each other.



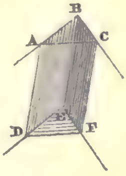
Let the lines AB, CD, be each of them parallel to the third line EF, though not in the same plane with it; then will AB be parallel to CD.

For, from any point G in the line EF, let GH, GI, be drawn perpendicular to EF, in the planes EB, ED, of the parallels AB, EF, and EF, CD.

Then, since the line EF is perpendicular to the two lines GH, GI, it is perpendicular to the plane GHI of those lines (*th.* 98). And because EF is perpendicular to the plane GHI, its parallel AB is also perpendicular to that plane (*cor. th.* 103). For the same reason, the line CD is perpendicular to the same plane GHI. Hence, because the two lines AB, CD, are perpendicular to the same plane, these two lines are parallel (*th.* 103).

THEOREM XII. (107.)

If two lines, that meet each other, be parallel to two other lines that meet each other, though not in the same plane with them; the angles contained by those lines will be equal.



Let the two lines AB, BC, be parallel to the two lines DE, EF; then will the angle ABC be equal to the angle DEF.

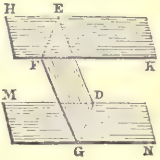
For, make the lines AB, BC, DE, EF, all equal to each other, and join AC, DF, AD, BE, CF.

Then, the lines AD, BE, joining the equal and parallel lines AB, DE, are equal and parallel (*th.* 24). For the same reason, CF, BE, are equal and parallel. Therefore

AD, CF, are equal and parallel (*th.* 106); and consequently also AC, DF (*th.* 24). Hence, the two triangles ABC, DEF, having all their sides equal, each to each, have their angles also equal, and consequently the angle ABC is equal to the angle DEF.

THEOREM XIII. (108.)

The sections made by a plane cutting two parallel planes are parallel to one another.



Let the parallel planes HK, MN, be cut by the plane EG in the lines EF, GD; then EF will be parallel to GD.

For if EF, GD, be not parallel, since they are in the same plane EG, they would, if produced, meet: but as EF is in the plane HK, and GD in MN, these planes would in that case also meet. But these planes cannot meet, since, by hypothesis, they are parallel: and hence EF, GD, cannot meet, or they are parallel.

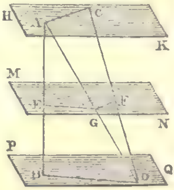
Cor. 1. The parallels ED, FG, comprehended between parallel planes HK, MN, are equal.

Let the plane in which the parallels ED, FG, lie, cut the parallel planes in EF, GD. Then these are also parallel: hence EG is a parallelogram, and its sides FG, ED, therefore equal.

Cor. 2. Hence two parallel planes are every where equi-distant. For in this case FG, ED, are perpendicular to the two planes (*th.* 100), and hence are parallel (*th.* 101), and hence again (*cor.* 1, *th.* 108) are equal to one another.

THEOREM XIV. (109.)

Straight lines being cut by parallel planes are divided proportionally.



Let there be, for instance, two straight lines AB, CD, cut in A, E, B, C, F, D, by the three parallel planes HK, MN, PQ: then we shall have AE : EB :: CF : FD.

Draw AD meeting the plane MN in G, and join AC, EG, GF, BD; the intersections EG, BD, of the plane ABD with MN, PQ, are parallel (*th.* 108); and hence AE : EB :: AG : GD. In like manner, AC, GF, are parallel, and hence AG : GD :: CF : FD. Hence AE : EB :: CF : FD. In the same way the property is established if there be more lines or more planes, or both.

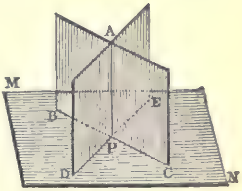
Scholium.

If three or more straight lines meeting a plane be divided proportionally, and in the same order, all the sets of corresponding points of section will lie in planes parallel to the first: but if there be only two lines, the planes through the corresponding points may or may not be parallel (p. 360). When the planes are not parallel, however, their sections with each other will either coincide or be parallel to each other.

THEOREM XV. (110.)

1. *If through a line which is perpendicular to a plane another plane be made to pass, this plane will be perpendicular to the former.*
2. *If two planes be perpendicular to one another, and in one of them a line be drawn perpendicular to the common section, it will be perpendicular to the other plane.*
3. *If a plane be perpendicular to a plane, and if at a point in their intersection a perpendicular be erected to the former plane, it will lie wholly in the latter.*
4. *If each of two planes be perpendicular to a third plane, their intersection will be perpendicular to it also.*

First. If the line AP be perpendicular to the plane MN, any plane through AP will be perpendicular to MN.



Let the planes AB, MN intersect in BC, and in the plane MN draw DE perpendicular to BP. Then the line AP, being perpendicular to the plane MN, will be perpendicular to each of the straight lines BC, DE: but the angle formed by the two perpendiculars PA, PD, at the common intersection measures the angle of the two planes, (*def.* 91); and hence (*def.* 90), since the angle is right, the two planes are perpendicular to each other.

Second. If the plane AB is perpendicular to the plane MN, and in AB the line AP is drawn perpendicular to the common section PB of the planes MN, AB, it will be perpendicular to the plane MN.

For, in the plane MN draw PD perpendicular to PB; then, because the planes are perpendicular, the angle APD is a right one: therefore the line AP is perpendicular to the two straight lines PB, PD. Hence it is perpendicular to their plane MN.

Third. If the plane AB be perpendicular to the plane MN, and if at a point P of the common intersection a perpendicular be erected to the plane MN, that perpendicular will be in the plane AB.

For if not, then in the plane AB a perpendicular AP might be drawn to the common intersection PB, which at the same time would be perpendicular to the plane MN. Hence two perpendiculars may be drawn from the same point to the same plane, which is impossible (*th.* 109).

Fourth. If each of two planes be perpendicular to a third plane, their common intersection will be perpendicular to it also.

That is, if the planes AB, AD, are perpendicular to a third plane MN, their common intersection AP will be also perpendicular to MN.

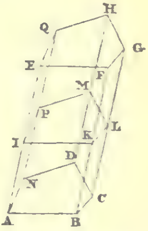
For at the point P, let a perpendicular be drawn to the plane MN. That perpendicular must be at the same time in the plane AB and in the plane AD, and hence it is their common intersection AP.

Scholium.

The properties in this theorem are the foundation of the method of *co-ordinates in space,* and of the principles and practice of *Descriptive Geometry.*

THEOREM XVI. (111.)

If any prism be cut by a plane parallel to its base, the section will be equal and similar to the base.

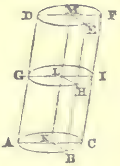


Let AG be any prism, and IL a plane parallel to the base, AC; then will the plane IL be equal and similar to the base AC, or the two planes will have all their sides and all their angles equal.

For, the two planes AC, IL, being parallel by hypothesis: and two parallel planes, cut by a third plane, having parallel sections (*th.* 108); therefore IK is parallel to AB, and KL to BC, and LM to CD, MP to DN, and IP to AN. But AI and BK are parallels (*def.* 95); consequently AK is a parallelogram; and the opposite sides BA, IK, are equal (*th.* 22). In like manner, it is shown that KL = BC, and LM = CD, MP = DN, and IP = AN, or the two planes AC, IL, are mutually equilateral. But these two planes having their corresponding sides parallel, have the angles contained by them also equal (*th.* 107), namely, the angle A = the angle I, the angle B = the angle K, the angle C = the angle L, and the angle D = the angle M. So that the two planes AC, IL, have all their corresponding sides and angles equal, or they are equal and similar.

THEOREM XVII. (112.)

If a cylinder be cut by a plane parallel to its base, the section will be a circle, equal to the base.



Let AF be a cylinder, and GHI any section parallel to the base ABC; then will GHI be a circle, equal to ABC.

For, let the planes KE, KF, pass through the axis of the cylinder MK, and meet the section GHI in the three points H, I, L; and join the points as in the figure.

Then, since KL, CI, are parallel (*def.* 102); and the plane KI, meeting the two parallel planes ABC, GHI, makes the two sections KC, LI, parallel (*th.* 108); the figure KLIC is therefore a parallelogram, and consequently has the opposite sides LI, KC, equal, where KC is a radius of the circular base.

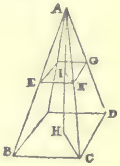
In like manner it is shown that LH is equal to the radius KB; and that any other lines, drawn from the point L to the circumference of the section GHI, are all equal to radii of the base; consequently GHI is a circle, and equal to ABC.

Scholium.

Had the base been any other curve whatever, it may be shown in the same manner, that the section parallel to the base will be a figure equal and similar to the base.

THEOREM XVIII. (117.)

In any pyramid, a section parallel to the base is similar to the base; and these two planes are to each other as the squares of their distances from the vertex.



Let ABCD be a pyramid, and EFG a section parallel to the base BCD, also AIH a line perpendicular to the two planes at H and I; then will BD, EG, be two similar planes, and the plane BD will be to the plane EG, as AH^2 to AI^2.

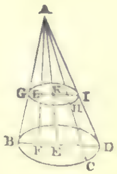
For, join CH, FI. Then, because a plane cutting two parallel planes, makes parallel sections (*th.* 108), therefore the plane ABC, meeting the two parallel planes BD, EG, makes the sections BC, EF, parallel. In like manner, the plane ACD makes the sections CD, FG, parallel. Again, because two pairs of parallel lines make equal angles (*th.* 107), the two EF, FG, which are parallel to BC, CD, make the angle EFG equal the angle BCD. And in like manner it is shown, that each angle in the plane EG is equal to each angle in the plane BD, and consequently those two planes are equiangular.

Again, the three lines AB, AC, AD, making with the parallels BC, EF, and CD, FG, equal angles (*th.* 14), and the angles at A being common, the two triangles ABC, AEF, are equiangular, as also the two triangles ACD, AFG, and have therefore their like sides proportional, namely, AC : AF : : BC : EF : : CD : FG. And in like manner it may be shown, that all the lines in the plane FG, are proportional to all the corresponding lines in the base BD. Hence these two planes, having their angles equal and their sides proportional, are similar, by *def.* 68.

But, similar planes being to each other as the squares of their like sides, the plane BD : EG :: $BC^2 : EF^2$, or : : $AC^2 : AF^2$, by what is shown above. Also, the two triangles AHC, AIF, having the angles H and I right ones (*th.* 98), and the angle A common, are equiangular, and have therefore their like sides proportional, namely, AC : AF : : AH : AI, or $AC^2 : AF^2 :: AH^2 : AI^2$. Consequently, the two planes BD, EG, which are as the former squares AC^2, AF^2, will be also as the latter squares AH^2, AI^2, that is, BD : EG : : $AH^2 : AI^2$.

THEOREM XIX. (118.)

In a cone, any section parallel to the base is a circle; and this section is to the base, as the squares of their distances from the vertex.



Let ABCD be a cone, and GHI a section parallel to the base BCD; then will GHI be a circle, and BCD, GHI, will be to each other, as the squares of their distances from the vertex.

For, draw ALF perpendicular to the two parallel planes, and let the planes ACE, ADE, pass through the axis of the cone AKE, meeting the section in the three points H, I, K.

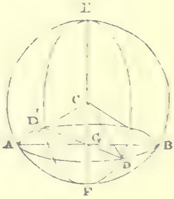
Then, since the section GHI is parallel to the base BCD, and the planes CK, DK, meet them, HK is parallel to CE, and IK to DE (*th.* 108). And because the triangles formed

by these lines are equiangular, KH : EC :: AK : AE :: KI : ED. But EC is equal to ED, being radii of the same circle; therefore KI is also equal to KH. And the same may be shown of any other lines drawn from the point K to the perimeter of the section GHI, which is therefore a circle (*def.* 44).

Again, by similar triangles, AL : AF :: AK : AE, or :: KI : ED, hence $AL^2 : AF^2 :: KI^2 : ED^2$; but $KI^2 : ED^2$ :: circle GHI : circle BCD (*cor. th.* 93); therefore $AL^2 : AF^2$ :: circle GHI : circle BCD.

THEOREM XX. (121.)

*If a sphere be cut by a plane, the section will be a circle *.*



Let the sphere AEBF be cut by the plane ADB; then will the section ADB be a circle.

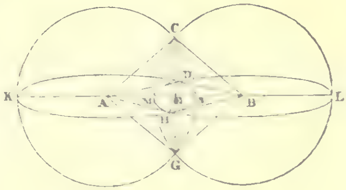
If the section pass through the centre of the sphere, then will the distance from the centre to every point in the periphery of that section be equal to the radius of the sphere, and consequently such section is a circle. Such, in truth, is the circle EAFB in the figure.

Draw the chord AB, or diameter of the section ADB; perpendicular to which, or the said section, draw the axis of the sphere ECGF, through the centre C, which will bisect the chord AB in the point G (*th.* 41). Also, join CA, CB; and draw CD, GD, to any point D in the perimeter of the section ADB.

Then, because CG is perpendicular to the plane ADB, it is perpendicular both to GA and GD (*def.* 89). So that CGA, CGD are two right-angled triangles, having the perpendicular CG common, and the two hypothenuses CA, CD, equal, being both radii of the sphere; therefore the third sides GA, GD, are also equal (*cor.* 2, *th.* 34). In like manner it is shown, that any other line, drawn from the centre G to the circumference of the section ADB, is equal to GA or GB; consequently that section is a circle.

THEOREM XXI.

If two spheres intersect one another, the common section is a circle.



Let A, B, be the centres: draw any two planes through AB cutting the spheres in the circles KCNG, MCLG, and KDNH, MHLD.

Join CG meeting AB in E; and join AC, CB, AG, GB, AD, DB, AH, HB, DE, EH.

Then, since CA = AD, and CB = BD, and AB common, the angles ACB, ADB, are equal; as are likewise the angles CAB, DAB, and the angles CBA, DBA.

* The section through the centre, having the same centre and diameter as the sphere, is called a great circle of the sphere: the other plane sections being called *less* circles of the sphere.

Again, the angles CAE, DAE, being equal, the sides CA and DA being also equal, and AE common; hence the angle DEA = CEA.

But the line joining AB bisects CG at right angles in E; hence also the angle DEA is a right angle.

In the same manner NEH is a right angle, and EH = EG. Hence the lines DE, EH, being in the same plane, and making the angles AED, BEH, right angles, they are in the same straight line at right angles to AB.

Also, since the two lines CG, DH, are perpendicular to AB, the plane in which they are is also perpendicular to AB.

Hence all the intersections, C, G, &c. are in a plane perpendicular to AB, and are, therefore, in a circle.

Again, since CE = EG, DE = CE, and EH = EG, the point E is the centre of the circle CDGH.

Theorems on the foregoing subjects for the student to demonstrate.

1. If two planes be parallel to the same plane, or to the same straight lines, they are parallel to one another.

2. If a plane and a straight line be parallel to the same plane, or to the same straight line, they are parallel to one another.

3. Two parallel straight lines make equal angles with the same or with parallel planes: and two parallel planes make equal angles with the same or with parallel straight lines.

4. When two parallel planes are cut by a third plane, or by a straight line, the exterior angle is equal to the interior opposite, the alternate angles are equal, and the two interior angles are together equal to two right angles: and show whether *generally* the converse of either of these three conditions takes place, in the case of the planes being cut by a *plane* and by a *line*, the planes would be parallel.

5. When two straight lines are not parallel, the planes which are drawn perpendicular to them will intersect one another.

6. When a straight line and a plane intersect one another, every straight line perpendicular to the plane will intersect every plane which is drawn perpendicular to the straight line.

7. If a plane bisect a dihedral angle, and from any point in it perpendiculars be drawn to the planes which contain the dihedral angle, these perpendiculars will be equal, and the plane passing through them will be perpendicular to the line in which all the planes meet.

8. If the perpendiculars from a point to two planes forming a dihedral angle be equal, the plane passing through this point and the line in which the planes containing the dihedral angle intersect, will bisect the dihedral angle.

9. If a plane bisect a line at right angles, lines drawn from any point in the plane to the extremities of the bisected line, will be equal.

10. If from the extremities of a line, two equal lines be drawn to meet in a point, then a plane drawn through this point perpendicular to the first line will bisect this line.

11. If through the middle of a given line a plane be drawn at right angles to that line, it shall pass through the vertex of every isosceles triangle having that line for a base.

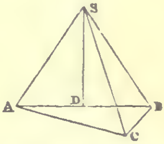
12. If on a given line as base, any number of equal triangles be constituted, having their equal sides terminated in the same extremity of the line, their

vertices will all lie in the circumference, or one cube whose plane is perpendicular to the common base *.

SECTION II.—OF SOLID ANGLES.

THEOREM XXII. (123.)

If a solid angle be contained by three plane angles, any two of them together are greater than the third; and the difference of any two of them is less than the third.



Let the solid angle at S be contained by the three plane angles ASB, BSC, CSA: then any two of them, as ASC, CSB, shall be greater than the third BSA.

In the plane ASB make the angle BSD equal to BSC, and draw any line ADB from A in the same plane, meeting SB in B. Take SC = SD, and join AC, BC.

Because the two sides BS, SD, are equal to the two BS, SC, and the angle BSD to the angle BSC, the bases BD, BC, are equal. But AC + CB are greater than BA (*th.* 10); and hence taking the equals BD, BC, from these, AC is greater than AD.

But the sides AS, SD, being equal to AS, SC, and the base AC greater than AD, the angle ASC is greater than ASD. Hence adding to the unequals ASC, ASD, the equals DSB, BSC, the angles ASC and SCB are together greater than ASB.

In the same way may the other part of the theorem be proved by means of *th.* 11.

---

* Verbal analogy, and analogy in the forms of enunciation, often leads to error. The great similarity in the *general form* of the enunciation of several theorems respecting lines in one plane, and of the lines and planes variously disposed in space, has often caused propositions respecting the latter to be tacitly assumed by the inexperienced student as truths, merely in consequence of the analogy in the form of expression, but which a little attention to the circumstances involved in the hypothesis would have shown at once to be fallacious. A few such are annexed: and the student should be required to distinctly demonstrate *why* they are false, and under what limited circumstances they are true.

1. When two planes intersect one another, straight lines which are perpendicular to them will also intersect.

2. When two planes intersect one another, any planes which are perpendicular to these will also intersect.

3. When two straight lines intersect, any straight lines perpendicular to these will also intersect.

4. When a straight line meets a plane, every straight line perpendicular to the given straight line will intersect every plane perpendicular to the given plane.

5. Two straight lines equally inclined to the same plane are parallel to one another.

6. Two planes equally inclined to the same straight line are parallel to one another.

7. Two planes equally inclined to the same plane are parallel to one another.

8. Two straight lines which make equal angles with the same straight line are parallel to one another.

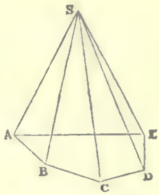
9. Two straight lines parallel to the same plane are parallel to one another.

10. Two planes which are parallel to the same straight line are parallel to one another.

11. When two straight lines are cut proportionally by three planes, these three planes are parallel.

THEOREM XXIII. (124.)

Every solid is contained by plane angles, which are together less than four right angles.



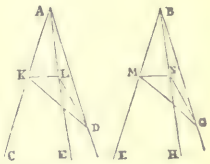
Let the solid angle at S be contained by the plane angles ASB, BSC, CSD, DSE, ESA: then the sum of these angles is always less than four right angles.

Let the planes which contain the several plane angles be cut by a plane ABCDE, these letters representing its intersections with the sides of the several plane angles, or with the edges of the solid angle.

Because the solid angle at A is contained by the three plane angles SAB, BAE, EAS, any two of which are greater than the third, the angles SAB + EAS are greater than EAB. For a similar reason, the two plane angles at B, C, D, E, which are the bases of the triangles having the common vertex S, are severally greater than the third angle at the same point, which is one of the triangles of the polygon ABCDE. Hence all the angles at the bases of the triangles SAB, SBA, &c. are together greater than all the angles EAB, &c. of the polygon ABCDE. And because all the angles at the bases of the triangles, viz. SAB, SBA, &c. together with the plane angles at S, are equal to all the angles of the polygonal base, together with four right angles, (being in each case equal to twice as many right angles as there are sides AB, BC, . . .), the angles at S are less than four right angles.

THEOREM XXIV. (125.)

If each of two solid angles be contained by three plane angles equal to one another, each to each: then the planes in which the angles are have the same inclination to one another.



Let the solid angles at A and B be contained by three plane angles CAD, DAE, EAC, and FBG, GBH, HBF, respectively; and let CAD = FBG, DAE = GBH, and EAC = HBF: then shall the dihedral angle formed by DAC and EAC be equal to the dihedral angle formed by GBF and HBF.

In the lines CA and FB take AK = BM, and from the points K and M draw the perpendiculars KD, KL, MG, MN, in the planes of the angles whose edges are AC and FB. Then the dihedral angles made by these planes are measured by the angles DKL and GMN respectively. Join LD, NG.

Then the triangles KAD, MBG, having KAD = MBG, and AKD = BMG, and the included sides equal, viz. AK = BM, therefore KD = MG, and AD = BG. In like manner, in the triangles KAL, MBN, it may be proved that KL = MN, and LA = NB.

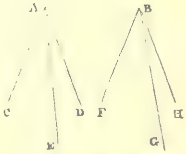
Again, the triangles LAD, NBG, having LA = NB, AD = BG, and angle LAD = angle NBG, the bases are equal, viz. LD = NG.

Lastly, in the triangles KLD, NMG, since the sides are equal, viz. KL = MN, KD = MG, and the bases also equal, viz. LD = NG; therefore the angles included, viz. LKD and NMG are also equal. But these are the measures of the inclination of the faces of the solid angle, or the measure of the dihedral

angle specified in the enunciation. In like manner may the other dihedral angles be shown to be equal.

THEOREM XXV. (126.)

If two solid angles be contained, each by three plane angles which are equal to one another, each to each, and follow each other in the same order, these solid angles are equal.

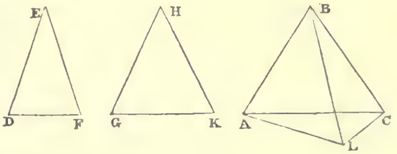


Let there be two solid angles at A and B, contained by the three plane angles CAD, DAE, EAC, taken in order, and FBH, HBG, GBF, also taken in the same order, such that CAD = FBH, DAE = HBG, and EAC = GBF. Then will the solid angle at A be equal to the solid angle at B.

Let the solid angle at A be applied to the solid angle at B, so that the plane angle CAD coincide with the plane angle FBH. Then since CA coincides with BF, and the dihedral angle made by CAD, CAE, is equal to the dihedral angle made by FBH and FBG (*th.* 125), the plane CAE will coincide with the plane FBG. Also, since the angles CAE, FBG, are equal, the line AE coincides with BG. Wherefore the plane angle EAD coincides with the plane angle GBH; and the solid angles at A and B thus coinciding, are equal to one another.

THEOREM XXVI. (127.)

If every two of three plane angles be greater than the third, and if the straight lines which contain them be all equal, a triangle may be made of the straight lines which join the extremities of those containing lines.



Let ABC, DEF, GHK, be the plane angles, any two of which are greater than the third; and take AB = BC = DE = EF = GH = HK: then the three lines AC, DF, GK, will form a triangle.

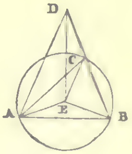
At the point B, in the straight line AB, make the angle ABL equal to GHK, and BL equal either of the equal lines AB, &c. Join AL, LC.

Then the triangles GHK, ABL, are equal, and therefore AL = GK. But the angles DEF and GHK, being greater than ABC, and GHK = ABL, therefore DEF is greater than LC, and the base DF is greater than LC. But AL and LC are together greater than AC; much more then are AL and DF (that is, GK and DF) together greater than AC. In the same way it may be shown, that AC + DF are greater than GK, and AC + GK greater than DF. Hence of the three lines AC, DF, GK, any two are greater than the third; and therefore a triangle can be constituted of them.

Cor. If from the three edges of a triangular pyramid a perpendicular be drawn to the plane of the base, it will meet the base in the centre of the circle which circumscribes the base. *See the next figure.*

THEOREM XXVII.

If about a triangle ABC a circle be described, and from its centre E a perpendicular ED be drawn perpendicular to the plane of the base, any point D in this perpendicular will be the vertex of a triangular pyramid, whose three edges are equal.



Let the three edges DA, DB, DC, of the tetrahedron DABC be all equal; and from D draw DE perpendicular to the plane ABC of the base. Then E will be the centre of a circle described about the triangular base ABC.

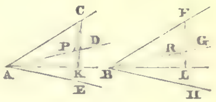
For join EA, EB, EC. Then since DE is perpendicular to ABC, the three angles DEA, DEB, DEC, are right angles.

Whence $AD^2 - DE^2 = EA^2$, $BD^2 - DE^2 = EB^2$, and $CD^2 - DE^2 = EC^2$.

But by hypothesis $AD^2 = BD^2 = CD^2$, and hence also $AE = BE = CE$; and E is the centre of the circle passing through the points A, B, C.

THEOREM XXVIII.

If two solid angles be contained each by three plane angles which are equal two and two: then in one solid angle any edge will have the same inclination to the plane of the two others, that the homologous edge of the other has to the homologous plane of the other two.



Let the solid angles A and B be contained by plane angles CAE, EAD, DAC, and FBH, HBG, GBF, which are two and two equal respectively; then the edge AC will be inclined to the plane EAD in the same angle that the homologous edge BF is to the homologous plane HBG.

Take AC equal to BF, and draw the perpendiculars CK, FL, to the planes DAE, GBH, meeting them in K, L; and from K, L, draw KP, LR, perpendicular to the homologous lines AD, BG; and join CP, FR.

Then (*th.* 102) CP is perpendicular to AD, and FR to BG. Hence CPK, FRL, are the inclinations of the planes CAD, DAE, and FBG, GBH, respectively; and these inclinations (*th.* 125) are equal to one another, and therefore the angles CPK, FRL, are equal.

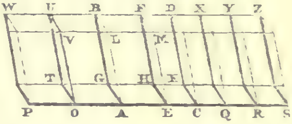
Again, since in the right-angled triangles CAP, FBR, the homologous sides AC, BF, are equal, and the angles CAD, FAG, also equal, the sides CP, FR, are also equal. Whence in the right-angled triangles CKP, FLR, the homologous sides CP, FR, are equal, and the angles CPK, FRL, also equal, the side CK is equal to the side FL.

Hence, we have in the right-angled triangles CKA, FLB, the two sides AC, CK, equal to the two BF, FL, each to each, the angle CAK is equal to FBL: and these are the inclinations of AC to DAE, and of FB to GBH. Hence these inclinations are equal.

SECTION III.—THE VOLUMES OF SOLIDS *.

THEOREM XXIX. (114.)

If a prism be cut by a plane parallel to the ends, it will be divided in the same ratio as any one of its parallel edges is divided by the same plane.



Let the plane EF cut the prism AD into two parts AF, ED, and any one of its parallel edges AC into the parts AE, EC: then

prism AF : prism ED :: AE : EC.

For produce the planes AK, AN, BN, BK; and in the produced edge AC take any number of lines AO, OP, each equal to AE, and any number CQ, QR, RS, each equal to EC, and through the several points O, P, Q, R, S, let planes OU, PW, QX, RY, SZ, be drawn parallel to AB, CD, or EF.

Then since TO is parallel to AG, and OV parallel to AL, the three plane angles TOV, VOA, AOT, are equal to the three GAL, LAE, EAG, each to each. Hence if the point O were applied to A, the line OA to the line AE, and the plane AT to the plane EG, the line OT would coincide with the line AG, and the line OV with the line AL.

For a similar reason, the lines TU, UV, UB, would coincide with GB, BL, BF; the lines BL, BG, with FM, FH, and TG, GA, with GH, HE. Hence, all the edges of the prism OB would coincide with all the homologous edges of the prism AF; and therefore all the faces of the one coincide with all the faces of the other, and the two prisms are equal.

In the same manner, it may be proved that OW is equal to AF; and similarly, that each of the prisms CX, QY, RZ, is equal to ED. Wherefore, whatever multiple the line PE is of the line AE, the same multiple is the prism EW of the prism EB; and whatever multiple the line ES is of the line CE, the same multiple is the prism EZ of the prism ED.

Again, if the line PE be greater than ES, the prism EW is greater than the prism EZ; if equal, equal; if less, less: and these are any equimultiples of the first and third AE, AF, and any equimultiples of the second and fourth EC, ED, each of each. Hence

prism AF : prism ED :: AE : EC.

---

* The following propositions are mainly proved by means of the method of Cavallerius, which he calls the *arithmetic of infinites.* It consists in assuming that all plane figures are made up of an infinite number of lines parallel to each other, and connected by a certain law according to the particular figure under consideration; and similarly, solids are assumed to be composed of an infinite number of indefinitely thin laminæ, or mere plane figures, all parallel to each other, and connected by the properties of parallel sections of the solid. This method is not rigorously conclusive: but the great length of the proofs by the *method of exhaustions*, renders them unsuitable to the space allowed to the subject in this course; and, as they all admit of a ready and brief investigation by means of the *integral calculus*, it does not appear to be essential to give a more rigid system of investigation in this place.

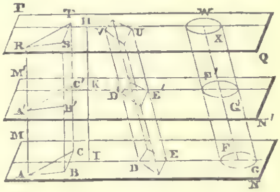
Scholium.

This demonstration being general for all prisms, the particular case of the rectangular parallopiped is included in it. The property as often used in reference to this particular figure is:—*rectangular parallelopipeds of the same altitude are to one another as their bases.*

THEOREM XXX. (113.)

Prisms and cylinders of the same altitude, or between the same parallel planes, are equal to one another.

Let ABT, DEV, FGXW, be prisms and a cylinder on equal bases ABC, DE, FG, respectively, and between the same parallel planes MN, PQ (or of the same altitude HI); they shall be equal to each other.



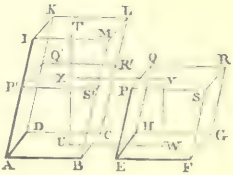
Parallel to the plane MN draw any plane M′N′ cutting the prisms and cylinder in the sections A′B′C′, D′E′, F′G′. Then these are respectively equal to the sections ABC, DE, FG; and as these latter are equal by hypothesis, the former are also equal. But the prisms and cylinder being respectively made up of these equal laminæ, and equally numerous, they must also be equal.

Cor. Every prism and cylinder is equal to a rectangular parallelopiped, of equal base and altitude with it.

THEOREM XXXI. (115.)

Prisms of equal bases are to one another as their altitudes.

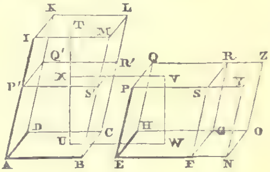
Let AL, ER, be two prisms on equal bases AC, EG; they shall be to one another as their altitudes TU, VW.



For, if the bases be not similar as well as equal, make the base EG similar to AC, and the prism ER upon it equiangular to the prism AL. Make AP′ equal to EP, and draw the plane P′R′ parallel to AC. Then the prism AR′ is equal to ER; and since the prism AL is cut by the plane P′R′ parallel to AC into parts proportional to AP′, P′I, the whole prism AL : prism AR′ :: AI : AP′ :: UT : UX :: UT : VW :: altitude of prism AL : altitude of prism ER. That is, the prisms upon the equal bases AC, EG, are to one another as their altitudes.

THEOREM XXXII. (115. *Cor.* 2.)

Prisms, neither whose bases nor altitudes are equal, are to one another in the ratio compounded of the ratio of their bases and the ratio of their altitudes.



LET AL, EZ, be two prisms, neither of whose bases AC, EO, nor altitudes TU, VW, are equal; they will be to one another in the ratio compounded of the ratio of AC to EO, and the ratio of TU to VW.

For take AP′ equal to EP, and draw the plane P′R′ parallel to AC; and take EF equal to AB, and draw the plane FR parallel to EQ. Then the prism AR′ will be equal to the prism ER′. Hence

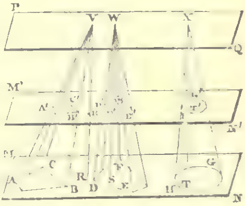
$$\frac{\text{prism ER}}{\text{prism EZ}} = \frac{\text{base EG}}{\text{base EO}}, \text{ and } \frac{\text{prism AL}}{\text{prism AR}'} = \frac{\text{UX}}{\text{TU}}, \ (th.\ 114,\ 115;)$$

whence, compounding these ratios, recollecting the equality of the prisms ER, AR′, and of the altitudes UX, VW, we have

$$\frac{\text{prism AL}}{\text{prism EZ}} = \frac{\text{base EG}}{\text{base EO}} \cdot \frac{\text{UX}}{\text{TU}};$$ and hence the theorem itself.

THEOREM XXXIII. (119.)

All pyramids and cones upon equal bases, and between parallel planes, or having equal altitudes, are equal.



LET the pyramids VABC, WDEF, and cone XGH, have equal bases ABC, DEF, GH, and have equal altitudes VR, WS, XT, or lie between the same parallel planes PQ, MN; they will be equal to each other.

For, draw any plane M′N′ parallel to the plane in which their bases are situated, cutting them in the sections A′B′C′, D′E′F′, and G′H′, respectively, and the perpendiculars VR, WS, XT, from their vertices to their bases in R′, S′, T′.

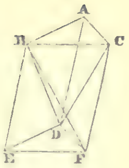
Then (*th.* 109) the perpendiculars are divided proportionally in R′, S′, T′; and (*th.* 117) the sections have to the bases the duplicate ratio of the perpendiculars VR′, VR; and, therefore, since the bases are equal, the sections are equal.

Now this is true for all the sections that can be made parallel to the base, and hence for the sums of all such sections: but these taken together make up the entire solids; and hence the solids themselves are equal.

THEOREM XXXIV. (120.)

Every pyramid is a third part of a prism of equal base, and lying between the same parallel planes.

LET, first, the prism and pyramid have triangular bases; viz. the prism whose ends are BAC, EDF, and the pyramid whose base is DEF and vertex B: then the pyramid will be one-third part of the prism.



In the planes of the faces of the prism draw CD, DB, BF; and the planes BDF, BCD, divide the whole prism into three pyramids BDEF, DABC, and DBCF.

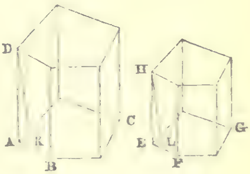
Since the opposite ends of the prism are equal to each other, the pyramid whose base is ABC and vertex D, is equal to the pyramid whose base is DEF and vertex B (*th.* 119), being pyramids of equal base and altitude. But the latter pyramid, whose base is DEF and vertex B, is the same solid as the pyramid whose base is BEF and vertex D, and this is equal to the third pyramid whose base is BCF and vertex D, being pyramids of the same altitude and equal bases BEF, BCF. Consequently all the three pyramids, which compose the prism, are equal to each other, and each pyramid is the third part of the prism, or the prism is triple of the pyramid.

In the second place, if the base have four or more sides, it can be divided into triangles, and the prism and pyramid on that base into corresponding triangular prisms and pyramids; it will follow, since each partial pyramid is one third part of the corresponding partial prism, that the entire partial pyramids will be one third part of the entire partial prisms. Hence, generally, every pyramid is one third part of the prism, having the same base, and lying between the same parallel planes.

Cor. Any cone is the third part of a cylinder, or of a prism, of equal base and altitude; since it has been proved that a cylinder is equal to a prism, and a cone equal to a pyramid, of equal base and altitude.

THEOREM XXXV. (116.)

Similar solids are to one another in the triplicate ratio of their homologous edges.

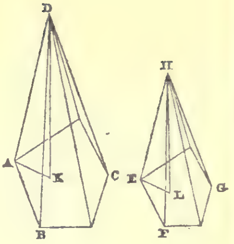


First. Let DABC, HEFG, be two similar prisms, the corresponding letters being at homologous points; they shall be to one another in the triplicate ratio of the homologous sides.

From D, H, homologous points of the ends opposite to ABC and EFG, draw the perpendiculars DK, HL, to those bases, and join KA, EL.

Then, since DABC, HEFG, are similar prisms, the angles at A are equal to those at E, and the angle DAK is equal to the angle HEL, (*th.* XXVIII.) and the angles DKA, HLE, are right angles, since DK, HL, are perpendicular to the bases. Hence the triangles DKA, HLE, are similar; and DA : HE :: DK : HL. Again, the two prisms are to each other in the ratio compounded of the ratio of their bases and the ratio of their altitudes; and hence, in the ratio compounded of the ratio of their bases and homologous sides AD, HE.

Now the bases themselves, being by hypothesis similar figures, are to one another in the duplicate ratio of their homologous sides AB, EF; and the sides AD, HE, are in the ratio of the sides AB, EF. The prisms, therefore, are in the ratio compounded of the ratio of AB to EF, and the duplicate ratio of AB to EF; that is, in the triplicate ratio of AB to EF.



Second. Let ABCDK, EFGHL, be two similar prisms: they shall be to one another in the triplicate ratio of their homologous sides AD, EH.

Draw the perpendiculars DK, HL, from the vertices to the bases, and join AK, EL. Then it may be proved, as in the last case, that AD : EH :: DK : HL, and the remaining part of the demonstration will be exactly similar.

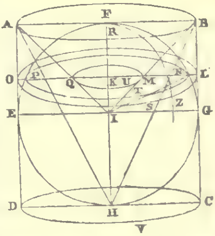
Lastly. As any two similar polyhedrons are divisable into the same number of similar triangular pyramids, these partial pyramids will have the same ratios as the entire polyhedrons: but the partial pyramids have the triplicate ratio of any one of the homologous sides; hence the entire polyhedrons are in the same ratio. Like reasoning applies also to similar cones and similar cylinders.

Cor. 1. Since cubes are included in the demonstration for the prism, the cubes described on two homologous edges of the polyhedrons will be to each other in the triplicate ratio of those edges. Hence any two similar polyhedrons have the same ratio as the cubes of the homologous sides.

Cor. 2. Similar cones and similar cylinders are also to each other as the cubes of the diameters of their bases. For if in and about the two similar cylinders, similar prisms upon regular polygonal bases be described, of 4, 8, 16,.... 2^m sides successively, the circumscribed prisms will be diminished in magnitude continually as m becomes greater and greater; whilst under the same circumstances, the inscribed prisms will be increased in magnitude. There is, also, no conceivable limit to the diminution of the magnitude of the circumscribed prism, nor to the increase of the inscribed prism, besides the cylinders themselves. The cylinder, therefore, is the limit towards which the inscribed and circumscribed prisms continually tend, and ultimately to be equal to each of them. Whatever, therefore, is proved respecting similar prisms is true, whatever be the number of their faces, and therefore is ultimately true, when they, by their number becoming infinite, resolve themselves virtually (so far as magnitude is concerned) into cylinders. The same reasoning is applicable to cones.

THEOREM XXXVI. (122.)

Every sphere is two thirds of its circumscribing cylinder.



LET CDR be a cylinder circumscribing the sphere EFGH; the sphere shall be two-thirds of the cylinder.

Let FH be the axis of the cylinder, which will also be a diameter of the sphere, since the sphere is inscribed in the cylinder; and let the centre of the sphere I be made the vertex of a cone whose base is ARB. Through the axis FH let any plane pass cutting the cylinder in the square AC, the sphere in the circle FGH, and the cone in the triangle AHB. Also draw a plane parallel to the base of the cylinder, cutting the cylinder in the circle OLS, the sphere in the circle PNT, and the cone in the circle QMU; and join IN.

Now the triangles IFB, IKM, being similar, and IF equal to FB, the side IK is equal to KM; and the triangles IKM, IKN, IKL, are right angle, and KL equal to IG equal to IN, we have

$$IK^2 = KM^2, \text{ and } KL^2 = IN^2 = IK^2 + KN^2 = KN^2 + KM^2.$$

Let now the circle whose radius is r be denoted by $r^2\pi$ (*th.* 92): then, multiplying the terms of the last equation by π, we have $\pi.KL^2 = \pi.KN^2 + \pi.KM^2$, or the circle OTL equal to the two circles PTN and QUM. Hence the corresponding sections of the sphere and cone will be equal to the corresponding section of the cylinder: and as this is the case in all the parallel sections, it is true of the sums of all the corresponding sections, that is, of the figures themselves; and the cylinder AG is equal to the hemisphere EFG and cone AIB together.

Again, the cone HABR is double the cone IABR, (*th.* 126, 129, *Schol.*,) the sphere EFGHZ is double the hemisphere EFGZ, and the cylinder AC is double the cylinder AG. Hence, the cylinder AG is equal to the sphere EFGHZ and the cone AHB together: but the cone AHB is one third of the cylinder AC (*th.* 129), and hence the sphere is two thirds of the same cylinder.

Cor. 1. A cone, hemisphere, and cylinder of the same base and altitude, are to each other as the numbers 1, 2, 3.

Cor. 2. All spheres are to each other as the cubes of their diameters; all these being like parts of their circumscribing cylinders.

Cor. 3. From the foregoing demonstration it also appears that the spherical zone or frustrum EGNP is equal to the difference between the cylinder EGLO and the cone IMQ, all of the same common height IK. And that the spherical segment PFN is equal to the difference between the cylinder ABLO and the conic frustrum AQMB, all of the same common altitude FK.

Theorems for exercise in demonstration.

1. If two great circles of the sphere intersect one another, and tangents be drawn to them at the point of intersection, these tangents will contain an angle equal to the dihedral angle of the planes of the great circles.

2. The square of the diagonal of any rectangular parallelopipedon is equal to the sum of the squares of the three edges.

3. If a plane be drawn to touch a right cone in one of its edges, and through the axis and this edge another plane be drawn: this plane will be at right angles to the plane which touches the cone.

4. Planes bisecting the dihedral angles of a tetrahedron meet in one point; and that point is the centre of a sphere inscribed in the tetrahedron.

5. If each two of four given spheres be enveloped by tangent-cones, the vertices of the six cones thus formed will lie in one plane.

6. If a cone have its base coincident with a circular section of a sphere, it will again cut the sphere in another circular section.

7. If the three edges about any solid angle of a tetrahedron be equal to one another, and a perpendicular be drawn to the plane of the opposite face, it will meet that plane in the centre of the circle which circumscribes that face.

8. Through the centre of a sphere draw three lines, each at right angles to the other two: then the six points of intersection will be the angles of a regular octahedron; and the lines joining each of them to its adjacent points will be the edges, and the three diameters of the sphere its diagonals.

9. If a sphere be inscribed in a right cone, its centre is in the axis of the cone, and the surface of the sphere touches the centre of the base of the cone.

10. Lines drawn on the face of any tetrahedron from the angles to the middles of the opposite edges, all meet in one point: and if lines be drawn from the four points thus determined to the opposite solid angles, these four lines intersect in the same point, and divide one another in the same ratio.

11. If three straight lines intersect at any point within a sphere, each at right

angles to the plane of the other two: then the sum of the squares of their six segments is equal to the square of the diameter of the sphere, together with twice the rectangle of the segments of the diameter made at the point of intersection.

12. If lines be drawn joining the centres of the faces of a cube: these will be the edges and diagonals of a regular octahedron; and the square of the diagonal is double the square of the edge.

13. If the edges of a regular tetrahedron be bisected, and the four solid angles cut off by planes passing through these points, the nucleus left will be a regular octahedron.

14. Draw lines from the middle of each side of the base of a triangular pyramid to the middles of the opposite edges: the three lines thus drawn meet in one point and bisect each other.

15. The sum of the squares of the three faces of a triangular pyramid whose plane angles at the vertex are all right angles, is equal to the square of the base.

16. If four lines form a quadrilateral figure, but not in the same plane: then the lines which bisect the opposite sides, and that which bisects the diagonals, all pass through in one point; and the sum of the squares of the four sides is equal to the sum of the squares of the diagonals together with four times the square of the line joining their middle points.

Also, the sum of the squares of the four sides and of the two diagonals, is equal to four times the sum of the squares of the lines which join the middles of the opposite sides, and of the line which joins the middles of the diagonals.

17. Planes drawn perpendicularly from the middles of those six lines, viz. the four sides and two diagonals, all intersect in one point.

18. Let a right pyramid be erected on any parallelogram as its base, and be cut by a plane: then the sum of the reciprocals of the edges, reckoned from the vertex, which are opposite to one another, is to the sum of the reciprocals of the other two opposite edges, in a determinable constant ratio.

19. If a circle of the sphere be made the base of a cone, whose vertex is anywhere in the superficies of the sphere, and if a tangent plane to the sphere be drawn at the vertex of the cone: then any plane parallel to this tangent plane will cut the cone in a circular section.

20. If any three unequal lines be placed parallel to one another in space, then lines joining the extremities of these, two and two, (forming three trapezoids, or one trapezoid and the diagonals of two others,) the three points of intersection will all be in one straight line: and if four such lines be taken, the six points of intersection of the lines so drawn will lie in one plane.

21. If three spheres mutually intersect, they will do so in a straight line at right angles to the plane passing through their centres.

PRACTICAL GEOMETRY.

The preceding part of this *Course* contains the most important theorems of plane and solid geometry, demonstrated as briefly as the nature of the subject would admit of, on valid principles: this section will comprise the constructions of the most usually required problems that occur in geometrical practice. For the most part, the demonstrations are either omitted altogether, or only indicated; so that they may serve as additional exercises in demonstration. One or

two definitional remarks, however, appear to be necessary in this place, in order to enable the student to proceed systematically.

A *problem** is a proposition in which certain things (points, lines, circles, or other curves, or any combination of them) are given or exhibited; and from these some other things are required to be found, so as to fulfil certain specified conditions.

The complete statement and solution of a problem comprises the following parts:—

1. The *enunciation*, or statement in words, of the things which are given and required.

When the enunciation is given in words only, it is called the *general enunciation;* and when in reference to a particular figure, the *particular enunciation.*

2. The operations to be performed to obtain the things sought (or *quæsita*) from those given (the *data*); which is called the *construction of the problem.*

3. The *demonstration* of the construction consists in proving, that if all the operations spoken of be performed, the result will be that which constitutes the quæsitum of the problem. This assumes, as the hypothesis of a theorem, the data of the problem and all the subsequent operations; the conclusion of the theorem being the same with the quæsitum of the problem.

Certain preliminary problems, which are called *postulates*, are assumed as possible to be constructed. They merely imply the separate and independent use of the ruler and compasses: but in the following constructions, other instruments are used for the sake of facility; though, in all cases, other methods are

---

* At p. 290 another kind of proposition besides the Theorem and the Problem was spoken of—viz. a PORISM. It is not proposed, here, to treat of this class of propositions, which are in some degree intermediate between the two. It constituted a very important and difficult branch of the Greek Geometry; but for two thousand years the true view of the subject was totally lost; and it was only during the last century that it was re-discovered by Dr. Robert Simson, the editor of *Euclid*, and of some other ancient geometrical works. His researches are contained in his *Opera Reliqua*,—a book which is now extremely scarce and valuable. An excellent dissertation on the same subject was published by the late Professor Playfair, in the Transactions of the Royal Society of Edinburgh, Vol. III. to which the inquiring student is especially referred.

Playfair's definition is:—"A porism is a proposition affirming the possibility of finding such conditions [amongst the data] as shall render a certain problem indeterminate or capable of innumerable solutions."

Simson's is:—"Porisma est propositio in qua proponitur demonstrare rem aliquam, vel plures datas esse, cui, vel quibus, ut et cuilibet ex rebus innumeris non quidem datis sed quæ ad ea quæ data sunt eandem habent relationem, convenire ostendendum est affectionem quandam communem in propositione descriptam."—*Op. Rel.* p. 323.

Dugald Stewart, Professor of Moral Philosophy (and formerly of Mathematics) in the University of Edinburgh, defines it:—"A proposition affirming the possibility of finding one or more of the conditions of an indeterminate theorem."

The note given at p. 290 of this edition should be cancelled, as the *history* there referred to, though written, was not printed. That history was drawn up by the editor of this work, as an appendix to the very incomplete article which was actually printed; and included a description of every theory respecting the nature and mode of investigating porisms, yet made public. The following is the definition extracted from those unpublished papers; and it applies equally to the geometrical and algebraical mode of treating the subject.

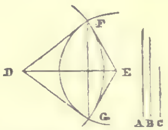
A porism is a proposition in which is affirmed the possibility of finding such relations amongst the data of a problem as shall render the general solution indeterminate; and which also requires the investigation of those relations, and the construction of the problem subject to those conditions of relation.

given dependent only upon the postulates. These postulates are the three following: —

1. That a straight line may be drawn from any one point to any other point.
2. That a terminated straight line may be produced to any length in a straight line.
3. That a circle may be described about any point as a centre, and at any distance from that centre.

PROBLEM I.

Three straight lines, A, B, C, each two of which are greater than the third, being given, to construct a triangle whose sides shall be respectively equal to them.

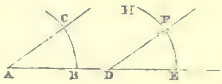


MAKE DE equal to A, and with centres D and E and radii equal to B and C respectively, describe circles intersecting in F, G. Join DF, FE, and likewise DG, GE. Then either of the triangles DFE or DGE will be that required *.

Scholium. When any two of the three lines are equal, the triangle is *isosceles*, and when all three are equal, then it is *equilateral*. These particular cases are, therefore, comprised in the general construction.

PROBLEM II.

At a given point D in a given line DE to make an angle equal to the given angle BAC.

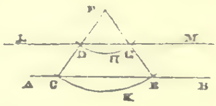


WITH any radius describe arcs from the centres A and D; the first BC meeting AB, AC, in B and C, and the second EH meeting DE in E. With centre E, and radius equal to BC, describe an arc to meet HE in F. The line DF being drawn, will make the angle EDF equal to BAC.

This is only an application of the last problem, and the equality of the angles will be evident from Geom. *th.* 8 †.

PROBLEM III.

Through a given point D to draw a line parallel to a given line AB.

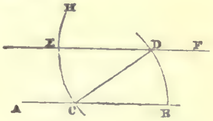


First method. Draw any line CD through C to meet AB in C, and produce it backwards, till DF is equal to DC. With centre F describe the arcs CKE and DHG, the former meeting AB in E. Join FE meeting the arc DHG in G. Then the line DG being drawn and produced if necessary, will be the parallel required.

* In the drawing, only the small portions of the circles in the estimated regions of F and G need be actually traced.

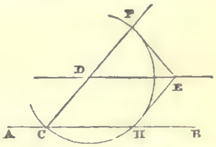
† The arc BC need not be drawn at all, it being sufficient that the points B, C, E, be *marked*, and the circles meeting at F need only be drawn in the estimated region of their intersection.

For the line DG divides the sides of the triangle CFE proportionally, (bisects them,) and hence it is parallel to the base CE. Geom. *th.* 82.



Second method. Draw any line DC, meeting AB in C, and with centres C and D, and radius CD, describe arcs DB, CH, the former meeting AB in B. Then with centre C and radius BD describe an arc to meet CH at E. The line CE being drawn will be parallel to AB.

For by the construction of the second problem, the alternate angles EDC, DCB, are equal; and hence, Geom. *th.* 13, the line EF is parallel to AB *.

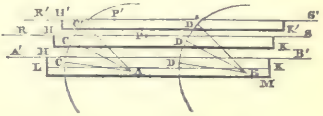


Third method. Draw any line CD through D, and with centre D describe the circle CHF cutting AB in H and CD in F, and with centres H and F, and the same radii as before, describe arcs cutting in E. Then DE will be parallel to AB.

For, since CDH is isosceles, each of the angles at C and H is half the external angle FDH. But since FD, DE, are equal to HD, DE, and FE equal to HE, the angles FDE, HDE, are equal, and hence each of them is equal to half the angle FDH. Whence FDE is equal to DCH, and therefore the lines CH, DE, are parallel †.

Scholium. The frequent occurrence of this, a paper problem, has given rise to the construction of instruments for facilitating the operation. They are, however, reducible, as to general principle of construction, to two,—the *parallel ruler* and the *parallel scales.* A brief description of them is annexed.

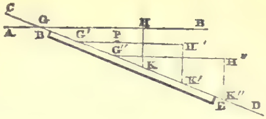
1. *The parallel ruler.* It has been proved, Geom. *th.* 23, that when a quadrilateral has each pair of opposite sides equal, they will also be parallel. Let, then, AB = BD: hence, whatever be the angle BAC, the line CD will be parallel to AB. If now we suppose the lines AB, CD, to be respectively traced on two flat rectangular rulers HK, LM, along the middles of them, and of the same length, and then cross pieces AC, BD, fastened by means of axes to them, as in the figure, then it is clear that the edges of these rulers will be parallel in all positions which the rulers so united can possibly take. Whence, if the upper edge of the upper ruler were placed along a line A′B′, and the lower held firmly whilst the upper is pushed forward to the given point P ‡; then the line RS drawn through P by the edge of the ruler in this position, will be parallel to A′B′. In like manner, if the upper edge be still moved forward to another point P′, a line R′S′ drawn along the edge of the ruler in its new position, will also be parallel to A′B′. And so on for any number of lines.



* None of the intersecting lines or arcs need be drawn, it being sufficient to mark the several points of intersection: and the same remark applies throughout this entire series of problems.

† The advantage of this method is, that it requires only a single opening of the compasses, and the entire use of it completed without intermediate operations.

‡ In the figure the ruler HK adjoining the ruler LM indicates the relative position of the two rulers when the instrument is closed to be laid aside. The descriptive part of the work refers to the upper position of HK.



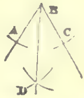
2. *The marquois, or parallel scales.* This is merely a right-angled triangle and a flat ruler, altogether unconnected with each other mechanically. They are generally of box-wood, and the scales are variously graduated at the edges. The triangle has no marks except one for the middle of the hypothenuse: and its dimensions are usually the longer and shorter sides about the right angle nine inches and three inches respectively.

In using the *marquois*, the longer leg GH of the triangle is laid along the line to which another is to be drawn parallel, and whilst held in this position, the ruler BE is placed against the hypothenuse GK. The ruler being held in this position, the triangle is slid with the right hand, up or down, as the case may require, till the edge GH passes through the given point, as in the position G′H′. Then both ruler and triangle being held firmly with the left hand in this position, the line is traced along G′H′, and this is the parallel required. If there be more parallels through more points required, again hold the ruler firmly, and again slide the triangle up or down till the edge GH passes through a second point, as in the position G″H″, and draw the second parallel as before.

When the point K″ is arrived so far in advance of the end E of the ruler as to give an unsteadiness to the instrument, slide the ruler down before moving the triangle, and then proceed with the triangle as before *.

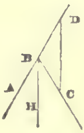
PROBLEM IV.

To bisect a given angle ABC.



First method. Take any equal distances BA, BC, in the sides containing the angle, and with A, C, as centres, and any equal radii describe circles cutting in D. Then BD being drawn will bisect the angle ABC.

For if AD, CD be drawn, the three sides of the triangle ABC are equal each to each to those of the triangle CBD. Hence, Geom. *th.* 8, the angles ABD, CBD, are equal.



Second method. Produce one of the sides AB to D, and take BD equal to BC. Then a line BH through B, parallel to CD will bisect the angle ABC.

For by construction BD is equal to BC, hence the angles BDC, BCD, are equal (*th.* 3). But by the parallels DC, BH, the angles ABH, ADC, are equal, and the angles HBC, BCD, are equal (*th.* 12): whence the angles ABH, HBC, are equal †.

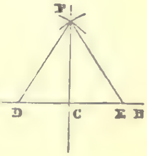
* The use of this instrument is easily acquired, though it requires a little more practice than the common parallel ruler to use it with complete facility. On the whole, especially in respect of its use in drawing perpendiculars, it is a more convenient instrument than the old parallel ruler: and it is matter of surprise that it has not obtained more attention from architectural and mechanical draughtsmen than it has yet done. Military draughtsmen, to whom time, as well as accuracy, is an important object, seldom use any other; and this is a good proof of its value as a parallel ruler.

† This latter method is the preferable one in drawings, where a parallel ruler is admitted, as the line DC need not then be drawn at all.

PROBLEM V.

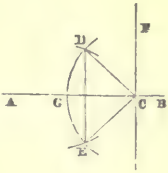
Through a given point C in a given line AB to draw a perpendicular.

First method. Take equal distances CD, CE, on each side of the given point, and with any convenient equal radii describe arcs meeting in F. Join FC; it is the perpendicular required.



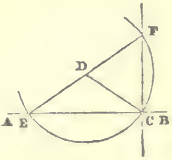
For, conceive DF, EF, joined: then the equality of the sides of the triangles DCF, ECF, give the angles at C equal (*th.* 8), and hence CF perpendicular to AB (*def.* 25) *.

Second method. With centre C and any radius describe a circle cutting the line AB in G. Set off the arcs GD, GE, equal to one another; and through C draw CF parallel to DE.



For DE is perpendicular to AB, and CF parallel to ED †.

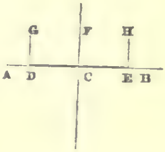
Third method. Take any point D with which as a centre, and DC as radius, describe a circle, cutting AB in E. Draw ED to meet this circle in F, and join FC. This will be the perpendicular required.



For the angle ECF being in a semicircle is a right angle (*th.* 52) ‡.

Scholium. These methods are often, in practice, superseded by the following one.

In every case of instruments there is a rectangular parallelogram of wood or ivory, DGHE called a scale, having, amongst other lines, one CF drawn from C, the middle of DE, at right angles to DE, or parallel to the ends DG, HE. The edge of the ruler is placed to coincide with the given line AB, and its middle point with the point C: then the opposite end of CF being marked on the paper, the scale is removed and CF drawn.



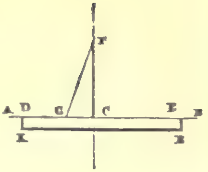
It is also often effected by means of the triangle and ruler described in the *Scholium to Prob.* 3.

---

* This method is applicable where there is sufficient space obtainable in the drawing on both sides for setting off CD and CE.

† This is adapted to the case where AB does not admit of prolongation sufficiently beyond C, as near the edge of the drawing.

‡ This method is adapted to the case where AB does not admit of sufficient prolongation to apply the first method, as when near the edge of the drawing. When C is near a corner, this is the only method applicable. It may, indeed, be generally advantageous to the student to practise each of these problems and modes of construction in all the corners and edges of the drawing, as he will then see at once the circumstances which determine the applicability of each method.

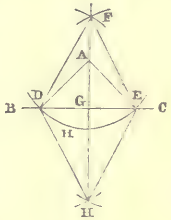


Place the ruler DEHK to coincide with the given line AB, and holding it in this position, place the right angle of the triangle at C. Hold the triangle firm, (the ruler being either retained or removed,) and draw the line FC by its edge.

If there be several perpendiculars to be drawn at equal distances, this method is convenient, if HD be a graduated scale adapted to those distances. However, in most cases, if the divisions be already made on AB, it will be more convenient to draw one perpendicular to AB, and draw parallels to it through the given point by the parallel ruler.

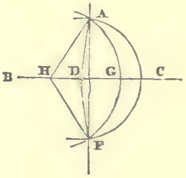
PROBLEM VI.

From a given point A without a given line BC to draw a perpendicular to BC.



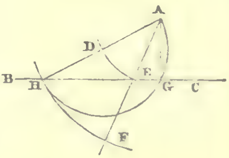
First method. With centre A, and any convenient radius AD, describe a circle cutting BC in two points D and E; and from D and E with any equal radii describe circles intersecting at F. Then FA being joined, cutting BC in G, will be the perpendicular required.

For conceiving the lines DA, AE, DF, FE, to be drawn, it may be proved, as before, that the angle DAE of the isosceles triangle is bisected by AG; and hence AG is perpendicular to DE *.



Second method. With any two centres H and D in BC describe circles passing through A and intersecting again in F. Join AF, cutting BC in G. It will be the perpendicular required.

For, as before, HD bisects the angle AHF of the isosceles triangle AHF, and is therefore perpendicular to AF; that is, AF is perpendicular to BC †.



Third method. Draw any line AE nearly perpendicular to BC by estimation, and produce it till EF = EA. With centres A and radii AF, AE, describe circles, the former meeting BC in H, and the latter to intersect the line drawn from A to H. With centre D and radius DA describe a semicircle cutting BC in G: then AG being drawn, will be the perpendicular required.

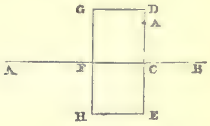
For since AE = EF by construction, and that AH = AF; and AE being

* This method can only be used when A is not near the edge of the drawing. When it is near the top or bottom of the drawing, (BC supposed horizontal,) the two triangles DAE, DFE, may be drawn on the same side of AB, but not with the sides of the one equal to those of the other.

† This method is very convenient when the point A is near the margin, but not near a corner of the drawing. The points H and D should be taken as remote as circumstances will allow, to prevent the arcs intersecting under too acute an angle.

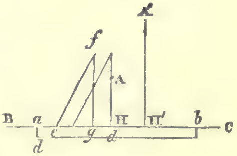
half AF, D is the middle of AH. Hence the semicircle passes through H, and AGH in the semicircle is a right angle *.

Scholium. In practice these methods are sometimes superseded by others analogous to those spoken of in the last Scholium, especially where A is not very remote from the line BC. The former method is more convenient than the latter of those processes.

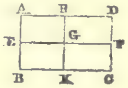


Adjust the scale so that the cross line CF shall coincide with AB, and the side DE shall pass through A; and draw AC by the edge DE.

Or, thus again, by the scale.



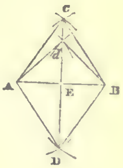
Place the ruler *abcd* along the given line, and the shorter side of the triangle *efg* against the ruler: then the longer one will be perpendicular to AB, however it be slid along the ruler. If the point A be not more distant from BC, the triangle in being slid along may be made to pass through A, and the line drawn along it will be the perpendicular required. If, on the contrary, the side *fg* will not reach A′, draw any perpendicular AH by means of it, and then through A′, a line A′H′ parallel to this perpendicular. It will be that required.



A very accurate rectangular ruler may be made by doubling a piece of stiff paper, ABCD, so as to obtain a straight line EF; and then carefully doubling again in GH, so that GF shall coincide with GE.

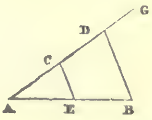
PROBLEM VII.

To bisect a given straight line AB.



First method. With centres A, B, and any convenient equal radii, describe circles intersecting in C and D. Join CD: then it will bisect AB in E.

For it may be shown, as before, that CE bisects the vertical angle CB of the isosceles triangle, and hence also it bisects the base (*th.* 3. *Cor.* 1) †.



Second method. From one extremity A draw any line AG, and take any equal distances AC, CD, in it. Join DB, and draw CE parallel to it, meeting AB in E, then E is the middle of AB.

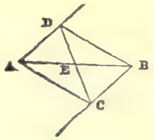
For the line CE being parallel to the base DB of the

* This method is generally given in the form of "take any line AH and bisect it in D," and so on. The present only differs as to the mode of finding the middle point D. The process is applicable, in this latter form, to the case where G falls near the corner of the drawing.

† It is not essential that the radii of the circles intersecting in C and of those in D should be equal; but when circumstances admit, it is convenient to take them so as the compasses require no alteration. They need not even be on different sides of the line AB; and hence, when AB is near an edge of the picture, it will be requisite to take these pairs of radii different, and obtain two intersections C and D on the same side of AB.

triangle BAD, it divides the sides AD, AB, proportionally. But AD is bisected in C; hence AB is bisected in E *.

Third method. From the two extremities A, B, draw parallel lines on alternate sides of AB. Take AD, BC, any equal lines, and join CD: it will bisect AB in E.

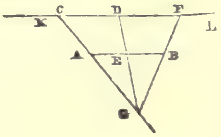


For, obviously, the triangles are equal, having the angles at A and D equal to those at B and C, each to each, and the side AD equal BC. Whence the sides AE, EB, are also equal †.

Scholium. When the common parallel ruler is used, a still better form of construction will be:—

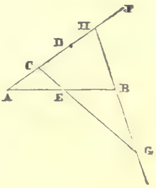
From A, B, draw any two pairs of parallels intersecting in C and D; then the diagonal CD will bisect the diagonal AB.

Fourth method. Draw any line KL parallel to AB, and taking any point D in it, set off CD, DF, on each side of any equal lengths. Join CA, FB, meeting in G or CB, FA, meeting in G′: draw the line DG or DG′, cutting AB in E. Then E is the middle of AB.



For by similar triangles, AE : EB :: CD : DF, and CD = DF; hence AE = EB. In the same way for the point G′ ‡.

Fifth method. Draw any line AF, and in it take any three equal distances AC, CD, DH; join HB, and produce it till BG = BH; join CG, cutting AB in E. This is the middle of AB.



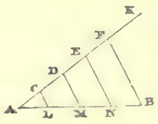
For, (*th.* 95,) AE : EB :: GH . AC : BG . CH
:: 2BG . AC : 2AC . BG.

Hence, since the last two terms are equal, the two former are so too: that is, AE = EB, and AB is bisected in E.

PROBLEM VIII.

To divide a given line AB into any given number of equal parts.

First method. Draw any straight line AK making an angle with AB, and in it set off AC, CD, DE.... all equal to one another (and as nearly as possible of the estimated length of one of the required parts of AB as can be estimated, will be convenient): join FB, and draw EN, DM, CL, ... parallel to FB. The sections of these lines with AB will be the points sought.



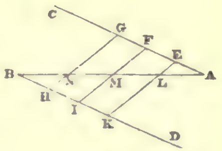
* This method is convenient when a parallel ruler is used, and AB is near and parallel to a margin of the drawing.

† This method is also convenient where AB is not nearly parallel and adjacent to a margin, if at the same time a parallel ruler be used.

‡ This method is very convenient when there is a line KL parallel to AB already in the drawing.

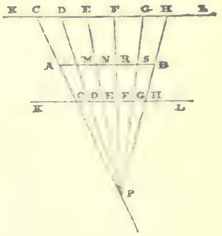
For by parallels, AC : CD : DE .... :: AL : LM : MN .... and as the former are all equal, the latter will be so too *.

Second method. Take any two parallel lines AC, BD; and in them respectively set off equal parts AE, EF, FG .... and BH, HI, IK .....; the number of parts in each being one less than the number of parts into which AB is to be divided. Draw EK, FI, GH ..., (the point E being the nearest to A, and K the most remote from B,) meeting AB in L, M, N .... Then these will be the points of division sought.



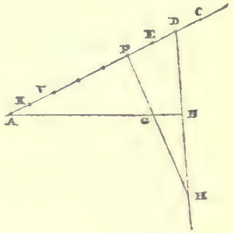
For by the parallels AC, BD, and the construction, EK, FI, GH ... are all parallel, and hence the line AB cutting them is divided proportionally †.

Third method. Draw any line KL parallel to AB, and in it take CD, DE, EF, FG, GH ... all equal to one another, and the same number as there are to be of parts formed of AB. Join AC, HB, the extremes meeting in P; and draw PD, PE, PF, PG .... meeting AB in M, N, R, S.... Then these will be the points of division sought.



For by the parallels, CD : DE : EF .... :: AM : MN : NR .... and the former being all equal, the latter are so too ‡.

Fourth method. Draw any line AC making an angle (a small one better than a large one) with AB. Set off equal distances AX, XV .... FE, ED, such that AE is as many times AX as AB is of the part required, and let D and F be the points of division one on each side of E adjacent to it. Join DB, and produce it till BH = BD; join FH, cutting AB in G: then BG is the part of AB required.



For, (*th.* 95,) AF . DH . BG = FD . HB . GA. Hence, if we denote by m the number of parts into which AB is to be divided, we have AE = m . AX, AF = $(m - 1)$ AX, FD = 2AX, and DH = 2DB. Whence $(m - 1)$ AX . 2BH . BG = 2AX . BH . AG or

AG = $(m - 1)$ GB; and therefore AG + GB = mGB: that is, AB = mGB, or GB is the mth part of AB §.

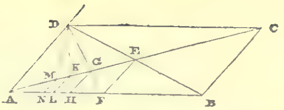
---

* This is a convenient method in practice if the parallel ruler or parallel scales are used. It, however, can also be used when AB is near and nearly parallel to the margin of the drawing.

† This is a convenient process when the compasses and ruler are alone used. In that case, the lines AC, BD, should be the sides of the equilateral triangles described on AB, one on each side. It requires, however, that AB should not be near and nearly parallel to the margin of the picture.

‡ When the line KL can be taken sufficiently long, it will be desirable to take it at least about twice the length of AB, as otherwise the point P will fall very remote from AB. When this cannot be done, and the line will allow of sufficiently distinct divisions, take it about or less than half the length of AB, as in the lower position of the figure, for the same reason. When it is possible to take it at some distance from AB, set off the divisions nearly equal to the estimated divisions of AB, and draw AH, CB, crossing *between* AB and KL, as in the corresponding construction of the last problem.

§ This process is due to M. Chenou, Professor of Mathematics in the Royal College of Douay,



Fifth method. On AB describe any parallelogram ABCD, and draw the diagonals AC, BD, intersecting in E; draw EF parallel to AD, meeting AB in F, and join DF, meeting AC in G; draw GH parallel to AD, meeting AB in H, and join DH meeting AC in K. Continue this process as far as may be necessary for the purpose. Then AF is the half of AB, AH is the third of AB, AL is the fourth of AB; and so on, as far as the operations may be carried on.

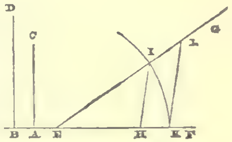
Since BD is bisected in E (*th.* 22), and EF is parallel to the base AD of the triangle ABD, we have AF = FB.

Since AD : EF :: AB : BF, we have AD = 2EF, and hence by parallels we have also AF : AH :: AE : AG :: DF : DG :: 3 : 2, or 2AF : AH :: 6 : 2, or again AB = 3AH.

In a similar manner may the truth of the succeeding divisions be proved: but it is more fully detailed in the demonstration of Prob. 7 of the next section on *Practical Geometry in the Field.*

PROBLEM IX.

To draw a third proportional to two given lines AC, BD.

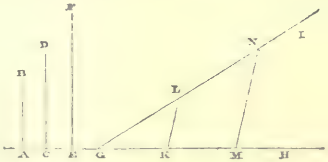


Take any two lines EF, EG, meeting in E; and in these respectively take EH = AB, and EK = EI = BD. Draw KL parallel to IH: then EL is a third proportional to AB and CD.

For by parallels EH : EI :: EK (= EI) : EL; and by construction, EH = AB, EI = EK = BD; whence AC : BD :: BD : EL *.

PROBLEM X.

To draw a fourth proportional to three given lines AB, CD, EF.



Take two lines GH, GI, meeting in G, and in GH take GK, GM, equal to the first and third lines AB, EF, and in GI take GL equal to the second line CD. Draw MN parallel to KL, meeting GI in N. Then GN is the fourth proportional to the three given lines AB, CD, EF. The proof is similar to that of the last proposition †.

---

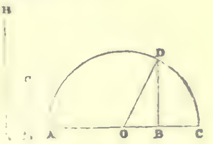
and is much used by the French draughtsmen. Its chief recommendation is, that it dispense[s] with the use of parallels; whilst its chief objection arises from the time and care required to ge[t] the compasses accurately set to the distance BG, so as to slip off the other points of division o[f] the line AB, the smallest amount of error being so multiplied in the process as to create a grea[t] difference between the several intermediate portions and the last towards A. It is well adapte[d] to the case, where only the *n*th part of a given line is required without the other points of th[e] line AB being sought; but where all are required, it is inferior in precision and simplicity, an[d] the preceding methods are preferable, especially if parallel instruments be allowed.

* The solution may be varied in several different but very obvious ways; but they are al[l] alike in principle.

† The same remark applies to this as to the last problem.

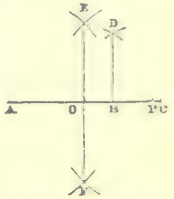
PROBLEM XI.

To find a mean proportional between two given lines AB, BC.



First method. Let them be placed in one straight line, as in the figure: on AC, as diameter, describe a semicircle, and from B draw BD perpendicular to AC, meeting the circle in D. Then BD is the mean proportional required.

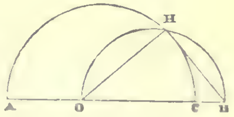
This is evident from *th.* 87 *.



Second method. Let them be placed in a straight line as before. With any equal radii from centres A and C describe arcs intersecting in E and F, and let O be the intersection of EF, AC. Make BP = BO, and from O and P as centres, with radii equal to AO describe arcs intersecting in D. Then DB will be the mean proportional between AB and BC.

For the first part of the construction finds O the middle of AB, and the second at the same time finds BD perpendicular to AC and OD = OA. The construction is therefore identical with the preceding one.

Scholium. The fifth method of bisecting the line AC also works well with the subsequent operations here employed.

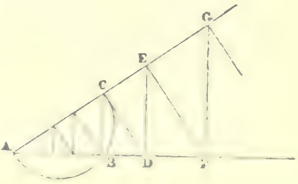


Third method. Let AB, BC, be placed as in the figure. Bisect AC in O, and describe semicircles on AC and BO, intersecting in H. Then BH is the mean proportional required.

For by the construction, the angle OHB is in a semicircle, and hence is a right angle, and AOH a radius of the circle AHC. Whence BH is a tangent to AHC, and hence AB : BH :: BH : BC †.

PROBLEM XII.

To find any number of continued proportionals to two given lines AB, AC.



On AC, the greater of the two lines, describe a semicircle, in which place the less line AB, and produce both lines indefinitely. Draw CD perpendicular to AC, meeting AB produced in D; draw DE perpendicular to AD, meeting AC produced in E; draw EF perpendicular to AE, meeting AD produced in F; and so on. Then AB, AC, AD, AE .... are a series of continued proportionals.

---

* This method *implies* more actual work than the statement of it might lead us to expect. The next method, which is in reality identical with this, comprises the entire operation.

† This method, though more laborious, is sometimes convenient: viz. when one or both the given lines are of considerable length. Its advantage in this case is that the circles do not require so much space, nor are they so difficult to describe, as in the other processes.

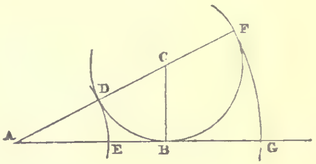
For by similar triangles ABC, ACD, ADE, ... we have

AB : AC :: AC : AD :: AD : AE :: AE : AF :: ......

This construction is adapted to the antecedent AB, being the less of the given lines: when AC the greater is the antecedent, the construction will be reversed, as is exemplified in the unlettered lines of the figure *.

PROBLEM XIII.

To divide a given line AB in extreme and mean ratio.



At one extremity B draw a perpendicular, and take in it BC $= \frac{1}{2}$ AB: with centre C and radius CB describe a circle, and join AC meeting it in D. With centre A and radius AD describe a circle cutting AB in E. Then AB is divided in E in extreme and mean ratio; or such that BA : AE :: AE : EB.

For, produce AC to meet the circle again at F. Then, since AB is at right angles to BC, therefore it is a tangent to the circle at B, and BA : AD :: AF : AB; or since AD = AE, and AB = 2BC = DF, we have AB : AE :: AB + AE : AB, and, dividendo, AB : AE :: AE : EB.

PROBLEM XIV.

(See preceding figure.)

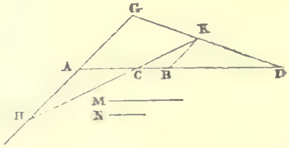
To extend a given line AB so that AB shall be the greater segment of the whole line divided in extreme and mean ratio; or, to extend a given line in extreme and mean ratio.

Construct as in the last problem, but set off AF in *prolongation* of AB to G. Then GB : BA :: BA : AG, and the line AB is produced in extreme and mean ratio.

For, since BA : AE :: AE : EB, by comp. BA + AE : BA :: AE + EB : AE; that is, GA : BA :: BA : EA.

PROBLEM XV.

To divide a given line AB harmonically in a given ratio, M : N.



Draw any two lines through A and B parallel to each other, and in them take AG, AH, each equal to M, and BK equal to N. Join GK, HK, meeting AB in D and C: then the line AB will be harmonically divided in C and D in the given ratio of M to N.

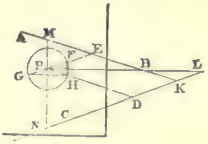
For AC : CB :: AH : BK :: AG : BK :: AD : DB.

* An instrument founded on this principle was proposed by Descartes for the construction of a series of mean proportionals, but it has never received any improvement, and is mechanically inadequate to the purpose, though theoretically correct in principle.

PROBLEM XVI.

Through a given point P to draw a line which shall make equal angles with two given lines AB, CD, whose intersection K would be beyond the limits of the picture.

Through P draw PD, PE, parallel to AB, CD, respectively, and take in them equal distances, PF, PG, PH, as in the figure, of which the dark lines represent the edges of the paper. Through P draw PL parallel to GH, and MN parallel to FH. These will be the lines sought.



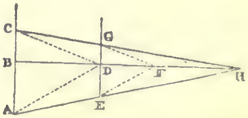
For, since PE is parallel to CD, the angle EPL is equal to PLD, and since PD is parallel to AK, the angle DPL is equal to EBP. But by problem iv. the angle EPL is equal to DPL; and hence EPB is equal to PLK.

In the same way we may prove that EMP is equal to DNP.

PROBLEM XVII.

To draw through a given point C a straight line which shall tend to the intersection of two given lines BD, AE, but whose point of intersection falls beyond the limits of the drawing.

First method. Through C draw any line whatever meeting the given lines BD, AE, in B and A, and any other line EG parallel to it meeting AE in E. Draw any two parallels AD, EF, meeting BF in D and F. Then if FG be drawn parallel to CD meeting EG in G, and CG be drawn, it will tend to the intersection of BD, AE.

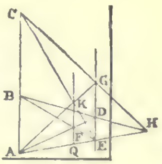


For, by parallels, DE : BA : : DF : DB : : DG : BC, and hence AB : BC : : ED : DG, and the lines AE, BD, CG, will, if produced, meet in the same point H *.

Scholium. The usual process for constructing this problem, hitherto adopted in England, is as follows :—

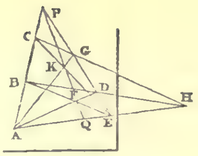
Draw any line AC through the given point C and any other line EG parallel to it. Find DG a fourth proportional to AB, BC, ED. The line CG being drawn is that sought. It is the method of finding DG adopted above, that constitutes the improvement of the process.

Second method. Draw any two parallels CA, GE, cutting the given lines as before. Join AD, BE, meeting in F, and through F draw QF parallel to AB or DE; and join CE meeting it in K, and AK meeting ED in G. Then CG will tend to the intersection of BD, AE.



For, AB : BC :: QF : FK :: ED : DG, and hence, as before, the three lines AE, BD, CG, tend to one point, H.

* This solution was first given, so far as he is aware, by the present editor of the Course, in the Monthly Magazine for August 1825. It is here divested of the technicalities of perspective, the original one having been published as the solution of a problem of frequent occurrence in drawings, and therefore given in a form exclusively adapted to that particular art. That is, however, almost the only case in which it particularly occurs in practice. It may be remarked, that the point D, to which the arbitrary line from A is drawn, may be taken *anywhere* in BH; and it is only taken in EG for the convenience of having as few marks as possible in the line BH.

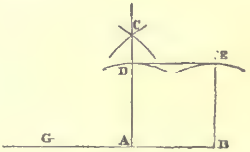


Third method. Through C draw any line AP, and from any point P in it draw any other line PE, and let the intersections be as in the figure. Join AD, BE meeting in F, and draw PFQ; and through the intersection K of CE and PQ draw AK meeting PE in G. Then CG being joined, will tend to the point H, in which AE, BD, intersect.

The truth of this construction follows from the demonstration of *th.* 97, p. 339 *.

PROBLEM XIX.

On a given line AB to describe a square.

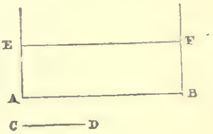


First method. At A draw AC perpendicular to AB, and take AD equal to AB. Draw DE, BE, parallel to AB, AD: then ABED is the square required (*def.* 39) †.

Second method. Produce AB till AG is equal to AB. With any radius greater than AG or AB describe arcs intersecting in C; join AC, and take AD equal to AB. With centres B and D describe arcs intersecting in E, and join DE, BE; then ADEB is the square required ‡.

PROBLEM XX.

To make a rectangle whose length and breadth shall be given lines AB, CD.



At A make AE perpendicular to AB and equal to CD, and draw the parallels as in the last problem.

Or a course of constructions analogous to the second construction of the last proposition may be employed.

---

* These methods all effect the purpose very completely, and without any bye-work beyond what appears in the figures. The *latter* methods are much used by the continental draughtsmen; and the last especially, being independent of parallels, is much valued by them. As, however, these problems are never performed but under advantageous circumstances for drawing, it seems to partake of affectation to reject the use of instruments so simple as the parallel ruler. Besides, in comparison of the work which *must be done* in executing the process by the different methods, it will appear that each of the two last has more than the first; and what is of greater consequence, it is work of that kind to injure the picture itself.

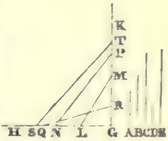
Mr Nicholson invented two instruments for the purpose of *practically* solving this problem, which he called *centrolineads*, and for which he received rewards from the Society of Arts. The parallel ruler does the work as effectively and rapidly by the first method, as the centrolinead does, and being a simple instrument, is less liable to derangement.

† This method is convenient when instruments for perpendiculars and parallels are available. The most simple is the Marquois.

‡ This method does not differ in principle from the last; the additional work prescribed being only that necessary for finding the perpendicular AC and the parallels DE, BE, when the ruler and compasses only are used.

PROBLEM XXI.

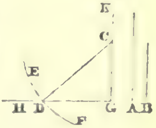
To make a square which shall be equal to any number of given squares, viz. those whose sides are A, B, C, D, E, . . .



Draw two lines HG, GK, at right angles to each other; in GH take GL equal to A, and in GK take GM equal to B, and join ML. Then in GH take GN equal to ML, and GP equal to C, and join PN. Again, in GH take GQ equal to NP, and in GK take GR equal to D, and join RQ. Proceed thus till all the given lines have been employed in the construction. The square described upon the last, as ST, is equal to the sum of the squares on A, B, C, D, E: as is evident from successive applications of *th.* 34.

PROBLEM XXII.

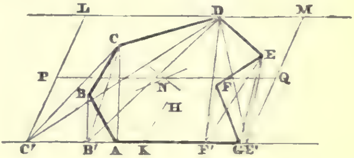
To describe a square equal to the difference of two squares, viz. of those on A and B.



Draw two lines HG, GK, at right angles to each other; in GK take GC equal to the less of the given sides, B; and with centre C and radius equal to A, describe an arc EF cutting GH in D. Then (*cor. th.* 34) the square of DG is the difference of the squares on A and B.

PROBLEM XXIII.

To describe a triangle equal to a given rectilineal figure ABCDEFG.



Having fixed upon the line, as AG, which is to be the base of the triangle required, find the highest point, D, in reference to the base (by sliding the parallel ruler is the most ready method), and draw line DA or DG, separating the polygon into two polygons of equal altitude.

Produce AG both ways. Through B draw BB′ parallel to AC, meeting AG in B′, and join B′C; through C draw CC′ parallel to B′D meeting AG in C′, and join C′D; and proceed thus till the highest point of the figure is attained. Then, commencing at the other end G of AG, draw FF′ parallel to GE meeting AG in F′, and join EF′; parallel to DF′ draw EE′ meeting AG in E′, and join DE′. Proceed thus till the highest point is attained, as at D. Then the triangle C′DE′ is that required.

For the triangle CB′A is equal to the triangle CBA (*th.* 25), and hence the quadrilateral DCB′A is equal to the quadrilateral DCBA. Again, the triangle DC′B′ is equal to the triangle DCB′, and hence the triangle DCA′ is equal to the quadrilateral DCB′A, or to the quadrilateral DCBA. In like manner, the triangle DAE′ is equal to the pentagon ADEFG: and hence, the triangle C′DE′ is equal to the given polygon.

PROBLEM XXIV.

To construct a parallelogram equal to a given polygon ABCDEFG, and which shall have an angle equal to a given angle HKG (see figure to last problem).

Construct the triangle C′DE′ as in the last problem. Through D draw LM parallel to AG, and make the angle GC′L equal to the given angle HKG, and draw E′M parallel to C′L. Draw the diagonals LE′, MC′, meeting at N, and through N draw PQ parallel to AG. Then PC′E′Q is the parallelogram required, as is obvious by *th.* 26, *cor.* 2.

PROBLEM XXV.

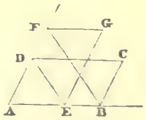
To find a square equal to a given parallelogram.

The side of a square is a mean proportional between the two sides of the rectangle (*th.* 87) ; and constructions for the mean proportional have been given in problem 11.

When the parallelogram is not a rectangle, a rectangle equal to it stands upon the same base and between the same parallels (*th.* 25), into which, therefore, the parallelogram may be converted, and the square equal to it found as above.

PROBLEM XXVI.

To describe a parallelogram one of whose angles BAD, and one of whose sides AE are given, and which shall itself be equal to a given polygon.

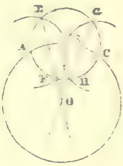


Construct a parallelogram ABCD, (*prob.* 24,) equal to the given polygon, and having an angle BAD equal to the given angle : and let AE be the given side of the parallelogram to be constructed. Join DE, and draw BF parallel to DE meeting AD in F ; and complete the figure FGEA. It is that sought.

For, (*th.* 82,) BA : AF :: AE : AD, or the sides about the angle A are reciprocally proportional ; and hence, (*th.* 81,) the parallelograms AG, AC, are equal *.

PROBLEM XXVII.

To find the centre and radius with which a given circle or segment of a circle was described.

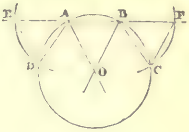


First method. Take any point B in the circumference, and with any radius BA describe a circle AEG ; and from the points A and C in which it cuts the given circumference, and the same radius as before, describe arcs cutting AEG in E and F, G and H. Then GH, LF, being drawn to meet in O, will give O the centre of the circle ; and the distance at which either of them, as GH, cuts the given circle from O, is the required radius.

* This problem most frequently occurs where the given and required figures are rectangles.

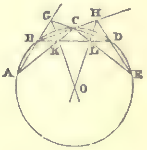
For, CB, BA, being drawn, GH, EF, bisect them at right angles, and hence pass through the centre; *dem.* of *th.* 41, and *prob.* 7 *.

Second method. Draw any line AB cutting the circle, and set off AE, AD, BF, BC, each equal to AB, the two former in the line, and the two latter in the circle. Parallel to DE and CF draw AO, BO, intersecting at O. Then O is the centre of the circle.



For this is only bisecting the equal angles at A and B, (*prob.* 4, *second method,*) and the bisecting lines pass through the centre.

Third method. Take any four equal distances AB, BC, CD, DE, in succession in the circumference of the circle: draw AB and CD meeting in G, BC and DE meeting in H, AC and BD meeting in K, and BD, CE meeting in L: then GK, HL, being drawn to meet in O, give this point as the centre of the circle.



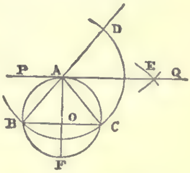
For it is easily shown that GK, HL, bisect BC, CD, at right angles, and hence they pass through the centre.

Scholium. Any of these methods enable us to find the centre and radii of the inscribed and circumscribed circles to a regular polygon. The chief value of the latter two consists in their ready adaptation to this purpose; especially the last, which requires the use of the ruler only.

PROBLEM XXVIII.

To draw a tangent to a given circle, from a given point A in the circumference.

First method. Find the centre O, and join OA; draw PQ through A perpendicular to AO: it will be the tangent sought, *th.* 46.

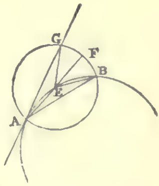


Second method. Draw any line BD through A, and with radius BA describe the circle BCD from centre A, cutting the circle in C, and BA produced in D; with centres D and C, and with equal radii describe arcs intersecting in E; then AE being drawn, it is the tangent required.

For, (*prob. iii. fourth method,*) AE is parallel to BC; and BC is perpendicular to the diameter through A, and hence it is a tangent at A.

Third method. Describe the arc BC from centre A, as in the last, and draw AQ parallel to it: then AQ is the tangent †.

Fourth method. Take any other point E in the given circumference, with which as centre, and with radius EA, describe a circle cutting AE produced in F, and the given circumference in B: make the arc FG equal to FB, and join GA, which will be the tangent required.



For, since BF is equal to FG, and E is the centre of the circle BFG, the angles BEF, FEG, are equal; and because BEA, GEA, are isosceles triangles, the angle

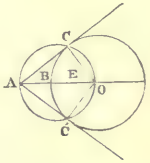
* It is only from convenience of working, and not from mathematical necessity, that AB is taken equal to BC.

† In the case where the parallel ruler is available, this is by far the neatest method.

FEB is double of EBA, and FEG is double of GAE: the angle GAE is equal to the angle EBA in the alternate segment. Whence AG touches the circle in A *.

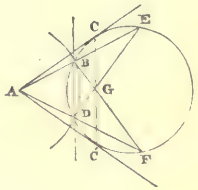
PROBLEM XXIX.

To draw a tangent to a given circle BCD from a given point A without it.

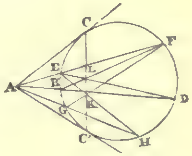


First method. Find the centre O of the given circle, join AO, and on it as diameter describe a circle ACOC′, meeting the given circle in C and C′: then if AC, AC′, be drawn, they will be tangents to the given circle.

For since ACO is a semicircle, the angle ACO is a right angle; and hence AC is perpendicular to the radius OC, and therefore a tangent. In a similar manner the other case is proved.



Second method. With A as centre, and any radius, describe an arc cutting the given circle in B and D; draw AB, AD, meeting the circle again in E and F, and draw BF, DE, meeting in G, and through G a line CC′ parallel to BD; it will cut the given circle in C, C′ the points of contact.



Third method. Through the given point A, draw any three lines cutting the circle in E, F, in B, D, and in G, H, respectively: draw the diagonals BF, ED, intersecting in L, and GF, EH, intersecting in K: then a line drawn through KL to cut the given circle in C and C′ determines the points of contact †.

---

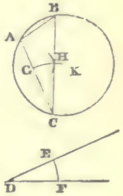
* This, or some modification of this, is the only method available when the centre is unknown or inaccessible, and the point A at or near one of the extremities of the given portion of the circumference.

† The first of these methods is the most usually employed in this country; but the others are more convenient when the centre of the given circle is not given. The last requires the use of the ruler only, and can be employed when only a part of the circle is given, provided it be not almost wholly one side of the point of contact. If, however, this last condition should not be fulfilled, we have no alternative but to find the centre, or at least in some way find other points in the circumference in the required region for completing the operation.

Respecting the two latter methods it may be generally remarked that they depend upon the principles intimated in theorems 70 and 71 of the Miscellaneous Exercises, pp. 346, 7, of this volume. They form a branch of a system of inquiries much cultivated by the continental writers under the name of *Geometry of the Rule*, and sometimes under the name of *transversals*. It forms one of the most interesting of all the departments of geometry in reference to practical utility. The same construction that is here used (the third) is also applicable to all the conic sections. For some further notice of these subjects the reader is referred to the second volume. The writings of Carnot, Garnier, Brianchon, Chasles, Servois, and De Gelder, may also be consulted with considerable advantage by the inquiring student.

PROBLEM XXX.

From a given point A in the circumference of a given circle ABC, to cut off a segment to contain a given angle EDF.

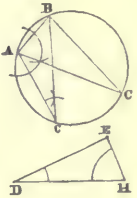


Draw any line AC from the given point to meet the circle in C: with any radius DE describe the arcs EF and GK, from centres D and C; and take GH equal to FE, and join CH cutting ABC in B: then AB being drawn will be the line, or the arc BCA will be that required.

For this construction makes the angles ACB, EDF, equal, and hence fulfils the condition.

PROBLEM XXXI.

In a given circle ABC to inscribe a triangle similar to a given triangle EDH, and having the angle which is equal to EHD at a given point A in the given circle.

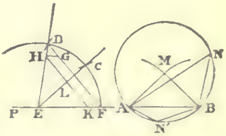


From A draw a line AB to cut off a segment containing one of the other given angles, as EDH, by the last problem; and make the angle BAC equal to the angle EHD. Then joining BC, the triangle ABC is that sought.

For it has by construction two of its angles BAC, ACB, equal to two of the angles EHD, EDH; and hence the third angles are equal and the triangles similar.

PROBLEM XXXII.

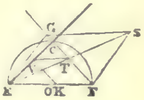
On a given line AB to describe a segment of a circle to contain a given angle CEF.



First method. If CEF be less than a right angle, make CED equal to it, and take ED equal to EF, (the particular length of ED being any whatever,) and join FD. In FD take FG equal to the given line AB, and draw GH parallel to EF to meet ED in H. Then with radii equal to EH, and centres A, B, describe arcs meeting in M; and M will be the centre of the circle sought, and the segment ANB may be described.

If the given angle PEC be greater than a right angle, construct as before with its adjacent acute angle CEF; and the segment AN′B will be that required.

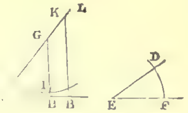
For, suppose HK drawn parallel to GF: then HK = GF = AB; and hence the sides of HEK are equal each to each to those of AMB, and consequently the angle AMB is equal HEK. But HEK is double of CEF by construction, and AMB is double of the angle ANB in the segment. The angle in the segment is, therefore, equal to the given angle CEF.



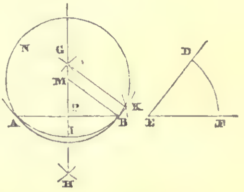
Second method. Take any point O in EF and describe a semicircle from O as centre, and with OE as radius, cutting EC in C: join FC, and produce it till FG is equal to the given line AB: through G and F draw parallels to EF, EG, meeting in S, and join SE cutting

FG in T: draw TV parallel to EF meeting EC in V, and VK parallel to GF meeting EF in K. Then EK is the radius of the segment, with which proceed as in the last construction *.

Third method. Take any line AL making any angle with AB, and at any point G in it make the angle AGH equal to the given angle DEF, and through B draw BK parallel to HG meeting AL in K. Then a circle described through the three points A, K, B will evidently be that which has the required segment cut off by AB †.

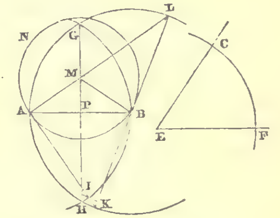


Fourth method. Draw the perpendicular GH to bisect the given line AB, and make the angle HGK equal to the given angle DEF, and through B draw BM parallel to GK, meeting GH in M. Then M is the centre from which the segment is to be described.



For join MA: then, since M bisects the base AB, of the triangle AMB, it bisects the angle AMB. Whence the angle AMB is double of BMP, that is of KGP, that is of DEF; and AMB is likewise double of ANB: whence ANB is equal to DEF ‡.

Fifth method. With radius equal to the given line AB, and with centres A, B, E, describe circles, the two former mutually intersecting in G and H, and the latter cutting CE, FE, in C and F. Make BI equal to CF, draw AI meeting the circle HAL in K, and draw the diameter KBL. Join AL, GH, intersecting at M: then M is the centre from which the segment is to be described.



For, GH is by this construction a perpendicular to the middle AB of the base of the triangle, and hence the centre of any circle through A and B is in this line. Also, the angle BAK being equal to the given one at E, and KAL, being an angle in a semicircle, is a right angle; and therefore, BAK and BAM are together equal to one right angle: but MPA being a right angle, PAM and AMP are together equal to one right angle; and hence AMP is equal to BAK or to CEF. Again, AMB is double of AMP, and therefore double of CEF; whence M is the centre of the circle required, in the same manner as in the former constructions §.

* This is but an obvious variation of the last construction. With the parallel ruler, however, it is a convenient one.

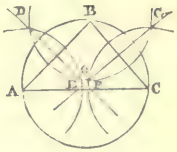
† This method anticipates the construction for a circle through three given points; but this is immaterial, as that problem is quite independent of this. It has, however, when carried out, more actual work than either of the former two; and has, moreover, the disadvantage of performing the work in the important part of the drawing, and thereby rendering the paper liable to injury.

‡ A few of the lines viz. AM, AN, NB, not requisite in the *construction*, are omitted, intentionally, in the figure.

§ This is essentially the same construction as that given by Euclid, but having all the implied operations detached, so as to suit the particular circumstances of the problem.

PROBLEM XXXIII.

Through three given points A, B, C, to describe a circle.



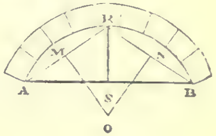
From B describe a circle with a radius greater than half the distance AB or BC, and from A and C circles with the same radii, cutting the former in D, E, and F, G: then DE, FG, being drawn to intersect in O, will give the centre required, and hence the circle may be described.

For if AB, BC, be joined, DE and FG would bisect them at right angles, and hence pass through the centre.

1. *Scholium.* The construction is the same when a circle is required to be described about a given triangle ABC.

2. *Scholium.* Sometimes a particular case of this problem arises in practice under the following form:—

Given the span AB and rise SR of a circular arch to describe it.

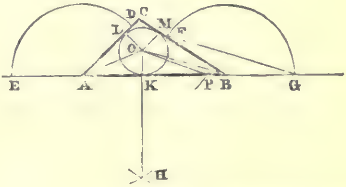


Join AR, RB, and bisect them by the perpendiculars MO, NO: then, as in the problem, O is the centre.

The joints between the stones, or *voussoirs*, are only continuations of the radii from the centre O.

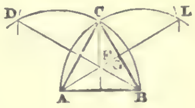
PROBLEM XXXIV.

To inscribe a circle in a given triangle ABC.



Produce AB both ways, and take AE, AD, BG, BF, all equal: parallel to ED and FG draw AO, BO, intersecting in O: then O is the centre of the inscribed circle. With centre O and radius AO describe an arc cutting AB in P, and with centres A and P and equal radii describe arcs intersecting in H. Join HO cutting AB in K: then OK is the radius of the inscribed circle, whence the circle may be drawn.

For, draw OL, OM, perpendicular to AC and BC: then, since AO bisects the angle CAB, and those at K and L are right angles, the two triangles AOK, AOL, are equiangular, and have the side AO opposite two equal angles common, they are equal in all respects, and KO is equal to OL *.



Scholium. When the given triangle is equilateral, and a parallel ruler not available, the following method is useful.

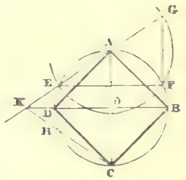
Find by intersections the vertices D and L of equilateral triangles on the sides AC, BC. Draw AL, DB, intersecting in F, and let one of them

* This method implies the use of the parallel ruler for bisecting the angle. When this is not available, bisect the angles at A and B by means of the first method of *prob. iv.* The rest of the construction is the same.

cut the opposite side in G. Then FG is the radius of the inscribed, and FA that of the circumscribing, circle.

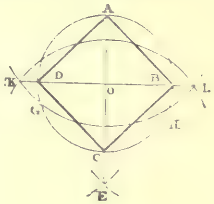
PROBLEM XXXV.

In a given circle to inscribe a square.



First method. At A with any convenient radius describe a circle EFG, cutting EA in G; and from A draw AC parallel to FG, meeting the circle again in C. From C set off CH equal to AE, and draw CH to meet EA in K; and through K a line parallel to EF, meeting the circle in D and B. Then A, B, C, D, are the angular points of the inscribed square.

For the construction (*prob.* 28) gives AC a diameter; and since the arcs EA, CH, are equal, the angles EAC, HCA, are equal; whence AKC is isosceles; and (*prob.* 6) KB is perpendicular to AC; and the two diameters AC, BD, are at right angles. Whence the four right-angled triangles AOB, BOC, COD, DOA, have all their sides about the right angles equal; and, therefore, AB, BC, CD, DA, are all equal. Also, since each of the angles ABC, BCD, CDA, DAB, is in a semicircle, it is a right angle. The figure, therefore, is a square *.



Second method. With centre A, and any radius greater than that of the circle, but less than its diameter, describe the circle KGHL, cutting the given circle in G and H: from G and H, with the same radius, describe arcs intersecting at E, and join EA, cutting the given circle in C: from C, with same radius, describe a circle meeting KGHL in K and L, and draw KL meeting the given circle in D and B. Then ABCD will be the square, as before †.

PROBLEM XXXVI.

To describe a square about a given circle.

(See the figures of preceding problem.)

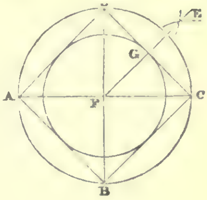
Find the diagonals of the inscribed square, viz. AC, BD: through A and C draw parallels to BD, and through D, B, draw parallels to AC. The intercepted portions of these parallels constitute the square required, as is obvious.

* This solution supposes that the centre of the circle is not given: but when the centre is already known, all we have to do is to draw the two diameters AC, BD, at right angles to one another for finding the angular points of the square. The construction above is only a combination of the process for finding the centre with that of drawing two diameters at right angles to one another. The use of the parallel ruler is, however, implied; but the next solution is by the ruler and compasses only.

† This is only another method of getting the rectangular diameters, drawn through the centre of the given circle. Other very obvious variations of these constructions might easily be given; but these, when the centre is not given, and the method intimated in the preceding note when the centre is given, are sufficient for all practical purposes. When, however, only a part of the

PROBLEM XXXVII.

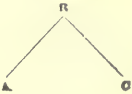
To inscribe and circumscribe circles to a given square ABCD.



Draw the diagonals AC, BD, intersecting in F, and draw FE bisecting the angle DFC (or parallel to AD): then F is the common centre of the two circles sought: FD is the radius of the circumscribed one, and FG of the inscribed one: and the circles can be drawn.

PROBLEM XXXVIII.

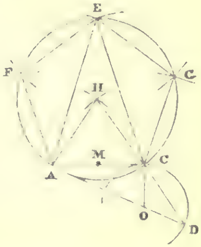
To inscribe and circumscribe circles to a given regular polygon, of which two sides AB, BC, and the angle ABC are constructed.



This problem is precisely the same as to describe a circle through three given points, that is, as *prob.* 33; where AB is equal to BC, which lessens somewhat the actual labour of construction.

PROBLEM XXXIX.

On a given straight line AC to describe a regular pentagon.



First method. At right angles to AC take CO equal to half AC, and with centre O and radius OC describe a circle: join AO, and produce it to cut the circle in D: with A and C as centres, and radius AD, describe arcs to intersect in E: and finally, with A, E, C, as centres, and AC as radius, describe arcs cutting in F and G respectively. Then A, C, G, E, F, are the angular points of the pentagon.

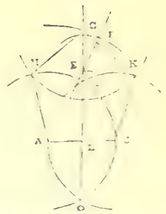
Second method. Construct as before to find D, and with centres A and C, and radius CD, describe arcs intersecting in H: then H is the centre of the circle in which the angles of the pentagon will be situated; and by setting off AF, FE, EG, all equal to AC, the angular points will be obtained.

As these two constructions are derived from the same process, it will be advisable to employ one as a check upon the accuracy of the other: and as they must flow from the same principle, their demonstrations may be conjoined in one course of reasoning, founded on *Ex.* 73 of the *Miscellaneous Exercises*, p. 347: but which, as it is presumed that the student has investigated for himself, it need not be given here.

---

circle is given, it will be requisite to find the radius of the circle, and on the diameter passing through the given point taken as a diagonal to describe the required square.

Third method. From A and C, with radii equal to AC, describe circles HCD, KAD, intersecting in B and D, and draw the indefinite line BD cutting AC in L. Make LE equal to AC, join AE and produce it till EF is equal to AL or LC. With centre A and radius AF describe an arc cutting BD in G; and from centre G with radius equal to AC describe a circle cutting HCD in H and KAD in K. Then the points A, C, K, G, H, are the angular points of the pentagon.

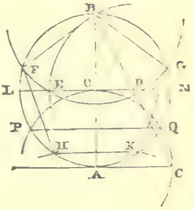


The proof of this is also dependent on the same principles as the two former constructions.

PROBLEM XL.

In a given circle ALBM to inscribe an equilateral and equiangular pentagon.

Let O be the centre, and B the position of one of the angles. Draw the diameter BOA: from centre A with radius AO describe the circle POQ, and parallel to PQ draw LM and AC through O and A respectively; then from centre D, where BC cuts LM, describe the circle BE, cutting LM in E. Lastly, with centre B describe the circle FEG cutting the given circle in F, G; and with the same radius and centres F, G describe arcs also cutting the given circle in H and K. Then B, F, H, K, G, are the angular points of the required pentagon *.



PROBLEM XLI.

About a given circle ABCDE to describe an equilateral and equiangular pentagon.

Find, by the last problem, the angular points A, B, C, D, E, of the regular inscribed pentagon. Parallel to each side, as C, D, (or to the line which joins the other extremity of the sides AB, AE, which meet in A,) draw lines as GH: similar operations being performed through all the angles, viz. HK drawn through B parallel to DE, KL through C parallel to AE, LM through D parallel to AB, and MG through E parallel to BC, the figure GHKLM formed by them will be the circumscribed pentagon required †.

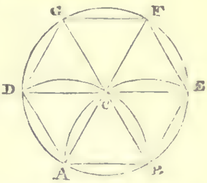
* The principle of this simple construction was given by Ptolemy in his *Almagest*; but it had never been demonstrated independently of the doctrine of proportion till about sixty years ago by Mr. Bonnycastle, formerly Professor of Mathematics in the Royal Military Academy. Its proof is best effected by the same principles as the other methods.

† The description being so simple, the student is left to construct his own figure. It may be remarked here once for all, in reference to circumscribed regular polygons, that their construction may be always effected by drawing tangents to the circle at the angular points of the inscribed polygon of the same number of sides; but the lines which form the circumscribed polygon may always be drawn by means of parallels, without reference to their tangency to the circle. When the polygon has an *odd* number of sides, draw the parallel to the most distant side; and when an even number, draw it parallel to the diagonal line, joining any two equidistant angular points of the inscribed polygon. The construction for the odd number is instanced as in the pentagon above, and for the even number in the hexagon of the next problem.

PROBLEM XLII.

To describe a regular hexagon: (1) *on the given line AB as side:* (2) *in a given circle: and* (3) *about a given circle.*

1. *When the hexagon is to be constructed on the given line AB.*



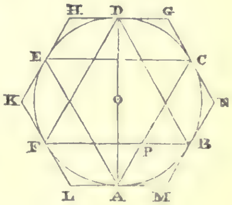
From centres A and B, with radii AB, describe segments of circles BCD, ACE, intersecting in C; and with the same radius describe from centre C another circle, ABD cutting the preceding segments in E and D respectively. Draw AC, BC, to meet the circle ABD in F and G. Then A, B, E, F, G, D, are the angular points of the hexagon.

For, the three arcs AD, AB, BE, are equal by construction, and hence the angles DCA, ACB, BCE, are equal: and the three triangles DCA, ACB, BCE, are equilateral triangles; and hence the three angles at C are equal to the three angles of one of the triangles, that is to two right angles (*th.* 17); and hence again, DC, CE, are one straight line, and each of the angles one-third of two right angles. The opposite angles to these at C are also equal (*th.* 5), and hence all the angles at C are equal. The lines AB, BE, EF, FG, GD, DA, are therefore equal, and the angles which they contain also equal; and the figure is, therefore, a regular hexagon.

2. *When the hexagon is to be inscribed within the given circle ABE.*

With the radius of the given circle, step the compasses round the circumference of the circle: it will, by the last case, mark out the angular points of the hexagon.

3. *When the hexagon is to be circumscribed about a given circle.*



Set off the six angular points in order A, B, C, D, E, F, of the inscribed hexagon by the last proposition. Parallel to FB or EC draw the lines GH, LM, through D and A: and proceed similarly for the lines HK, MN, parallel to DF or AC, and for the lines GN, KL, parallel to EA or DB. Then the six points G, H, K, L, M, N, will be the angular points of the circumscribed hexagon.

For, join DA, which by the preceding demonstrations will pass through the centre O of the circle. Also, since the arcs AF, AB, are equal, the line FB is perpendicular to AD, and hence also the lines LM and HG are perpendicular to AD. These, therefore, are tangents to the circle at A and D. In like manner, all the other lines are tangents, and the hexagon is circumscribed. Also KH being parallel to AC, and HG to FB, the angle KHG is equal to FPC, which is measured by half the sum of the arcs FEDC and BA, that is by one-third of the circumference; and in the same way each of the other angles at G, N, . . . is measured by one-third of the circumference. The angles are, therefore, all equal, and the figure is a regular hexagon.

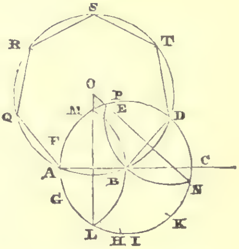
PROBLEM XLIII *.

To inscribe and circumscribe regular heptagons to a given circle.

Divide, *by trial,* the circumference of the circle into seven equal parts, and these will be the angular points of the inscribed heptagon, and the points of contact of the circumscribed one †.

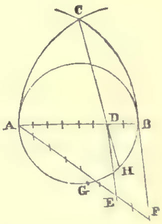
PROBLEM XLIV.

On a given line AB to construct a regular heptagon.



From centres A, B, with radius AB, describe circles intersecting in L and M, and produce AB to meet one of them (as that about B) in C: divide by trial the circle about B into seven equal parts, and let CD be one of these divisions: about D, with radius AB, describe a third circle cutting ADC in P and N: and draw LM, NP, to meet in O. About O, with radius OA or OB, describe a circle ASB. This will pass through D, and in this set off successively the points T, S, R, Q, with distances each equal to AB. The points thus determined are the angular points of the heptagon described upon the given straight line AB.

Scholium.



No geometrical method (that is, by means of the ruler and compasses) *can be given* for the construction of regular polygons, except in very limited cases. The polygons of 7, 9, 11, 13, 18, 19, 21,.... sides belong to this class; but it fortunately happens that they do not very often occur in drawing operations. The following has been often given by practical writers; but it is only an approximation, and generally a very rude one. It may serve, however, as a first step, or guess at the probable opening of the compasses to be taken, after which the distance may be easily corrected by the eye to any greater degree of accuracy.

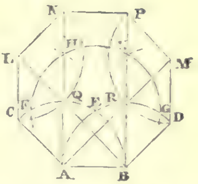
Let AB be a diameter of the circle, of which it is required to find the n^{th} part by a construction. From centres A and B, and with AB as radius, describe arcs intersecting in C: divide AB into as many parts (the figure is adapted to seven, and the first method of effecting the problem, p. 378,) as it is required by the problem to divide the circumference: through C and the second point of the division D, draw CD, to meet the circumference in H; then BH approximates to the required part of the circumference.†

* The same process applies to figures of 9, 11, 13,...... sides; and indeed, the whole series of figures not *geometrically* constructible might have been included under one enunciation; as there is, with too rare exceptions to be worthy of notice, no difference whatever in the manner of forming them. See, however, the *scholium* to the next problem.

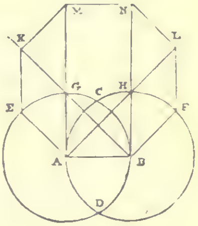
† This method was, I believe, invented by the elder Malton. and first published in his *Royal Road to Geometry.* A scrutinizing investigation of the degree of its approximation was given by Dr Henry Clarke, who proposed amendments in it; but these are also, besides being very troublesome, only one degree more close in their approximation.

PROBLEM XLV.

On a given line AB to construct a regular octagon.



First method, by the rule and compasses. From centres A and B, with radii equal to AB, describe the circles CQB, DRA, intersecting in E; from centre E, and radius AB, describe the circle FHKG, cutting the former circles in F and G; from centres F and G, and radius AB, describe arcs cutting FHKG in H and K; draw AH, BK, cutting the first two circles in Q and R; draw BQ, AR, and in them produced take QL, RM, each equal to AB; and lastly, with centres L and M, describe the circles NQC, PRD, cutting the lines AH, BK, in N and P, and the first circles in C and D. Then A, B, D, M, P, N, L, C, are the angular points of the octagon.

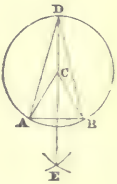


Second method, adapted to the use of the parallel ruler. Describe the circles from centres A and B, with radius AB intersecting in C and D; through A and B draw AM, BN, parallel to CD, cutting the circles in G and H; draw AH, BG, and parallel to them AE, BF, to meet the circles in E and F; through E, F, draw EK, FL, parallel to CD, AM, or BN, meeting BG in K, and AH in L; and lastly, draw KM, LN, parallel to AL, BK, meeting AG, BH, in M and N respectively, and join MN. The figure ABFLNMKE, is a regular octagon on AB.

The principles of these constructions are too obvious to need detail here.

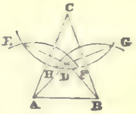
PROBLEM XLVI.

To construct an isosceles triangle, whose vertical angle shall be the half or the double of the vertical angle ACB of a given isosceles triangle ABC on the same base AB.



1. With centre C describe the circle ABD, and with the same radius, and centres A, B, arcs intersecting in E; join EC, and produce it to meet the circle in D: then AD, DB, being drawn, ADB is the triangle required.

For, the construction gives EC, perpendicular to AB, and bisecting it; hence also ADB is an isosceles triangle; and since ADB is an angle at the circumference, and ACB an angle on the same arc at the centre, ADB is the half of ACB. The conditions are, therefore, fulfilled.

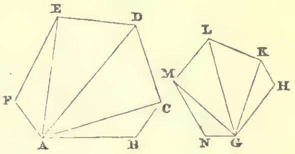


2. With centres A, B, C, describe circles with radius greater than half the side AC or BC, so that that about A intersects that about C in E and F, and that about B intersects it in G and H: then EF, GH, will intersect in the vertex D of the triangle sought.

For, D is the centre of the circle about ACB, and hence AD, DB, are equal; and the angle ADB at the centre is double ACB at the circumference.

PROBLEM XLVII.

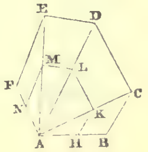
On a given line GH homologous to a given side AB of a given rectilineal figure ABCDEF, to construct a figure similar to the given one.



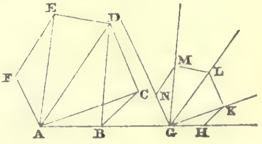
From one of the extremities of the line AB, which is homologous to GH, draw lines to all the angles of the figure; on GH construct a triangle GKH, equiangular to ACB; on GK a triangle KLG, equiangular to ACD; on GL a triangle LMG, equiangular to ADE; on MG a triangle GMN, equiangular to AEF; and continue the process as long as any triangles of the given figure remain: then GHKLMN is the figure required.

For GKH + GKL = ACB + ACD, KLG + GLM = CDA + ADE, and so on (*constr.*): hence the two figures are equiangular. Also BC : CA :: HK : KG and AC : CD :: GK : KL; and hence BC : CD :: HK : KL, or the sides about the equal angles at C and H are proportional; and the same may be shown of all the other homologous sides of the two figures. The figure GHKLMN is hence similar to the figure ABCDEF.

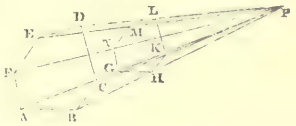
Scholia.



1. When the two sides AB, GH, are coincident, in the manner of AB, AH, the construction becomes simpler, since it consists merely in drawing HK parallel to BC, cutting AC in K, KL parallel to CD, cutting AD in L, and so on till MN is drawn parallel to EF, cutting AF in N. For, in this case the figures are composed of similar elementary triangles, and are therefore, similar to one another, as before.



2. When GH lies in the same line with AB, it is only requisite to draw GK, GL, GM, GN, parallel to AC, AD, AE, AF, respectively; then HK para'lel to BC, KL parallel to CD, and so on till we arrive at MN, parallel to EF, and join NG. The same similarity of figure as in the preceding construction obviously takes place.

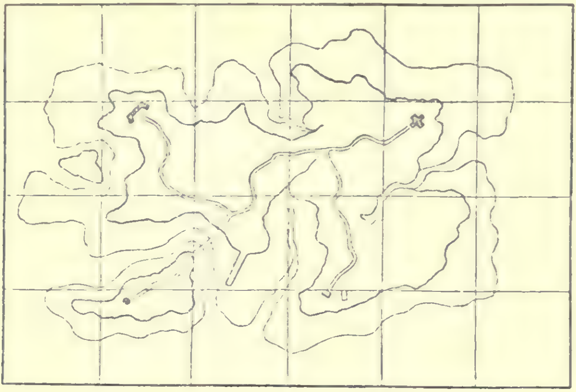


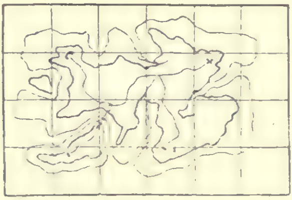
3. When the homologous sides AB, GH, are parallel, draw AG, BH (or AH, BG), meeting in P: then draw HK parallel to BC, meeting CP in K, KL parallel to CD, meeting DP in L, and so on. If the line GH be very distant from AB, or very nearly equal to it, then as the point P would fall at an inconvenient distance on the drawing, the point P may be taken *between* the lines, by joining BG and AH. The remaining part of the construction would be almost identical with that here given, but the order would be reversed.

4. Since equiangular triangles have the sides about the equal angles proportional, and triangles in general are most easily constructed by means of their sides, the *proportional compasses* are convenient in the general construction of this problem.

PROBLEM XLVIII.

To draw a complex figure similar to another figure, on the same or different scales, by means of squares





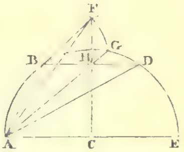
SURROUND the given figure by a square or a rectangle of convenient size, and divide it by pencil lines, intersecting perpendicularly, into squares, as small as may be deemed necessary. Generally, the more irregular the contour of the figure, or the more numerous the sinuosities or subdivisions, the more numerous the squares should be.

Then draw another square or rectangle, having its sides either equal to the former, or greater or less in the assigned proportion, and divide this figure into as many squares as there are in the original figure. Draw in every square of the new figure, right lines or curved to agree with what is contained in the corresponding square of the original figure; and this, if carefully done, will give a correct copy of the complex diagram proposed.

The *pentagraph* is also often used for the same purpose; but as there are great practical difficulties attached to it (especially its deficiency in easy, and consequently certain, motion), that it is not so valuable as its theoretical principles would lead us to anticipate. Dr. Wallace has obviated this and most other inconveniences by his *eidograph:* still its expense and liability to derangement have greatly militated, and perhaps ever will, against its introduction into general use. It is, however, but just to remark that Dr. Wallace's instrument is, in reality, but another, though much improved, form of Scheiner's pentagraph. Another form, somewhat intermediate to the common pentagraph and the most improved form of the eidograph, was exhibited to the Society of Arts in Scotland a few years ago, and which appears on the whole to answer the purpose of copying as well as the eidograph, and to be considerably less expensive.

PROBLEM XLIX.

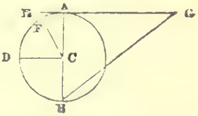
To draw a straight line nearly equal to the circumference of a given circle.



First method. Let AE be the diameter and C the centre of the circle, and let the semicircle be described upon AE. Set off AB, ED, each equal to radius AC. With centres A and E, and distances respectively equal to AD, EB, describe arcs intersecting in F. Then with centre B and radius BF describe a circle cutting the circumference (on the side D) in G. The chord AG is nearly equal to the quadrant of the circle.

For $AF = AD = 2 \sin 60^{\circ} = \sqrt{3}$, and $CF = \sqrt{AF^2 - AC^2} = \sqrt{2}$; and $FH = CF - CH = \frac{1}{2}\{2\sqrt{2} - \sqrt{3})\}$ Again, from the triangle BHF, we have $BG = BF = \sqrt{BH^2 + HF^2} = \sqrt{3 - \sqrt{6}}$; hence $\sin \frac{1}{2}BG = \frac{1}{2}\sqrt{3 - \sqrt{6}}$, $\cos \frac{1}{2} BG = \frac{1}{2}\sqrt{1 + \sqrt{6}}$, $\sin \frac{1}{2} AB = \frac{1}{2}$, $\cos \frac{1}{2} AB = \frac{1}{2}\sqrt{3}$. Therefore $2AG = 4 \sin \frac{1}{2} ABG = 4 \sin (\frac{1}{2} AB + \frac{1}{2} BG) = 4 \sin \frac{1}{2} AB \cos \frac{1}{2} BG + 4 \sin \frac{1}{2} BG \cos \frac{1}{2} AB = \sqrt{1 + \sqrt{6}} + \sqrt{9 - 3\sqrt{6}} = 3{\cdot}142399$....But to the same decimal extent the true value of the circumference is $3{\cdot}14159$; whence the degree of approximation is sufficiently close for most practical constructions.

This method, which may be performed by the aid of the compasses only, was invented by *Mascheroni*.



Second method. Let AB be the diameter of a circle, and C its centre. Draw an indefinite tangent at the point A, and a radius CD parallel to this tangent. Set off the radius DC towards A terminating in F, and draw CF to meet the tangent in E; and take upon this tangent from E on the side of A, $EG = 3CD$, the straight line BG is nearly equal to the semicircle.

For, $AE = \tan 30^{\circ} = \frac{1}{3}\sqrt{3}$: hence $AG = EG - AE = 3 - \frac{1}{3}\sqrt{3} = \frac{1}{3}(9 - \sqrt{3})$, and $BG = \sqrt{AB^2 + AG^2} = \sqrt{4 + \frac{1}{9}(9 - \sqrt{3})^2} = \frac{1}{3}\sqrt{6(20 - 3\sqrt{3})}$ $= 3{\cdot}1415334$.

This method, which is by *an anonymous German author*, gives a closer approximation than that of Mascheroni.

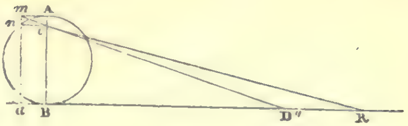
Third method. From any point A in an indefinite straight line draw a perpendicular AB equal to the given radius. Set off three times this radius from A to D, and draw BD. At the first of these divisions C of AD, draw the perpendicular CE. Set off DE in the prolongation of AD to F. Prolong AF beyond its extremities A and F by the lines AH and FG equal to radius AB. Take $FK = (\frac{1}{4} AB + \frac{1}{2}\,\frac{1}{4} AB) = \frac{3}{8} AB$; and make $AL = \frac{4}{5} AH$. Then KL is nearly equal to the circumference of the circle whose radius is AB.

$$\text{For } KL = AL + AD + FG + FD = \frac{4}{5} + 3 + \frac{3}{8} + DE = \frac{167}{40} + \frac{2}{3} BD$$

$$= \frac{167}{40} + \frac{2}{3}\sqrt{AB^2 + AD^2} = \frac{501 + 80\sqrt{10}}{120} = 3{\cdot}1415925534.$$

This method, which is remarkable for its extent of approximation, being true to six places, was invented by *M. Pioche*, an eminent statuary of Metz.

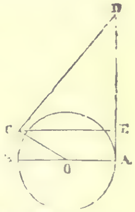
Fourth method. Take a circle whose diameter $BA = D$, and from the point B, in which the circle touches the indefinite line BR, take $Ba = \frac{1}{4}D$; and set off D three times from the point B to D″, and $aR = 15aB$. From a draw the perpendicular $am = BA = D$, and draw Rm, cutting AB in c; and finally draw $D'c$. This will be nearly equal to the circumference, whose diameter is AB.



For by similar triangles, RBc, cmn give $mn : nc :: ma : aR$; whence $mn = Ac = \frac{nc \cdot ma}{aR} = \frac{aB \cdot D}{15aB} = \frac{D}{15}$. Then $Bc = BA - Ac = D - \frac{D}{15} = \frac{14D}{15}$. Again, by the right-angled triangle $D''cB$, $D''c = D\sqrt{\left(\frac{14}{15}\right)^2 + 9} = \frac{D\sqrt{2221}}{15} = 3{\cdot}1418$.

This method is by *M. Quetelet* of Brussels, and though not extremely approximative, yet, being easy of application, is very convenient in practice.

Fifth method. Upon the circumference, whose centre is O, and radius $OB = 1$, take the arc $BC = 30°$ (which is found by the rules and compasses), draw the tangent BC, and by the other extremity A of the diameter BO, draw the indefinite tangent AD, upon which set off AD equal to three times the radius OB. Through C draw CE, parallel to BA, and join CD, which will represent very nearly the semicircle to radius OB.



For, the right-angled triangle DCE gives $DC = \sqrt{DE^2 + EC^2} = \sqrt{(DA - CB)^2 + CE^2}$; and $BC = \tan 30° = \frac{1}{\sqrt{3}}$, and $CE = 2$. Hence $DC = \sqrt{\left(3 - \frac{1}{\sqrt{3}}\right)^2 + 4} = 3\ 14153$.

This method is by *M. De Gelder* of Leyden, and is also convenient, from the simplicity of the work required.*

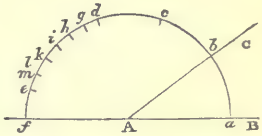
PROBLEM L.

To measure an angle by means of a pair of compasses only.

This will be easily comprehended by giving a single example. The method, in fact, consists in measuring an arc or angle proposed with a pair of compasses, without any scale whatever, except an undivided semicircle. Produce one of the sides of the angle backwards, and then with a pair of accurate compasses describe as large a semicircle as possible, from the angular point as a centre, cutting the sides of the proposed angle, and thus intercepting a part of the semicircle. This intercepted part is accurately taken between the points of the compasses, and stepped upon the arc of the semicircle, to ascertain how often it is contained in it; and the remainder, if, as usual, there be one, marked; then take this remainder in the compasses, and in like manner find how often it

* These methods of construction, almost unknown in this country, were first collected together and published by the editor in Leybourn's Mathematical Repository a few years ago.

is contained in the last of the integral parts of the first arc, with again some remainder; find in like manner how often this last remainder is contained in the former; and so on continually till the remainder becomes too small to be taken and applied as a measure. By this means we obtain a series of quotients, or fractional parts, one of another, which being properly reduced into one fraction, give the ratio of the first arc to the semicircle, or of the proposed angle to two right angles, or 180°, and consequently that angle itself nearly in degrees and minutes.



Thus, suppose the angle BAC be proposed to be measured. Produce BA out towards f; and from the centre A describe the semicircle $abcf$, in which ab is the measure of the proposed angle. Take ab in the compasses, and apply it four times on the semicircle as at b, c, d, and e; then take the remainder fe, and apply it back upon ed, which is but once, viz. at g; again take the remainder gd, and apply it five times on ge, as at h, i, k, l, and m; lastly, take the remainder me, and it is contained just two times in ml. Hence the series of quotients is 4, 1, 5, 2; consequently the fourth or last arc em is $\frac{1}{2}$ the third ml or gd, and therefore the third arc gd is $\frac{1}{5\frac{1}{2}}$, or $\frac{2}{11}$ of the second arc ef; therefore again this second arc ef is $\frac{1}{1\frac{2}{11}}$ or $\frac{11}{13}$ of the first arc ab; and consequently this first arc ab is $\frac{1}{4\frac{11}{13}}$ or $\frac{13}{63}$ of the whole semicircle af. But $\frac{13}{63}$ of 180° are $37\frac{1}{7}$ degrees, or $37^\circ\ 8'\ \frac{1}{2}$ nearly, which therefore is the measure of the angle sought. When the operation is carefully performed, this angle may be obtained within two or three minutes of the truth.

In fact, the series of fractions forms a continued fraction. Thus, in the example above, the continued fraction, and its reduction, will be as follows:—

$$\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{5\frac{1}{2}} = \frac{1}{4+1\frac{2}{11}} = \frac{1}{4\frac{11}{13}} = \frac{13}{63};$$

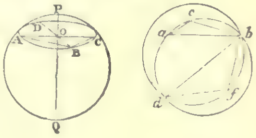
the quotients being the successive denominators, and 1 always for each numerator.

Scholium.

This method is due to *De Lagny*, and a corresponding process has been applied by *Adams* to finding what portion any given line is of another given line. The problems are so precisely alike, that any specific detail is altogether unnecessary. *Mr. Sankey* has also applied the metallic cycloid to the measurement of circular arcs: but of course its determinations are only rough approximations, though their being obtained with great facility, and near enough for most practical purposes, is a recommendation to its familiar usage.

PROBLEM LI.

To find the diameter of any solid sphere, as a ball or shell.



From any point P on the surface of the given sphere, and with any convenient radius (about the estimated chord of 60° will generally be best for accuracy) describe a circle ABC, and in it take any three points (nearly equidistant by estimation) A, B, C; and on

paper describe a triangle *abc* whose sides are equal to those of ABC: about *abc* describe a circle *abcd*, and draw its diameter *bd*: with centres *b*, *d*, and radii equal to the linear distance of P from A, B, C, describe arcs intersecting in *f*: and lastly, about *bfd* describe the circle *bfd*. Its diameter will be equal to that of the given sphere.

For let PO be drawn perpendicular to the plane of the circle ABC meeting it in O: then, being produced to Q, it will be a diameter of the sphere. If also BD be a diameter of the circle ABC, it is equal to the circle described about *abc*, since circles described about equal triangles are equal.

Again, if PD, PB, be joined, the three sides of the triangle DPB are equal to the three sides of the triangle *bfd*, each to each: and hence the circles about them are equal, and their diameters also equal. But PQ is the diameter of the sphere, and hence of the circle DPQ, the plane of which passes through PQ: and hence, again, the diameter of the circle *bfd* which is equal to that of the circle DPQ is also equal to the diameter of the given sphere.

Scholium.

This problem, which has often been proposed as a new one, owing to its not being inserted in books which are generally consulted, is yet as old, at least, as the first century B.C.: for it is found in the Spherics of Theodosius, and constructed almost exactly in the same way as above.

PRACTICAL GEOMETRY IN THE FIELD.

The absence of instruments in cases of exigence renders it of much importance to be possessed of means of determining *approximately* the positions of certain inaccessible points with regard to others that are accessible, by having recourse only to lineal measurements made in accessible places. A few of the more useful problems of this nature are annexed; but the want of sufficient space prevents the insertion of a greater variety.

All the solutions here given are effected by means of staves set up at particular and specified stations, together with the use of the chain or other lineal measure; as the determinations are practically made with greater certainty by this than by any of the means usually employed. So far, however, as other modes are concerned, there are solutions of the main part of these problems, though not specified in reference to this use, to be found amongst the problems in *Practical Geometry* already given.

PROBLEM I.

To continue a straight line on the ground, the two determining points, A, B, of which are given, there being no visual obstacles intervening.

Fix upright staves* at A and B, and walk as nearly as you can judge in the required prolongation to any point C′; which we will suppose to be a little to the right or left of the actual prolongation of AB. This will be known

---

* It is important in all these problems that the staves be placed as nearly perfectly vertical to the horizon as possible.

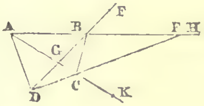
by the visual lines C′A, C′B, not coinciding. If the left side of the staves be visible, we are then on the left of the line AB, and if the right, we are on the right of AB. Move slowly *towards* the line, till the visual lines from A and B coincide, as at C; then C will be in the prolongation of AB. If we wish to place a mark as at H *between* two points A, B, in lineation, we must first lineate to C *beyond* one of them; and then by C, B, lineate H by a subsequent, and similar, process.

Scholium.

It will often in the following problems be necessary to find the intersection of two lineations; and though the process is very simple, it may be well to explain it in such a way as to be effected with the least possible trouble.

Case I. When the lineations AB, CD, do not intersect within the figure ABCD, bounded by lines joining each of the stations to the adjacent ones.

Let the point H be taken in lineation of AB *beyond* the probable point of intersection of AB, CD; then if the observer walk from H towards B in lineation, till the points C, D, also appear in lineation, to E, the point E will be that required.



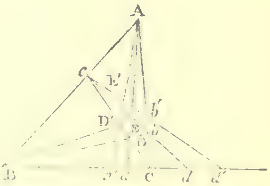
Case II. When the intersection G of AC, BD, are required. Set up two auxiliary marks, K and F, in lineation of AC and BD respectively; then by means of B and F let the observer walk in lineation of DB, till he arrives in lineation of AC, as indicated by the marks C, K. The point G, where this occurs is the intersection of AC, BD.

Many obvious facilities may be brought into the operation when there is more than one observer; but these solutions are adapted to the most unfavourable case as to assistance.

PROBLEM II.

To find the lineation of two points B, C, when obstacles intervene G, which render the points B, C, invisible.

First method. Take any point A without the line BC, from which B and C can be seen; also points *c* and *b* in them; find the intersection D of B*b* and C*c*, and then the intersection of *cb* and AD. Calculate the distance *bd* from the formula obtained below, and measure that distance in lineation of *bc*: the point *d* will be in lineation with BC.



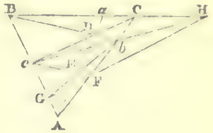
Produce AD to meet BC in *a*: then (*th.* 97) we have $c\text{E} : \text{E}b :: cd : db$, or $c\text{E} - \text{E}b : \text{E}b :: cd - ab\ (= cb) : bd$. Hence $bd = \dfrac{\text{E}b.bc}{c\text{E} - \text{E}b}$; and all the lines on the right side of the equation are measurable, and hence *bd* can be computed.

Scholium.

Should a second point be required for the purpose of continuing the lineation, we may repeat the process with another, or with the same triangle, ABC. The latter will be the better method, and is thus performed:—

Retaining one of the marks *c*, change the other from *b* to *b′*: then find the

point D′, the point E′, and finally the distance $b'd'$ from the above formula with the new values of the several parts of the line cb'.



Second method, without any calculation. Take any point A, and in the lineations AB, AC, take any points c and b: find the intersection D of the lineations Bb, Cc; take any point G in AB, and find the intersection E of Gb, AD, and the intersection F, of cE, AC: then the intersection H of bc, FG is in lineation with B and C.

For, produce AD to meet BC in a: then (*th.* 97) the lines cb and GF divide BC in prolongation in the same ratio that BC is divided in a. Hence they cut BC in the same point, and each other in lineation with B and C*.

PROBLEM III.

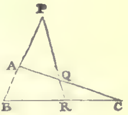
Through a given point B to lineate towards the invisible intersection H of two given lineations FG, bc. (Fig. prob. 2, second method.)

Take any point A and mark F, b, in any lineation through A, and the points G, c, in AB: find the intersection E of Gb, Fc, and D the intersection of AE, Bb: then cD, Ab intersect in a point C, which is in the same line with the given point B and the invisible point H.

This depends on the same principle as the last, and is proved in the same way.

PROBLEM IV.

To find the length of a line AP, inaccessible at one extremity.



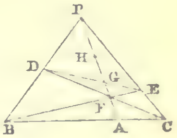
First method, when one end of the line is accessible. Take any convenient station B on the ground in lineation with A, P, and a station R out of that line; prolong BR to any convenient point C: then marking the point Q where the lines RP, AC, intersect, we shall find AP by the equation

$$\mathrm{PA} = \frac{\mathrm{QA\,RC\,AB}}{\mathrm{QA.RC} - \mathrm{QC.RB}}.$$

For, (*th.* 95) PA : PB :: QA.RC : QC.RB; or div°.

$$\mathrm{PA} : \mathrm{PB} - \mathrm{PA} :: \mathrm{QA.RC} : \mathrm{QC.RB} - \mathrm{QA.RC},$$

whence the theorem follows.



Second Method. Take any two points, B, C, lineating with A, and any point F in AP: find the intersection E of BF, CP; the intersection D of BP, CF; and the intersection G of DE, AP: and measure GF, FA. Then

$$\mathrm{PA} = \frac{\mathrm{AF.AG}}{\mathrm{AF - FG}}.$$

---

* On the same principle we may resolve the following problem, which at first sight may appear altogether impossible.

The directions of two lines whose point of concourse is invisible, and the directions of two others in like circumstances, are given: to find a point in lineation of the two invisible points of concourse.

Let Ac, bD, be two lines whose point of concourse B is invisible, and cD, Ab, two others whose concourse, C, is also invisible: to find a point d in lineation of the invisible points B, C.

Produce the four lines backwards till they form the quadrilateral AcdB; draw the diagonals meeting in E, and find bd by the equation of the text: then d is in lineation of B, C.

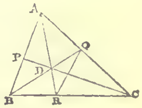
This problem is one of frequent occurrence in practical lineation, on account of visual obstacles occurring to prevent B and C from being seen from any point in lineation with them.

For, (*th.* 95) PA : PG : : AF : FG, or PA − PG : PA : : AF − FG : AF, or finally, $PA = \frac{AP - PG}{AF - FG} AF = \frac{FA.AG}{AF - FG}$.

One or other of these methods, where the relative positions of the fundamental points, from their entirely arbitrary character, admit of indefinite variation, will apply to all cases whatever.

PROBLEM V.

Through an accessible point Q to draw a line parallel to the accessible line AB.



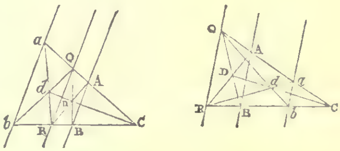
Take any three convenient equidistant points A, P, B, in the given line AB; take any point C in the lineation AQ, and find the intersection D of CP, BQ; and then the intersection R of BC, AD: the lineation QR will be parallel to the lineation AB.

For, (*th.* 96) AQ : QC : : AP.BR : PB.RC, and AP = PB (*by constr.*): hence AQ : QC : : BR : RC, and (*th.* 82) the line QR is parallel to AB.

When the line AB is inaccessible, the solution will be effected by *prob.* 9.

PROBLEM VI.

Through a given point A to lineate parallel to two given parallel lines QR and ab.



Through A lineate AC, cutting the parallels in Q and *a*, and from any point C in it lineate C*b*, cutting the same two parallels in R and *b*: find the intersection *d* of the lines *b*Q, *a*R; then the intersection D of C*d*, RA; and lastly the intersection B of QD, C*b*. The line AB will be parallel to each of the lines QR, *ab*.

For by a converse course of reasoning to that employed in the last problem, it may be shown that CD*d* would bisect *ab* and QR; and by similar reasoning to the last, that AB is parallel to either QR or *ab*, and hence to both of them.

Scholia.

This process is often applicable under circumstances where the last cannot be applied, arising from the intervention of practical obstacles. For instance, the lines might not be traceable so as to get the parallel through A to the line *ab*; whilst it might be possible (by the preceding problem) to trace a parallel to *ab* through Q, and thence by this to also trace a parallel to *ab*. This latter part, too, where two parallels already exist, is attended with some practical convenience, of which the preceding is destitute; as it requires not even the use of the chain: and if Q, R, *b*, are remarkable points, we may even solve the problem if they be inaccessible.

Two parallel lineations may be traced upon the ground by the following considerations:—

1. The lineations of a very distant object will be *sensibly* parallel.
2. The shadows of two upright staves taken at the same time are parallel; as are likewise the lineations of any star taken at the same time.

PROBLEM VII.

To divide a given line AB into two equal parts without measurement. (*Fig. prob.* 5.)

LINEATE any parallel QR to AB, and from the extremities A, B, of the given line, the lineations AC, BC, to any point C, without or between the parallels: find the intersection D of AR, BQ: then CD, being produced if necessary, will bisect AB in P.

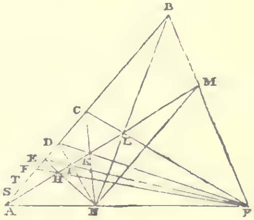
This rests on reasoning the converse of that employed in the demonstration of *prob.* 5.

PROBLEM VIII.

To cut off any part successively of a given lineation AB without the use of measures, supposing a line MN already drawn parallel to AB.

(1). TAKE any point P, and find M, N, the intersections of PB, PA, with the lineation MN; and mark the point L of intersection of AM, BN: then PL will cut off AC, one half the lineation AB, or AC $= \frac{1}{2}$ AB.

This is founded on the reasoning of *prob.* 5.



(2). Find K the intersection of AM, CN; then PK will cut off AD $= \frac{1}{3}$ AB.

For BC : CD :: BA : AD; or

AB — AC : AC — AD :: AB : AD; whence AD $= \frac{1}{3}$ AB.

(3.) Find I the intersection of AM, DN; then PI will cut off AE $= \frac{1}{4}$ AB.

For, as before, we have CD : DE :: CA : AE; and hence

CA — AD : DA — AE :: CA : AE, or AE $= \frac{1}{4}$ AB.

(4.) Find H the intersection of AM, NE; then PH will cut off AF $= \frac{1}{5}$ AB.

For again, as in the former cases, DE : EF :: DA : AF; and hence

DA — AE : AE — EF :: EA : AF, or AF $= \frac{1}{5}$ AB.

We may thus proceed successively to finding the n^{th} part of AB.

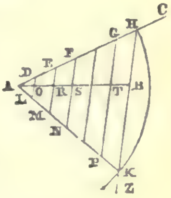
The demonstrations here given apply equally to *prob.* 7, *fifth method,* of the *Practical Geometry;* the two problems being in fact identical.*

---

* The following methods of constructing the corresponding problem in the *Practical Geometry* were accidentally omitted in their proper place: and are inserted here under the belief that they will be found quite as simple as any of the constructions there given. See p. 379, prob. 8.

To divide a given line AB into any number of equal parts.

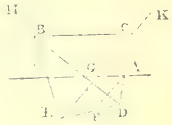
Fifth method. Draw any line AC, and take n equal portions AD, DE, ... GH; draw the line HZ through B, and from centre A describe the circle HK cutting HZ in K; set off AL, LM, ... PK, each equal to AD, and which will be n in number; lastly, join DL, EM, ... GP cutting AB in Q, R, ... T: then AB will be divided into n equal parts.



Sixth method. Through A and B (the reader can sketch the figure) draw parallels AK, BC, and in AK take n parts each equal to BC, viz. AD, DE, ... and draw CD, CE, ... cutting AB in d, e, ... : then Bd, Be, ... will be respectively the half, the third, etc. parts of AB.

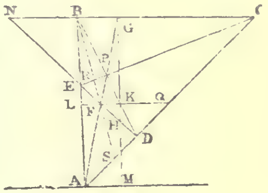
PROBLEM IX.

Through a given accessible point A to lineate parallel to a line, of which any two points B, C, are visible, but not accessible.



First method. Take any point D in lineation of A, C, and E any other convenient point: through A lineate AF parallel to the line (a part at least of) which is accessible, meeting ED in F: and through F lineate FG parallel to EB, intersecting BD in G: then GA will be parallel to BC.

For by similar triangles CED, AFD, and BED, GFD, we have CE : AF :: ED : DF :: BE : FG, or CE : EB :: AF : FG; and hence the triangles FGA, EBC, are similar, and the angles FGA, EBC, equal. Also the parts FGD, EBD, of these are equal: and hence AGD, CBD, are equal, and AG is parallel to BC *.



Second method. Take any points D, E, in AB, AC, and find P the intersection of EC, DB; and the intersection F of AP, ED: in AF take FG equal to FA, and in FD take FH equal to FE: find the intersection K of DB, GH, and produce GH, to M, till KM is equal to KG: then AM will be parallel to BC.

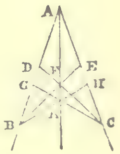
For, since AF, FE, are equal to GF, FH, and the angles EFA, HFG, are equal, AE is also equal to GH; and the remaining angles of the two triangles are equal, viz.: FGH to FAE, and FHG to AEF. Also, since in the triangles LFA, KFG, the angles LFA, FAL, are equal to the angles KFG, FGK, each to each, and the sides AF, FG, adjacent to the equal angles are equal; therefore, the sides LF, FK, are equal, and the line LK is bisected in F.

Again, produce DE to meet BC in N, and join BF: then (*th.* 97) the line DE is harmonically divided in F and N: and the line LK which cuts the three harmonical sectors BE, BF, BD, in L, F, K, and is bisected in F, is parallel to the fourth sector NBC (*conv of th.* 99.)

Lastly, since FK bisects the sides AG, GM, of the triangle GAM, it is parallel to AM; and hence AM is parallel to BC.

PROBLEM X.

To bisect an accessible angle BAC.



TAKE any two points B, D, in AB, and the distances AE, AC, equal to AD, AB, respectively: the intersection F of BE, DC, will be in the lineation which bisects the angle BAC.

For, BA, AE, are equal to CA, AD, (*constr.*,) and the angle at A common: hence the angles AEF, ADF, are equal; and likewise the two DBF, ECF. But since ADF, AEF, are equal, the angles FDB, FEC, are equal: and hence, in the two triangles CEF, BDF, there are two angles

* If E can be so taken that two distant objects H, K, are in lineation with B and C, then the parallels AF, FG, may be drawn, sensibly accurate, by the remark in the note on prob. 6, p. 4[illegible].

of the one equal to two angles of the other, and the sides adjacent to the equal angles equal; therefore DF is equal to FE.

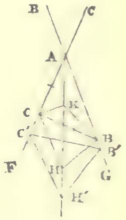
Again, the sides DA, AF, are equal to EA, AF, and the bases DF, FE, also equal; therefore, the angles DAF, EAF, are also equal, or BAC is bisected by the lineation AF.

Scholium.

Should any obstacle interpose to prevent our seeing A from F, we may take two other points G and H equidistant from A, and find the intersection K of CG, BH: then KF will be the bisecting lineation.

PROBLEM XI.

To bisect an inaccessible angle BAC.



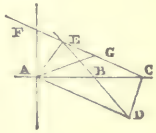
Take any points B, C, in the sides of the angle or their lineations, and join BC; bisect the accessible angles BCF, CBG; then H, their point of intersection, will be in the lineation bisecting the angle BAC.

If any obstacle prevent A being visible at H, bisect the angles ACB, ABC, by lineations meeting in K: then HK is the bisecting lineation *.

For K and H are the centres of the inscribed and escribed circles, and hence both HA and KA bisect the angle BAC (*th.* 99, *schol.* 6.)

PROBLEM XII.

To draw a perpendicular from a point A in a given lineation BC.



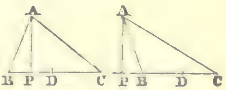
Take any point B in BC, and lineate BD in any direction through B: make BD equal to BA, and AC and DE each equal to AD: in CE take CG, GF, each equal to AB: then AF will be the perpendicular required.

For, join GA. Then, since AC is equal to DE, and AB, BD, parts of them are equal, the remaining parts EB, BC, are also equal. Hence the angles ABD, EBC, being equal, and AB : BD :: EB : BC, the triangles are similar, and the angles ECB, BAD, equal.

Again, in the triangles AGC, ABD, we have GC, CA, equal to BA, AD, and the angle GCA equal to the angle BAD: hence AG is equal to BD; and hence, again, the triangle AGC is isosceles, having AG equal to GC. Whence G is the centre of a semicircle passing through F, A, C; and the angle FAC, in it, is a right angle.

PROBLEM XIII.

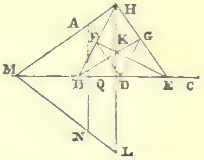
To lineate perpendicularly to a given lineation BC from a point A without it; all being accessible.



First method. Take any three equidistant points B, D, C, in the given lineation, and measure the distances AC, CB, BA: then the distance of the foot P of the perpendicular AP from D is found from the

* When the point K would fall too near to A, to render the operation by means of it practicable, we may either repeat the first part of the process with new points C′, B′, instead of C, B: or we may employ *prob.* 2 for finding the lineation of a point to the invisible intersection of the given lines AB, AC.

equation $DP = \frac{(CA + AB)(CA - AB)}{2BC}$, which being set off gives the point D. This is evident from *th.* 35, *Plane Geometry*, p. 311.



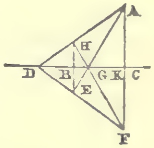
Second method. Take from any point D in the line BC, four equal lines DB, DE, DF, DG, the two former in the line BC, and the two latter in any directions whatever: find the intersections K, H, of BG, EF, and BF, EG: lineate HK cutting BC in Q, and in it take QL equal to QH: lineate HA cutting BC in M: and in ML take MN equal to MA. The lineation AN is perpendicular to BC.

For since DB, DE, DF, DG, are all equal, the points B, E, F, G, are in the circumference of a circle, of which D is the centre and BE the diameter: hence BG and EF are perpendicular to the sides of the triangle HBE, and therefore, also, HK is perpendicular to the base BE (*th.* 97, *schol.* 6.)

Again, since HQ is equal to QL, and MQ common to the two right-angled triangles MQH, MQL, the side HM is equal to LM; and (*constr.*) MA is equal to MN. Hence AM : MN : : HM : ML, and AN parallel to HL; that is, AN perpendicular to the lineation BC.

PROBLEM XIV.

To lineate perpendicularly to an accessible lineation BC from an inaccessible point A without it.

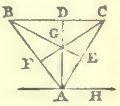


Find by either of the preceding constructions the perpendicular BH to the lineation BC, and make BE equal to BH: find D the intersection of AH, BC; G the intersection of AE, BC; and, lastly, the intersection F of DE, HG: then FA will be perpendicular to BC.

For, the right-angled triangles DBE, DBH, have their sides HB, BD, equal to EB, BD, and therefore HD, DE, are equal, as are also the angles HDB, EDB. Again, the triangles GHD, GED, have the sides GD, DH, equal to GD, DE, and their included angles equal; hence the angles DHG, DEG, are also equal. Again, since the triangles DHF, DEA, have the sides HD, DE, equal, the angles DHF, DEA, equal, and the angle HDE common, the sides DA, DF, are equal. Lastly, since AD : DF : : HD : DE, AF is parallel to HE, and therefore perpendicular to BC.

PROBLEM XV.

From an accessible point A to find a line perpendicular to an inaccessible line, two points B, C, of which are visible.



First solution. Form the inaccessible points B, C, lineate the perpendiculars to the lines AB, AC, (*prob.* 14,) to intersect in G: then AG will be perpendicular to BC.

The proof of this consists in showing, that if two lines drawn from the angles of a triangle be perpendicular to the opposite sides, the line from the third angle through the point of intersection will be perpendicular to the third side (*conv. of th.* 97, *schol.* 2.)

Second solution. Through the point A draw AH parallel to the inaccessible line BC (*prob.* 9), and from A in AH draw AD perpendicular to it (*prob.* 13). Then AD will be perpendicular to BC.

PROBLEM XVI.

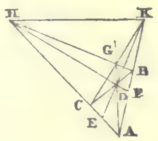
To find the length of a lineation BC which is inaccessible at both extremities.

(*See fig. prob.* 9, *second solution.*)

First method. Take any point A, and proceed as in the second solution of *prob.* 9, except the final one of finding M: find Q the intersection of KL, AC; and R, S, the intersections of BF with EC, CA: lastly, measure FS, FR, and QF: then $BC = \frac{QF.SR}{SF - FR}$.

For, by the reasoning of the second solution of *prob.* 3, we have at once BS : SF :: RS : SF — FR; and by the similar triangles, BSC, FSQ, we have BS : SF :: BC : FQ; and hence, also, SF — FR : RS :: FQ : BC, which gives the value stated above.

Second method. Let HK be the inaccessible line: take any point A: and in the lineations AH and AK, any points C and B: find G the intersection of CK, BH, and D, the intersection of CB, AG: find E the intersection of KD, AH, and F that of HD, AK: and, finally, measure the sides of the triangle ABC, and the parts AE, AF. The distance HK is found from the following calculation.



The lines AH, AK, are harmonically divided in their several points of section: and hence, reasoning as before, we have, putting a, b, c, for the sides of the triangle ABC, and putting m and n for the specified quotients:—

$$AK = \frac{AB.AF}{AF - FB} = mc; \text{ and } AH = \frac{AC.AE}{AE - EC} = nb,$$

$$\cos CAB = \frac{-a^2 + b^2 + c^2}{2bc}, \text{ and } HK = \sqrt{AH^2 - 2AH.AK \cos A + AK^2};$$

which, upon substitution, gives the following formula for HK,

$$HK = \sqrt{m(m-n)c^2 + mna^2 + n(n-m)b^2}.$$

Scholium.

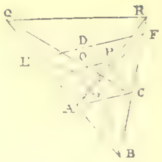
1. Though the calculation of the latter method of solution is longer than that of the preceding, yet as its field-work is shorter and less liable to error, it seems to deserve the preference.

2. We may also, generally, take AC equal to AB, in which case the formula becomes simplified, viz. $HK = \sqrt{(m-n)^2 c^2 + mna^2}$.

3. If by any means we can obtain a moveable equilateral triangle (as by joining three equal rods or staves) we may move along AH, with one side of it lineating with AH, till the other lineates with K. In this case, a, b, c, are all equal; and we have $HK = a\sqrt{m^2 - mn + n^2} = a\sqrt{(m-n)^2 + mn}$.

4. A better method, perhaps, where it can be applied than either of the preceding is annexed.

Third method. Let QR be the inaccessible line, and B any convenient point: in the lineations BQ, BR, take BA, AE, BC, CF, equal to one another: find the intersections G of AF, CE; D of BG, EF; O of AD, QC; and P of CD, AR: and, finally, measure the sides of the triangle DOP. Then calculate QR from the equation $QR = \frac{OP.AD^2}{OD.DP}$.



For, by similar triangles ODC, CQB, and DAP, ABR, DO : DC :: BC : QB, DA : DP :: BR : BA; and since DA = DC = BC = BA, we have also, DO : DP :: BR : QB; hence the triangles DOP, BRQ, are similar.

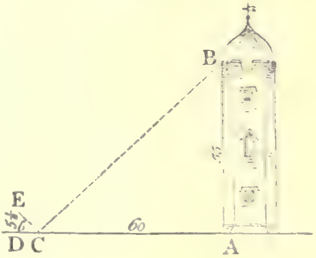
Again, by the similar triangles ODP, RBQ, we have OD : RB :: OP : RQ :: OD.DP : RB.DP; and DA : DP :: BR : AD, or $RB.DP = DA^2$; hence we have $OD.DP : AD^2 :: OP : RQ = \frac{OP.AD^2}{OD.DP}$.

PROBLEM XVII.

To show the use of the equality of the angles of incidence and reflection in the determination of altitudes of trees, buildings, etc.

When a ray of light falls upon a smooth or polished plane, as of quiescent water, or quicksilver, or a mirror, it is so reflected that the angle which it makes with the said plane after reflection is equal to the angle which it made before reflection. Of these angles, the first is usually called the *angle of incidence*, the other the *angle of reflection*. Sometimes the angles made respectively with the *perpendicular* to the plane of reflection receive those names. In either case, however, the practical application is the same.

Suppose AB in the annexed figure to be a tower, whose altitude is required, and which stands on a horizontal plane AD. At a convenient point C place a vessel of water or mercury; and when the surface is quite smooth and quiescent, recede a little way from it in the continuation of the line AC, until with your eye as at E you see the top of the building, or the point whose height you wish to ascertain, reflected from the surface. Then, by the similar triangles CDE, CAB, we have CD : DE :: CA : AB. Thus, if CD be 6 feet, $DE = 5\frac{1}{2}$, $AC = 60$, then will $AB = 55$ feet, the height required.



Scholium.

A considerable number of problems analogous to those which are solved in this chapter might easily be given: but as space could not be afforded for treating them with adequate expansion, this course has been confined to those of the greatest importance, and of the most frequent occurrence in the field. Some of them, with the aid of the sextant, will, however, be resumed in the second volume.

APPLICATION OF ALGEBRA

TO

GEOMETRY.

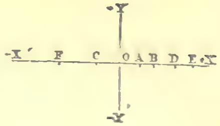
Algebra is essentially a numerical system; and in every case to which we apply it, we must be able to express the relations between the objects under consideration numerically. In the application of algebra to geometry, this is effected by considering every magnitude concerned in the investigation as containing some number of times another magnitude of the same species, which other magnitude may be called the standard-unit of the system. Thus, a foot, a yard, a mile, are each some number of times the length of an inch taken as the standard-unit; a pole or an acre are some number of times the square foot or the square yard taken as standard-units; the mass of the earth is some number of times the magnitude of a cubic foot, a cubic yard, or a cubic mile, taken respectively as the standard-units. The same mode of estimation may be applied to angular space, where any angles may be considered as the repetitions, each a certain number of times, of any assigned angle taken as the standard-unit of angular measure. The present chapter will not, however, include any problem relating to angles, as that subject properly belongs to the next treatise, Trigonometry, immediately following it.

It will be obvious from this statement, that the standard-unit for each species of magnitude must be of the same species as the magnitude which it measures; and that, though arbitrary in the outset as to its own magnitude, it must be kept constantly the same for all the quantities concerned in the same problem. In all expressions involving general symbols, this is presumed in the notation; but in the actual reduction to numbers, particular care must be taken to reduce all the numbers concerned to the same denomination in the final result. (*See prob.* 1, *p.* 415.)

It greatly conduces to convenience, though it is not absolutely essential to the nature of the system, to take as the standard-unit of surface, the square described upon the linear unit, and as the unit of volume, the cube whose edge is the linear unit. In practice, therefore, the units of surface and volume are ultimately referred to the linear unit.

In general symbols it is not necessary to specify the actual magnitude of the unit, as whatever magnitude we conceive it to be, the quantities measured by it are expressed in numbers which have amongst themselves the same ratios, and the final result is given in terms of the same unit. The symbols $a, b, c, d \ldots\ldots x, y, z$, then only express the number of times which the magnitudes they designate contain the standard-unit by which they are severally measured.

Geometry is not conversant with magnitude only: it also treats of *form* and *position*. The first of these implies the consideration of angles, and hence does not fall under our present discussion. Position may be determined several ways, most of which include a consideration of angular magnitude; but there is one mode which may be considered essentially linear; and it enters extensively into every view which can be taken of the application of algebra to geometry.



Suppose we had given the expression $a + b - c + d + e - f$, where each of these letters expresses a certain number of standard-units of length. We have first to draw the indefinite line X'X, and to take the point O in it as the origin from which we set off the lines in question. Let, also, the positive values tend towards X, then the negative ones, whose effect is to diminish the sum of those already set off, will be drawn in the contrary direction, or towards X'. So far our selection is perfectly arbitrary, (except previous data shall have fixed it,) both as to the position of the line X'X, the point O from which we are to measure, and the side of O which shall be estimated as the direction of positive measurement. The standard-unit, too, (under the same exception,) is also arbitrary.

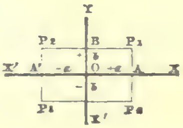
Having determined upon all these conditions, there is no longer any thing arbitrary, or assumable at pleasure, in the problem; and the problem itself, as a geometrical one, is not merely to find the length of the line, but the *position* of the final point resulting from the following operation, with respect to the origin O.

From O	set off OA	$= + a$,	towards X	pos. side.
.... A	.... AB	$= + b$,	 X	....
.... B	.... BC	$= - c$,	...... X'	neg. ...
.... C	.... CD	$= + d$,	 X	pos. ...
.... D	.... DE	$= + e$,	 X	....
.... E	.... EF	$= - f$,	...... X'	neg. ...

Then OF is the length of the line to be constructed, and F falling (in the case here supposed) on the negative side of O, shows that the negative values exceed the positive by as many standard-units as there are in OF.

It is thus that position combines itself with the simple consideration of magnitude in the application of algebra to geometry; and there are very few cases in which it cannot be distinctly traced, though in many of them our attention may not be specifically called to it by the forms of the results.

If we draw a line YY' perpendicular to XX' through the origin O, we may clearly in the same manner construct any given expression in the same way upon this line as we did upon X'X.*



In the next place, if we wish to express the position of a point with respect to two given lines at right angles to each other; it is sufficient if we can express its distance from each of them, and likewise on which side of each of them it lies. This will be done if we attend to the observations already made respecting + and — as signs of position to the right or left of O, and above or below O.

Taking, then, as before, OX, OY, for the lines which express positive directions from O, and OX', OY', for those which express negative directions, let the distance of the point P from the line YY' be $\pm a$, and the distance of P from XX' be $\pm b$. Then the values $+ a$, $+ b$ will express the point being at P_1;

* In fixing the position of these lines X'X and YY', if we commence from no preceding conditions, it is most usual, though rendered so from custom rather than from motive, to draw them horizontal and vertical, and to take the positive values to the right, and upwards from O, as we have here done.

$-a, +b$, that it is at P_2; $-a, -b$, that it is at P_3; and $+a, -b$, that it is at P_4.

The lines X'X, YY', are called axes of co-ordinates, and the two distances from them are called the co-ordinates of the point. As, however, the subject of co-ordinates as a system will not be entered upon till we come to the second volume, this brief notice of the nature of the notation for position will be sufficient in the present chapter.

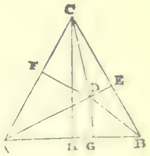
In respect of notation, the early letters of the alphabet in this section are mostly used to designate known or given magnitudes; and the latter, unknown ones. Of the unknowns, lines drawn horizontally are denoted by x, and those vertically by y; as, for instance, the unknown base and perpendicular of a right-angled triangle. When the sides of a triangle are given, they are denoted by a, b, c, the side a being opposite to the angle A, and so on of the others. Sometimes the initial letter of a name is used, as p for a given perpendicular, and s for semiperimeter, or $\frac{1}{2}(a + b + c)$; but these, though matters of usual practice, need not be dwelt on here.

The conditions of a problem, *formally given*, are seldom so many as to furnish as many equations as there are unknowns; but there are always as many geometrically-demonstrated properties *implied* (and which must be taken from geometry), as will complete that number of equations if it be properly proposed.

All the results may be *constructed* were it an object to do so; but as this is never required, it is unnecessary to say anything on the subject here. Most of the older works on this subject contain instructions on this head, but the construction is now a mere object of curiosity, and therefore not worthy to arrest the student's progress in the present stage of his studies.

PROBLEMS.

PROB. I. *In an equilateral triangle, having given the lengths of the three perpendiculars drawn from a certain point within it to the three sides, to determine the sides.*



Let ABC be the equilateral triangle, and DE, DF, DG, the perpendiculars from the point D upon the sides respectively. Denote these perpendiculars by a, b, c, in order, and the side of the triangle ABC by $2x$. Then if the perpendicular CH be drawn, $CH = \sqrt{AC^2 - AH^2} = \sqrt{4x^2 - x^2} = x\sqrt{3}$.

Now (*th.* 81. *cor.* 2) we have triangle $ADB = \frac{1}{2}AB \cdot GD = \frac{1}{2}cx$. Similarly, triangle $BDC = \frac{1}{2}ax$, triangle $CDA = \frac{1}{2}bx$, and triangle $ACB = \frac{1}{2}AB \cdot CH = x^2\sqrt{3}$. Also $BDC + CDA + ADB = ABC$: that is, in symbols $x^2\sqrt{3} = (a + b + c)x$, or $x = \dfrac{a + b + c}{\sqrt{3}}$, which is half the side of the triangle sought.

Suppose $a = 3$ ft, $b = 9$ in, and $c = 2$ yds 1 ft 6 in. Then, in ft, we have

$$x = \frac{3 + \frac{3}{4} + 7\frac{1}{2}}{\sqrt{3}} = \frac{45}{4\sqrt{3}};\text{ or, in yds, } x = \frac{15}{4\sqrt{3}}.$$

Cor. From the resulting equation we have $x\sqrt{3} = a + b + c$; and again, $CH = x\sqrt{3}$. Hence $CH = a + b + c$, or the whole perpendicular CH is equal to the three smaller perpendiculars from D upon the sides, wherever the point D is taken *within* the triangle. Had the point D been taken *without* the triangle, the perpendicular upon the side which subtends the angle within which the point

lies would have become negative. Thus, had it lain without the triangle, but between the sides AB, AC, *produced*, then CH = DF + DG − DE.

PROB. II. *A maypole was broken by the wind, and its top struck the ground twenty feet from the base, and being repaired was broken a second time five feet lower, and its top struck the ground ten feet farther from the base. What was the height of the maypole?*

Let AB be the unbroken maypole, C and H the points in which it was successively broken, and D and F the corresponding points at which the top B struck the ground. Then CAD and HAF are right-angled triangles.

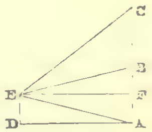
Put BC = CD = x, CA = y, AD = a, and AF = b, and CH = c. Then AB = $x + y$, BH = HF = $x + c$, and HA = $y - c$. Then in the triangles CDA, FHA, we have

$$y^2 + a^2 = x^2, \text{ and } (y - c)^2 + b^2 = (x + c)^2.$$

Expand the second and subtract the first equation from it, and we have, finally,

$$x + y = \frac{b^2 - a^2}{2c} = 50 \text{ feet, the height required.}$$

PROB. III. *A statue eighty feet high stands on a pedestal fifty feet high, and to a spectator on the horizontal plane they subtend equal angles; required the distance of the observer from the base, the height of the eye being five feet.*



Let AB = a the height of the pedestal;
BC = b the height of the statue;
DE = c the height of the eye from the ground; and
DA = EF = x, the distance of the observer from the base.

Then $EC^2 = EF^2 + FC^2 = x^2 + (a + b - c)^2$, and
$EA^2 = EF^2 + ED^2 = x^2 + c^2$.

But (*th.* 83) $EC^2 : EA^2 :: CB^2 : BA^2$; in which inserting the preceding values of these lines, we have, after easy reductions,

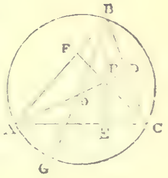
$$x = \pm \sqrt{\frac{a\,(a - c)^2 + b\,(a^2 - c^2)}{b - a}}$$

the double sign merely indicating that x may be measured either way, from A towards D, or from D towards A.

Inserting the given values of a, b, c, we have $x = \pm\, 5\sqrt{399} = 99{\cdot}874922$ feet.

PROB. IV. *Given the three sides, a, b, c, of a triangle, to find:—*

(1). *The three perpendiculars from the angles upon the opposite sides;*
(2). *The area of the triangle;*
(3). *The radius of the circumscribing circle;*
(4). *The radius of the inscribed circle;*
(5). *The radii of the escribed circles.*



Let ABC be the triangle, and BE a perpendicular from B to the opposite side AC. Let a, b, c, denote the sides opposite to the angles A, B, C, respectively, and p_1, p_2, p_3, the perpendiculars from A, B, C, viz. AD, BE, CF; Δ the area of the triangle; R the radius of the circumscribing circle; r that of the inscribed circle; and r_1, r_2, r_3, the radii of the three escribed circles which touch the sides a, b, c, externally.

(1). *The perpendiculars.* By *th.* 37 we have $BC^2 = BA^2 + AC^2 - 2CA.AE$

$$\text{or } AE = \frac{BA^2 + AC^2 - BC^2}{2CA} = \frac{-a^2 + b^2 + c^2}{2b}. \text{ Again,}$$

$$BE^2 = BA^2 - AE^2 = c^2 - \left\{\frac{-a^2+b^2+c^2}{2b}\right\}^2$$

$$= \frac{4b^2c^2 - \{-a^2 + b^2 + c^2\}^2}{4b^2}$$

$$= \frac{\{2bc + (-a^2+b^2+c^2)\}\{2bc - (-a^2+b^2+c^2\}}{4b^2}$$

$$= \frac{\{-a^2 + (b+c)^2\}\{a^2 - (b-c)^2\}}{4b^2}$$

$$= \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4b^2}$$

$$= \frac{\frac{a+b+c}{2} \cdot \frac{-a+b+c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{a+b-c}{2}}{\frac{1}{4}b^2}$$

Put $\frac{a+b+c}{2} = s$; then $\frac{-a+b+c}{2} = \frac{a+b+c}{2} - a = s - a$;

and similarly, $\frac{a-b+c}{2} = s - b$, and $\frac{a+b-c}{2} = s - c$.

Making these substitutions, we have the equation at once converted into

$$BE^2 = p_2{}^2 = \frac{s(s-a)(s-b)(s-c)}{\frac{1}{4}b^2}; \text{ and similarly,}$$

$$p_1{}^2 = \frac{s(s-a)(s-b)(s-c)}{\frac{1}{4}a^2}, \text{ and } p_3{}^2 = \frac{s(s-a)(s-b)(s-c)}{\frac{1}{4}c^2}.$$

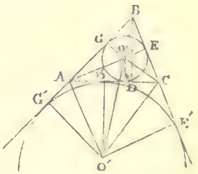
(2). *The area.* This by *th.* 81, *cor.* 2, is $\frac{1}{2}$ AC.BE $= \Delta$. But putting in this the value of BE, obtained in the last case, we have,

$$\Delta^2 = \tfrac{1}{4}b^2 . \frac{s(s-a)(s-b)(s-c)}{\frac{1}{4}b^2} = s(s-a)(s-b)(s-c).$$

(3). *The radius of the circumscribing circle.* Let the diameter BG be drawn: then *(th. 63, Geom.)*

$$R = \frac{AB \cdot BC}{2BE} = \frac{\frac{1}{4}abc}{\sqrt{s(s-a)(s-b)(s-c)}}.$$

(4). *The radius of the inscribed circle.* Let ABC be the triangle, O the centre of the inscribed circle, BD the perpendicular from B upon AC, and E, F, G, the points of contact. Then, *th.* 81, *cor.* 2, we have trian. BOA $= \frac{1}{2}rc$; trian. BOC $= \frac{1}{2}ra$; trian. COA $= \frac{1}{2}rb$; and trian. ABC $= \frac{1}{2}bp_2$. But BOA + AOC + COB = ABC; or in symbols,



$$r(a+b+c) = bp_2, \text{ and hence } r = \frac{bp_2}{a+b+c} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

(5.) *The radii of the escribed circles.* Let O′ be the centre of the escribed circle, touching AC *exteriorly*, and E′, F′, G′, the points of contact. Then ABC = ABO′ + CBO′ − AO′C; or in symbols, as before,

$$r_2 = \sqrt{\frac{s(s-a)(s-c)}{s-b}}; \text{ and in a similar manner we obtain}$$

$$r_1 = \sqrt{\frac{s(s-b)(s-c)}{s-a}}, \text{ and } r_3 = \sqrt{\frac{s(s-a)(s-b)}{s-c}}.$$

Cor. 1. By multiplying the values of the four radii of the circles of contact together, we have $r\ r_1\ r_2\ r_3 = s\ (s - a)\ (s - b)\ (s - c) = \Delta^2$, a remarkable theorem discovered by *Lhuillier*.

Cor. 2. Also, taking their reciprocals, we get $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.

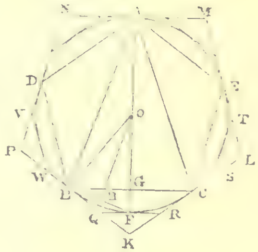
Cor. 3. By multiplying together the values of R and r, we get $2Rr = \frac{abc}{a + b + c}$; and similarly, $2Rr_1 = \frac{abc}{-a + b + c}$, and so on.

Many other curious properties may be seen in the supplement to the "Ladies' Diary" for 1835 and 1836, by the editor of this work.

Prob. V. *Given the radius r of a circle to find the sides of the inscribed and circumscribed pentagons and decagons.*

1. *The inscribed pentagon.* Let ADBCE be the pentagon inscribed in the circle, (*prob.* 40, or *Euc.* iv. 11), and let O be the centre of the circumscribing circle. Join AB, AC, and draw AF perpendicular to BC. Then by known properties AC is bisected in G, and the line AF passes through the centre O of the circle; and likewise,

AB : BC :: BC : AB — BC.



Put $BC = 2x$, or $BG = x$, $BA = y$, and $BF = z$. Then the preceding proportion becomes $y : 2x :: 2x : y - 2x$, and hence we have, $y = (1 + \sqrt{5})\,x$.

Now we have $OG = \sqrt{r^2 - x^2}$, $AG = \sqrt{y^2 - x^2}$, and hence $AG = AO + OG$, gives $\sqrt{y^2 - x^2} = r + \sqrt{r^2 - x^2}$, or squaring $y^2 - 2r^2 = 2r\sqrt{r^2 - x^2}$, and squaring again, $y^4 - 4r^2y^2 + 4r^2x^2 = 0$. Also we have seen that $y = (1 + \sqrt{5})x$, which, inserted and the expression reduced, gives for the side of the inscribed pentagon, $2x = \frac{1}{2}\,r\,\sqrt{10 - 2\sqrt{5}}$.

Cor. 1. $y = x(1 + \sqrt{5}) = \frac{1}{2}\,r\sqrt{10 + 2\sqrt{5}}$; a value which will be useful in the other parts of the problem, though no part of the quæsita in any one.

Cor. 2. $OG = \sqrt{r^2 - x^2} = \sqrt{r^2 - \frac{1}{8}(5 - \sqrt{5})\,r^2} = \frac{1}{4}\,r\,(1 + \sqrt{5})$.

2. *The inscribed decagon.* Join BF: then, since AF bisects the line BC at right angles, it bisects the arc BFC in F, and hence BF is the side of the inscribed decagon.

But ABF being a right angle, since it is inscribed in a semicircle, we have $BF^2 = FA^2 - AB^2$: or adopting the preceding notation, and denoting as above BF by z, we have it converted into

$z^2 = 4r^2 - y^2 = 4r^2 - \frac{1}{4}(10 + 2\sqrt{5})r^2 = \frac{1}{2}(3 - \sqrt{5})r^2$; or extracting $z = \frac{1}{2}\,r(\sqrt{5} - 1)$, as the length of the side of the inscribed decagon.

3. *The circumscribing pentagon.* The inscribed and circumscribing pentagons being regular, are similar figures, and their sides are as the perpendiculars from the centre upon the sides. That is, if PK be a side of the circumscribing pentagon, OG : OB :: BC : PK; which put into symbols, gives the value.

$$PK = \frac{OB.BC}{OG} = \frac{r\cdot\frac{1}{2}r\sqrt{10 - 2\sqrt{5}}}{\frac{1}{2}r\sqrt{\frac{1}{2}(3 + \sqrt{5})}} = 2r\sqrt{5 - 2\sqrt{5}}.$$

4. *The circumscribing decagon.* Let QR be one of the sides; and draw OH perpendicular to BF, which it bisects in H. Also, by similar triangles, ABF,

OHF, we have $OH = \frac{1}{2} AB = \frac{1}{2} y = \frac{1}{4} r \sqrt{10 + 2\sqrt{5}}$. Also, as in the last case, $OH : OF :: BF : QR$, and hence,

$$QR = \frac{OF \,.\, BF}{OH} = \frac{r \,.\, \frac{1}{2} r (\sqrt{5} - 1)}{\frac{1}{4} r \sqrt{10 + 2\sqrt{5}}} = 2r \sqrt{\frac{5 - 2\sqrt{5}}{5}}.$$

Further problems for exercise.

6. In a right-angled triangle, having given the base (3), and the difference between the hypothenuse and perpendicular (1); to find these sides.

7. In a right-angled triangle, having given the hypothenuse (5), and the difference between the base and perpendicular (1); to determine these sides.

8. Having given the area, or measure of the space, of a rectangle, inscribed in a given triangle; to determine the sides of the rectangle.

9. In a triangle, having given the ratio of the two sides, together with the segments of the base, made by a perpendicular from the vertical angle; to determine the sides of the triangle.

10. In a triangle, having given the base, the sum of the other two sides, and the length of a line drawn from the vertical angle to the middle of the base; to find the sides of the triangle.

11. In a triangle, having given the two sides about the vertical angle, with the line bisecting that angle, and terminating in the base; to find the base.

12. To determine a right-angled triangle; having given the lengths of the lines drawn from the acute angles, to the middle of the opposite sides.

13. To determine a right-angled triangle; having given the perimeter, and the radius of its inscribed circle.

14. To determine a triangle; having given the base, the perpendicular, and the ratio of the two sides.

15. To determine a right angled triangle; having given the hypothenuse, and the side of the inscribed square.

16. To determine the radii of three equal circles, described in a given circle, to touch each other and also the circumference of the given circle.

17. In a right-angled triangle, having given the perimeter, or sum of all the sides, and the perpendicular let fall from the right angle on the hypothenuse; to determine the sides of the triangle.

18. To determine a right-angled triangle; having given the hypothenuse, and the difference of two lines drawn from the acute angles to the centre of the inscribed circle.

19. To determine a right-angled triangle; having given the side of the inscribed square, and the radius of the inscribed circle.

20. To determine a right-angled triangle; having given the hypothenuse, and the radius of the inscribed circle.

21. To divide a line of ten inches in extreme and mean ratio.

22. To add to it a segment, such that the rectangle under the whole line thus increased, and the part of it increased, shall be to the square of the difference of the two segments into which the line is now divided, as 12 : 5.

23. A circle AFB and a point D are given, the distance DC of the point from the centre C of the circle being b, and the radius r; to draw through D a line EF, terminated both ways by the circle in E and F, so that its length shall be $2a$.

24. Given the adjacent sides a, b, and the diagonal, c, of a parallelogram, to find the other diagonal.

25. Given the chords of two arcs of a given circle, to find the chord of their sum, and the chord of their difference.

26. To divide the base, a, of a triangle into two segments proportional to the sides, b, c.

27. Two circles being given which touch one another inwardly; to describe a third circle that shall touch both the former, and also the right line passing through their centres.

28. Having given the lengths of two chords which intersect at right angles, and the distance of their point of intersection from the centre; to find the diameter of the circle.

29. Given, to determine the area of the triangle, and the lengths of its sides, the three perpendiculars from the angles upon the opposite sides.

30. If a, b, c, d, taken in order, be the sides of a quadrilateral inscribed in a circle, and x, the diagonal, joining the extremities of the sides, a, d, or b, c, and y the other diagonal; it is required to show that

$$xy = ac + bd,\ x : y :: ad + bc : ab + cd,$$

and thence to find x and y, together with the area, and the radius of the circle circumscribing the quadrilateral.

31. Supposing the town A to be 30 miles from B, B 25 miles from C, and C 20 miles from A; if a house be erected to be equally distant from each of those towns, what will its distance from them be? Ans. 15·11856 miles.

32. The chords of three arches completing a semicircle being given, 3, 4, and 5 respectively; required the diameter. Ans. 8·05581.

Theorems for Exercise.

1. Show that $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$, and $r_1 + r_2 + r_3 = 4\,\mathrm{R} + r$.

2. Prove that $\dfrac{1}{p_1} + \dfrac{1}{p_2} + \dfrac{1}{p_3} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$; and $\dfrac{\mathrm{R}}{2r} = \dfrac{r_1 r_2 r_3}{p_1 p_2 p_3}$.

3. Establish the relation $\frac{1}{2}\,\mathrm{R}\,r = \dfrac{\Delta^2}{p_1 p_2 + p_2 p_3 + p_3 p_1}$.

4. Prove that $p_1 = \dfrac{2r_2 r_3}{r_2 + r_3}$, and $\dfrac{1}{p_1} + \dfrac{1}{p_2} - \dfrac{1}{p_3} = \dfrac{1}{r_3}$.

5. If a, b, c, d, be the sides of a quadrilateral, and a be parallel to c, and h the distance of these parallels: then,

$$4h^2 (a - c)^2 = 2 (a - c)^2 (b^2 + d^2) - (a - c)^4 - (b^2 - d^2)^2$$

6. If a circle be inscribed in an equilateral triangle, and a triangle in this circle, and again a circle in the triangle, and so on, *ad inf.*; prove that $r = r_1 + r_2 + r_3 + \ldots$ where r, r_1, r_2, . . are the radii of the successive circles.

7. Show that the side of a hexagon inscribed in a circle is mean proportional between the sum and difference of the sides of the inscribed pentagon and decagon.

8. If l_1, l_2, l_3, be the lines drawn to bisect the angles of a triangle, whose sides are a, b, c; show that $\dfrac{l_1 l_2 l_3}{abc} = \dfrac{4 (a + b + c)\ \Delta}{(a + b) (b + c) (c + a)}$:

$$16 (l_1^4 + l_2^4 + l_3^4) = 9 (a^4 + b^4 + c^4), \text{ and}$$

$$16 (l_1^2 l_2^2 + l_2^2 l_3^2 + l_3^2 l_1^2) = 9 (a^2 b^2 + b^2 c^2 + c^2 a^2.)$$

9. If m_1, m_2, m_3, be the lines drawn from the angles to bisect the opposite sides: then $m_1^2 + m_2^2 + m_3^2 = \frac{3}{4} \{a^2 + b^2 + c^2\}$.

10. If a, b, c, be the distances of a point from three of the angles A, B, C, of a square, B being the nearest or most distant angle, show that the area of the square is $\frac{1}{2} \{a^2 + c^2 \pm \sqrt{4b^2 (a^2 - b^2 + c^2) - (a^2 - c^2)^2}\}$; and that if the four distances from a point to the angles of a rectangle, be a, b, c, d, taken in order then will $a^2 + c^2 = b^2 + d^2$.

PLANE TRIGONOMETRY.

Plane Trigonometry is that particular portion of the application of algebra to geometry, in which the angles of a triangle occur either in the data, the quæsita, or the intermediate equations, or in all of these.

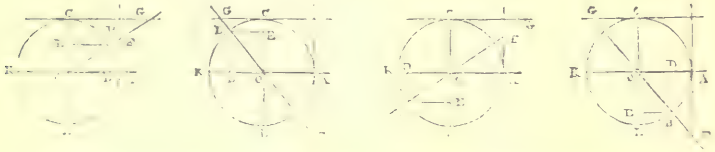
It has been established (*Euc. vi.* 33, or *Geom. th.* 94), that in the same or equal circles, angles at the centre are to one another in the same ratio as their subtending arcs. It hence follows, that in any algebraic expression of the angle in terms of a given standard angular unit, we may substitute the expression of the subtending arc of a circle in terms of a corresponding standard circular unit. If we take the right angle as the angular unit, the quadrant which subtends the right angle will be the corresponding circular unit: if four right angles be the angular unit, the whole circumference will be the corresponding circular unit; and so on for any other corresponding angular and circular units. Whatever, therefore, is stated or proved with respect to the arc, may be stated or proved with respect to the angle; and the converse. We may, therefore, indifferently employ the terms *arc* or *angle* to express the inclination of two lines to one another. In the higher departments of mathematical science, however, it is not a matter of indifference which term we use, or which thing we reason from; whether it be viewed in reference to the conception of the things, or the manner in which we reason upon them. The advantage is in all cases in favour of the arc, but especially in all general investigations respecting the unlimited extension of magnitude, which, without departing from the principle of continuity, we may suppose the arc or angle to admit of. In the mere solution of problems, however, relating to triangles or polygons, it is usual to employ the word angle in our phraseology, notwithstanding that all our investigations turn upon its measure, the arc.

Since the relation established (p. 367), already referred to, is true, whatever may be the radius of the circle, it is obvious that our conclusions will not be affected by changing the radius, so long as all the subtending arcs involved in the same investigation are measured by circles having the same radius. In other words, the radius-unit of the circle is altogether independent of the linear unit by which the linear magnitudes of the same problem are measured. It is found to be most generally convenient to take the radius, in calculation, as 1, and in any case where a different radius is required in connecting by an equation any parts of two different circles, to multiply the part of the circle belonging to the unit radius by the new radius, to obtain the corresponding part of the new circle. This is founded on *th.* 93, or *Euc. xii.* 2; or rather, on an obvious corollary from it.

Facility of investigation and calculation, however, is better consulted by the use of certain lines drawn in uniform and specific ways with relation to the subtending arcs, than by the direct introduction of the arcs themselves into the equation. These lines are called the trigonometrical functions of the arc or angle: and they have been calculated by methods which will be hereafter explained for very minute divisions of the quadrant, and arranged in tables for convenience of use. In all cases these lines are tabulated for the radius unity. These lines or functions we shall now proceed to describe, and then to lay down their fundamental geometrical relations.

I. DEFINITIONS AND NOTATION.

Let AOB be an angle, and with the unit of length, as the radius, describe a circle, commencing at A, and proceeding towards and through B (in all the figures annexed passing upwards from the line OA) till it again arrives at A. Through O draw the diameter CL perpendicular to OA, and produce OA to meet the circle at K. From B, the intersection of AB with the circle, draw BD, BE, perpendicular to OA, OC, (produced if necessary,) and from A and C draw tangents to the circle meeting OB in F and G.



1. Then denoting the arc AB by a, BC is called the *complement* of a, and BK the *supplement* of a.

2. The semicircle ACK is usually denoted by π, and consequently AC the quadrant by $\frac{1}{2}\pi$. Whence $\frac{1}{2}\pi - a$, and $\pi - a$, are respectively the complement and supplement of a.

3. The perpendicular BD is called the *sine* of a, and written for brevity $\sin a$.

4. The portion of the tangent AF cut off by the other line OB of the angle AOB, is called the *tangent* of a, and written $\tan a$.

5. The portion OF of the line OB intercepted between O and the tangent is called the *secant* of a, and written $\sec a$.

6. The distance between the sine and the arc, estimated on OA, viz. AD, is called the *versed sine* of a, and written $\text{vers}\, a$.

7. The line BE = OD is the *sine of the complement* of a, or $\sin(\frac{1}{2}\pi - a)$ It is called for brevity of expression, the *cosine* of a, and is written $\cos a$.

8. The lines CG, OG, CE, are in like manner the tangent, secant, and versed sine of the complement CB of the arc AB; and are hence respectively called, for the same reason as before, the *cotangent, cosecant,* and *coversed sine* of a; and written $\cot a$, $\text{cosec}\, a$, and $\text{covers}\, a$.

9. When we have occasion to calculate *numerically* the arcs concerned as measures of the angles, the unit by which they are commonly estimated is a *degree,* of which 360 make up the entire circumference. Hence a semicircle contains 180, and a quadrant 90 degrees. These degrees are subdivided into minutes, of

which 60 make a degree, and each minute again into 60 seconds. All subdivisions of seconds are expressed as decimals of a second. The notation for degrees, minutes, and seconds, is, the marks $^\circ$, $'$, $''$, written in the place usually assigned in algebra to the indices of powers. Thus $37^\circ\ 15'\ 18''\cdot279$ signifies 37 degrees, 15 minutes, and 18·279 seconds *.

10. For the general investigation of theorems, angles are denoted by A, B, C, ... or a, b, c ...., or α, β, γ, ... : but in the investigation of the solutions of general problems, not specifically relating to mere triangles, the data are expressed by α, β, γ, ... and the unknowns by later letters of the Greek alphabet, as $\phi, \theta, \chi, \omega$.

11. When the investigation relates to a triangle, whether of a theorem or problem, the angles are usually denoted by A, B, C, and the sides respectively opposite them by a, b, c.

II. RELATIONS AMONGST THE TRIGONOMETRICAL FUNCTIONS OF A SINGLE ARC.

In the several figures we have by right-angled triangles (*Euc. i.* 47, or *th.* 34), and similar triangles (*Euc. vi.* 2, or *th.* 72).

Geometrical properties.	*Corresponding equations.*
$DB^2 + OD^2 = OB^2$	$\sin^2 a + \cos^2 a = 1$ .... (1)
$OA^2 + AF^2 = OF^2$	$1 + \tan^2 a = \sec^2 a$ .... (2)
$OC^2 + CG^2 = OG^2$	$1 + \cot^2 a = \operatorname{cosec}^2 a$ ...... (3)
$OD : DB :: OA : AF = \dfrac{OA.DB}{OD}$	$\tan a = \dfrac{\sin a}{\cos a}$ .... (4)
$OE : EB :: OC : CG = \dfrac{OC.EB}{OE}$	$\cot a = \dfrac{\cos a}{\sin a}$ .... (5)
$OD : OB :: OA : OF = \dfrac{OA.OB}{OD}$	$\sec a = \dfrac{1}{\cos a}$ .... (6)
$OE : OF :: OC : OG = \dfrac{FO.OC}{EO}$	$\operatorname{cosec} a = \dfrac{1}{\sin a}$ .... (7)
$AF : AO :: OC : CG = \dfrac{AO.OC}{AF}$	$\cot a = \dfrac{1}{\tan a}$ .... (8)
$AD = AO - OD$	$\operatorname{vers} a = 1 - \cos a$ .... (9)
$CE = OC - OE$	$\operatorname{covers} a = 1 - \sin a$ .... (10)

By the established principle respecting the signification of + and — as signs of geometrical position, (see p. 414,) we shall be able to determine in which of the four successive quadrants the point B is situated, from the value of any two of the functions being given with their proper signs prefixed.

---

* The French division of the quadrant is into 100 parts or *grades*, each grade into 100 *centimes*, and so on, each part being successively divided into 100 of the next. This has the advantage of rendering the work entirely decimal: but its disadvantage is, that it requires all preceding tables and astronomical observations to be transformed to suit this division—a work of too much labour to encourage even a hope of it ever being performed. It would also render a great number of excellent instruments almost useless. The conversion in any given case is easily performed by the equation $E = \frac{9}{10}F$, or $F = \frac{10}{9}E$; where E signifies any given number of English degrees, and F the corresponding of French grades: or again, in this form, n degrees $= (n + \frac{1}{9}n)$ grades, and n grades $= (n - \frac{1}{10}n)$ degrees.

In order to give unity to the entire system, let all the functions of arcs less than a quadrant, that is, of AB in the first figure, be taken positive. Then the sine is positive whilst it lies above the line KA, and consequently negative when below. The cosine is positive whilst it lies to the right of CL, and negative when to the left. The tangent is positive when it is above the point A, and negative when below. The cotangent is positive when to the right of C, and negative when to the left. The secant is positive when the line proceeds from O through B to meet the tangent, and negative when it proceeds from B through O to meet the tangent; or in other words, it is positive when B lies between O and F, and negative when B is on the opposite side of O from F. The cosecant is positive when B is between O and G, and negative when on the opposite of O from G. The versed sine and coversed sine are + in all the quadrants, since they do not change the direction in which they are estimated. All these relations are consistent with those in the preceding section, and the sine and cosine being given with their proper signs, the same conclusions may be obtained by means of those relations.

It also appears from this, that if we take the arc *negatively*, or proceeding from A in a contrary direction round the circle from that assumed as positive, the several functions will be the same as those of $2\pi - a$. Hence

$$\begin{array}{ll}
\sin(-a) = \sin(2\pi - a) = -\sin a & \cos(-a) = \cos(2\pi - a) = +\cos a \\
\tan(-a) = \tan(2\pi - a) = -\tan a & \cot(-a) = \cot(2\pi - a) = -\cot a \\
\sec(-a) = \sec(2\pi - a) = +\sec a & \operatorname{cosec}(-a) = \operatorname{cosec}(2\pi - a) = -\operatorname{cosec} a \\
\operatorname{vers}(-a) = \operatorname{vers}(2\pi - a) = +\operatorname{vers} a & \operatorname{covers}(-a) = \operatorname{covers}(2\pi - a) = +\operatorname{covers} a
\end{array}$$

The fifth and subsequent quadrants, being as to position only repetitions of the first four, the signs of position of the several repetitions will be identical with those of the first four quadrants taken in order. It will therefore be necessary to tabulate only the first four.

	sin a	cos a	tan a	cot a	sec a	cosec a	vers a and covers a
1st, 5th quadts.	+	+	+	+	+	+	+
2nd, 4th......	+	−	−	−	−	+	+
3rd, 6th......	−	−	+	+	−	−	+
4th, 8th......	−	+	−	−	+	−	+

For the particular values of these functions at the particular values of a, viz. 0°, 90°, 180°, 270°, 360°,.... we have, by a simple observation of the several figures, taking into account their known geometrical properties:—

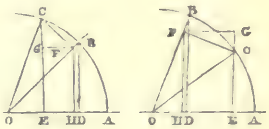
when $a = 0^\circ$	when $a = 90^\circ$	when $a = 180^\circ$	when $a = 270^\circ$
$\sin a = 0$	$\sin a = 1$	$\sin a = 0$	$\sin a = -1$
$\cos a = 1$	$\cos a = 0$	$\cos a = -1$	$\cos a = 0$
$\tan a = 0$	$\tan a = \frac{1}{0}$	$\tan a = 0$	$\tan a = -\frac{1}{0}$
$\cot a = \frac{1}{0}$	$\cot a = 0$	$\cot a = -\frac{1}{0}$	$\cot a = 0$
$\sec a = 1$	$\sec a = \frac{1}{0}$	$\sec a = -1$	$\sec a = \frac{1}{0}$
$\operatorname{cosec} a = \frac{1}{0}$	$\operatorname{cosec} a = 1$	$\operatorname{cosec} a = -\frac{1}{0}$	$\operatorname{cosec} a = -1$
$\operatorname{vers} a = 0$	$\operatorname{vers} a = 1$	$\operatorname{vers} a = 2$	$\operatorname{versin} a = 1$
$\operatorname{covers} a = 1$	$\operatorname{covers} a = 0$	$\operatorname{covers} a = 1$	$\operatorname{covers} a = 2$

and the values at the end of the fourth, fifth, and successive quadrants, repetitions of these severally.

We might also here find the values for some specific angles, as of 30°, 45°, 60°; but a more convenient place will occur hereafter. (*Chap*. vi.)

III. ON FUNCTIONS OF TWO ARCS.

To find the sine and cosine of the sum and difference of two arcs, in terms of the sines and cosines of the arcs themselves.



Let AB, BC, be two arcs; draw BD, CE, perpendicular to OA, and CF perpendicular to OB. Then BD, CF, are the sines, and OD, OF, are the cosines of the arcs AB, BC; and CE is the sine, and OE the cosine of their sum in fig. 1, and of their difference in fig. 2. Draw FH perpendicular to AO, and FG perpendicular to CE.

In both figures, the triangles OFH, OBD, CGF, are similar; and therefore

$$\begin{aligned}
&HF\ (= GE) : FO :: DB : BO;\ \text{or}\ GE.OB = OF.BD\\
&GC : CF :: DO : OB;\ \text{or}\ CG.OB = CF.OD\\
&HO : OF :: DO : OB;\ \text{or}\ HO.OB = FO.OD\\
&GF\ (= HE) : FC :: DB : BO;\ \text{or}\ HE.OB = CF.BD
\end{aligned}$$

By adding and subtracting the first pair of these, we get the following forms,

$$CE.OB = OF.BD + CF.OD \text{ in } \textit{fig. } 1.$$
$$CE.OB = OF.BD - CF.OD \text{ in } \textit{fig. } 2.$$

Similarly, subtracting and adding the last pair, we get the following forms,

$$OE.OB = FO.OD - CF.BD \text{ in } \textit{fig. } 1.$$
$$OE.OB = FO.OD + CF.BD \text{ in } \textit{fig. } 2.$$

Put AB $= a$, and BC $= \beta$: then, writing the results just obtained in terms of a and β, and recollecting that the radius OB $= 1$, we have at once

$$\sin(a+\beta) = \sin a \cos\beta + \cos a \sin\beta \quad \ldots\ldots \quad (1)$$
$$\cos(a+\beta) = \cos a \cos\beta - \sin a \sin\beta \quad \ldots\ldots \quad (2)$$
$$\sin(a-\beta) = \sin a \cos\beta - \cos a \sin\beta \quad \ldots\ldots \quad (3)$$
$$\cos(a-\beta) = \cos a \cos\beta + \sin a \sin\beta \quad \ldots\ldots \quad (4)$$

Combining these by addition and subtraction we obtain the following values:

$$\sin(a+\beta) + \sin(a-\beta) = 2\sin a \cos\beta \quad \ldots\ldots \quad (5)$$
$$\sin(a+\beta) - \sin(a-\beta) = 2\cos a \sin\beta \quad \ldots\ldots \quad (6)$$
$$\cos(a+\beta) + \cos(a-\beta) = 2\cos a \cos\beta \quad \ldots\ldots \quad (7)$$
$$\cos(a-\beta) - \cos(a+\beta) = 2\sin a \sin\beta \quad \ldots\ldots \quad (8)$$

Again, $a = \frac{1}{2}(a+\beta) + \frac{1}{2}(a-\beta)$ and $\beta = \frac{1}{2}(a+\beta) - \frac{1}{2}(a-\beta)$. Hence applying formula (1, 3,) to these values of a, β, we get,

$$\sin a = \sin\tfrac{1}{2}(a+\beta)\cos\tfrac{1}{2}(a-\beta) + \cos\tfrac{1}{2}(a+\beta)\sin\tfrac{1}{2}(a-\beta) \ldots (9)$$
$$\sin\beta = \sin\tfrac{1}{2}(a+\beta)\cos\tfrac{1}{2}(a-\beta) - \cos\tfrac{1}{2}(a+\beta)\sin\tfrac{1}{2}(a-\beta) \ldots (10)$$

By adding and subtracting these two equations, there results at once,

$$\sin a + \sin\beta = 2\sin\tfrac{1}{2}(a+\beta)\cos\tfrac{1}{2}(a-\beta) \ldots\ldots (11)$$
$$\sin a - \sin\beta = 2\cos\tfrac{1}{2}(a+\beta)\sin\tfrac{1}{2}(a-\beta) \ldots\ldots (12)$$

Also applying (2, 4,) to the values of a, β, as before,

$$\cos a = \cos\tfrac{1}{2}(a+\beta)\cos\tfrac{1}{2}(a-\beta) - \sin\tfrac{1}{2}(a+\beta)\sin\tfrac{1}{2}(a-\beta) \ldots (13)$$
$$\cos\beta = \cos\tfrac{1}{2}(a+\beta)\cos\tfrac{1}{2}(a-\beta) + \sin\tfrac{1}{2}(a+\beta)\sin\tfrac{1}{2}(a-\beta) \ldots (14)$$

And by addition and subtraction of these we obtain,

$$\cos a + \cos\beta = 2\cos\tfrac{1}{2}(a+\beta)\cos\tfrac{1}{2}(a-\beta) \ldots\ldots (15)$$
$$\cos\beta - \cos a = 2\sin\tfrac{1}{2}(a+\beta)\sin\tfrac{1}{2}(a-\beta) \ldots\ldots (16)$$

Dividing the results already obtained, as indicated on the left side of the page, we get successively,

$$\frac{(11)}{(12)} = \frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \frac{\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)} = \frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)} \quad \ldots\ldots \quad (17)$$

$$\frac{(15)}{(16)} = \frac{\cos\alpha + \cos\beta}{\cos\beta - \cos\alpha} = \frac{\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)} = \frac{\cot\frac{1}{2}(\alpha+\beta)}{\cot\frac{1}{2}(\alpha-\beta)} \quad \ldots\ldots \quad (18)$$

$$\frac{(11)}{(15)} = \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \frac{\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)} = \tan\tfrac{1}{2}(\alpha+\beta) \quad \ldots\ldots \quad (19)$$

$$\frac{(12)}{(16)} = \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \frac{\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)} = \cot\tfrac{1}{2}(\alpha+\beta) \quad \ldots\ldots \quad (20)$$

Again, since these equations are true for all values of α and β, they are true when any given values of α and β are doubled; that is, when 2α and 2β are written for α and β. This substitution being made in the equations we find,

$$\sin 2\alpha + \sin 2\beta = 2\sin(\alpha+\beta)\cos(\alpha-\beta) \quad \ldots\ldots \quad (21)$$

$$\sin 2\alpha - \sin 2\beta = 2\cos(\alpha+\beta)\sin(\alpha-\beta) \quad \ldots\ldots \quad (22)$$

$$\cos 2\alpha + \cos 2\beta = 2\cos(\alpha+\beta)\cos(\alpha-\beta) \quad \ldots\ldots \quad (23)$$

$$\cos 2\beta - \cos 2\alpha = 2\sin(\alpha+\beta)\sin(\alpha-\beta) \quad \ldots\ldots \quad (24)$$

$$\frac{(21)}{(22)} = \frac{\sin 2\alpha + \sin 2\beta}{\sin 2\alpha - \sin 2\beta} = \frac{\tan(\alpha+\beta)}{\tan(\alpha-\beta)} \quad \ldots\ldots \quad (25)$$

$$\frac{(23)}{(24)} = \frac{\cos 2\alpha + \cos 2\beta}{\cos 2\alpha - \cos 2\beta} = \cot(\alpha+\beta)\cot(\alpha-\beta) \quad \ldots\ldots \quad (26)$$

$$\frac{(21)}{(23)} = \frac{\sin 2\alpha + \sin 2\beta}{\cos 2\alpha + \cos 2\beta} = \tan(\alpha+\beta) \quad \ldots\ldots \quad (27)$$

$$\frac{(22)}{(24)} = \frac{\sin 2\alpha - \sin 2\beta}{\cos 2\beta - \cos 2\alpha} = \cot(\alpha+\beta) \quad \ldots\ldots \quad (28)$$

The formulæ here given enable us, in connection with those of *chap.* II., to obtain any of the other functions of the sum and difference of two arcs, as well as some useful formulæ relating to two arcs which have specified relations. Also by supposing one of the arcs to be itself the sum or difference of two arcs, some useful results are obtained; though from the complexity of the formulæ this inquiry has never been carried to great extent. Indeed, transformation is the main object of all these researches; inasmuch as they enable us to put the expression under different forms, better adapted to facilitate investigations and calculations, than they appear originally in the proposition. The diversity of forms that may be given would evidently be very great; but in an elementary work only those which are of frequent utility can possibly find a place.

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta} = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \quad (29)$$

$$\cot(\alpha \pm \beta) = \frac{\cos(\alpha \pm \beta)}{\sin(\alpha \pm \beta)} = \frac{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta}{\sin\alpha\cos\beta \pm \cos\alpha\cos\beta} = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha} \quad (30)$$

$$\tan\alpha \pm \tan\beta = \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta} = + \frac{\sin(\alpha \pm \beta)}{\cos\alpha\cos\beta} \quad \ldots\ldots \quad (31)$$

$$\cot\alpha \pm \cot\beta = \frac{\cos\alpha\sin\beta \pm \sin\alpha\cos\beta}{\sin\alpha\sin\beta} = \pm \frac{\sin(\alpha \pm \beta)}{\sin\alpha\sin\beta} \quad \ldots\ldots \quad (32)$$

$$\frac{\tan\alpha \mp \tan\beta}{\tan\alpha \pm \tan\beta} = \frac{\sin\alpha\cos\beta \mp \cos\alpha\sin\beta}{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta} = + \frac{\sin(\alpha \mp \beta)}{\sin(\alpha \pm \beta)} \quad \ldots\ldots \quad (33)$$

$$\frac{\cot\alpha \mp \cot\beta}{\cot\alpha \pm \cot\beta} = \frac{\cos\alpha\sin\beta \mp \cos\beta\sin\alpha}{\cos\alpha\sin\beta \pm \cos\beta\sin\alpha} = - \frac{\sin(\alpha \mp \beta)}{\sin(\alpha \pm \beta)} \quad \ldots\ldots \quad (34)$$

Again we have the following very useful transformations,

$$\sin(\alpha+\beta)\sin(\alpha-\beta) = (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$
$$= \sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta = \sin^2\alpha - \sin^2\beta \quad \ldots\ldots \quad (35)$$

$$\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta = \cos^2\alpha - \sin^2\beta \quad (36)$$

IV. PARTICULAR RELATIONS AMONGST THE ARCS.

1. Let $a = n\beta$. Then we have $a \pm \beta = (n \pm 1)\beta$. Whence

$$\sin (n \pm 1)\beta = \sin n\beta \cos \beta \pm \cos n\beta \sin \beta \qquad (1)$$

$$\cos (n \pm 1)\beta = \cos n\beta \cos \beta \mp \sin n\beta \sin \beta \qquad (2)$$

Adding and subtracting the two forms of (1) we have,

$$\sin (n+1)\beta + \sin (n-1)\beta = 2 \sin n\beta \cos \beta \qquad (3)$$

$$\sin (n+1)\beta - \sin (n-1)\beta = 2 \cos n\beta \sin \beta \qquad (4)$$

Similarly from equation (2) we get,

$$\cos (n-1)\beta + \cos (n+1)\beta = 2 \cos n\beta \cos \beta \qquad (5)$$

$$\cos (n-1)\beta - \cos (n+1)\beta = 2 \sin n\beta \sin \beta \qquad (6)$$

2. Let $a = \beta$, or $n = 1$. Then $a + \beta = 2a$; and we get,

$$\sin 2a = 2 \sin a \cos a \qquad (7)$$

$$\cos 2a = \cos^2 a - \sin^2 a \qquad (8)$$

But (8) may be changed into $(1 - \sin^2 a) - \sin^2 a$ or $\cos^2 a - (1 - \cos^2 a)$, and hence,

$$\cos 2a = 1 - 2 \sin^2 a; \text{ whence } \sin^2 a = \frac{1 - \cos 2a}{2} \qquad (9)$$

$$\cos 2a = 2 \cos^2 a - 1; \text{ whence } \cos^2 a = \frac{1 + \cos 2a}{2} \qquad (10)$$

$$\frac{(9)}{(10)} \text{ gives } \frac{\sin^2 a}{\cos^2 a} = \tan^2 a = \frac{1 - \cos 2a}{1 + \cos 2a} \qquad (11)$$

And the same holds good if for a we write $\frac{1}{2}a$ in the three last formulæ, giving,

$$\sin^2 \tfrac{1}{2}a = \frac{1 - \cos a}{2}, \cos^2 \tfrac{1}{2}a = \frac{1 + \cos a}{2}, \tan^2 \tfrac{1}{2}a = \frac{1 - \cos a}{1 + \cos a} \qquad (12)$$

But we may proceed differently, and get another pair of useful forms for $\tan \frac{1}{2}a$; thus, put $\beta = \frac{1}{2}a$, then $a - \beta = \frac{1}{2}a$.

$$\sin (a - \tfrac{1}{2}a) = \sin \tfrac{1}{2}a = \sin a \cos \tfrac{1}{2}a - \cos a \sin \tfrac{1}{2}a: \text{ whence,}$$

$$(1 + \cos a) \sin \tfrac{1}{2}a = \sin a \cos \tfrac{1}{2}a, \text{ or } \tan \tfrac{1}{2}a = \frac{\sin a}{1 + \cos a} \qquad (13)$$

$$\cos (a - \tfrac{1}{2}a) = \cos \tfrac{1}{2}a = \cos a \cos \tfrac{1}{2}a + \sin a \sin \tfrac{1}{2}a: \text{ whence,}$$

$$(1 - \cos a) \cos \tfrac{1}{2}a = \sin \tfrac{1}{2}a \sin a, \text{ or } \tan \tfrac{1}{2}a = \frac{1 - \cos a}{\sin a} \qquad (14)$$

Two other formulæ may be thus obtained which are often useful. We have $\sin^2 a + \cos^2 a = 1$, and $2 \sin a \cos a = \sin 2a$. Hence, by addition and subtraction,

$$\sin^2 a + 2 \sin a \cos a + \cos^2 a = 1 + \sin 2a.$$

$$\sin^2 a - 2 \sin a \cos a + \cos^2 a = 1 - \sin 2a.$$

Extracting the roots, and adding and subtracting the results,

$$\sin a = \tfrac{1}{2} \left\{\sqrt{1 + \sin 2a} \pm \sqrt{1 - \sin 2a}\right\} \qquad (15)$$

$$\cos a = \tfrac{1}{2} \left\{\sqrt{1 + \sin 2a} \mp \sqrt{1 - \sin 2a}\right\} \qquad (16)$$

Many other formulæ respecting double and half arcs are easily obtained; but as they are not of frequent use in elementary study, they are left to the student's choice to pursue or not.

3. Multiple arcs are generally most elegantly expanded by Demoivre's theorem, hereafter to be given; but as for small multiples, they frequently occur in early stages of trigonometry. One method, that of successive deduction, is indicated, rather by example than precept, but sufficient for the present purpose. It will be kept in mind that $\sin 2a$ and $\cos 2a$ have been found in (7, 8).

Then $\sin 3a = \sin(2a + a) = \sin 2a \cos a + \cos 2a \sin a$
$= 2 \sin a (1 - \sin^2 a) + \sin a (1 - 2\sin^2 a) = 3 \sin a - 4 \sin^3 a.$
$\cos 3a = \cos(2a + a) = \cos a \cos 2a - \sin 2a \sin a = -3 \cos a + 4 \cos^3 a.$
Similarly $\sin 4a = \sin(3a + a)$, $\cos 4a = \cos(3a + a)$, and so on.

V. THE EXPRESSIONS FOR $\sin(a \pm \beta)$ AND $\cos(a \pm \beta)$ WHEN a IS SOME WHOLE NUMBER OF QUADRANTS.

Let $a = \frac{\pi}{2}$; then, $\sin\frac{\pi}{2} = 1$, and $\cos\frac{\pi}{2} = 0$; and we have,

$$\sin\left\{\frac{\pi}{2} - \beta\right\} = \sin\frac{\pi}{2}\cos\beta - \cos\frac{\pi}{2}\sin\beta = +\cos\beta,$$

$$\cos\left\{\frac{\pi}{2} - \beta\right\} = \cos\frac{\pi}{2}\cos\beta + \sin\frac{\pi}{2}\sin\beta = +\sin\beta,$$

$$\sin\left\{\frac{\pi}{2} + \beta\right\} = \sin\frac{\pi}{2}\cos\beta + \cos\frac{\pi}{2}\sin\beta = +\cos\beta,$$

$$\cos\left\{\frac{\pi}{2} + \beta\right\} = \cos\frac{\pi}{2}\cos\beta - \sin\frac{\pi}{2}\sin\beta = -\sin\beta.$$

Let $a = \pi$: then $\sin\pi = 0$ and $\cos\pi = -1$; and we have,

$$\sin(\pi - \beta) = \sin\pi\cos\beta - \cos\pi\sin\beta = +\sin\beta$$
$$\cos(\pi - \beta) = \cos\pi\cos\beta + \sin\pi\sin\beta = -\cos\beta$$
$$\sin(\pi + \beta) = \sin\pi\cos\beta + \cos\pi\sin\beta = -\sin\beta$$
$$\cos(\pi + \beta) = \cos\pi\cos\beta - \sin\pi\sin\beta = -\cos\beta$$

Let $a = \frac{3\pi}{2}$; then $\sin\frac{3\pi}{2} = -1$, and $\cos\frac{3\pi}{2} = 0$; and we have,

$$\sin\left\{\frac{3\pi}{2} - \beta\right\} = \sin\frac{3\pi}{2}\cos\beta - \cos\frac{3\pi}{2}\sin\beta = -\cos\beta$$

$$\cos\left\{\frac{3\pi}{2} - \beta\right\} = \cos\frac{3\pi}{2}\cos\beta + \sin\frac{3\pi}{2}\sin\beta = -\sin\beta$$

$$\sin\left\{\frac{3\pi}{2} + \beta\right\} = \sin\frac{3\pi}{2}\cos\beta + \cos\frac{3\pi}{2}\sin\beta = -\cos\beta$$

$$\cos\left\{\frac{3\pi}{2} + \beta\right\} = \cos\frac{3\pi}{2}\cos\beta - \sin\frac{3\pi}{2}\sin\beta = +\sin\beta$$

Let $a = 2\pi$: then $\sin 2\pi = 0$, and $\cos 2\pi = 1$; and we have similarly,

$$\sin(2\pi - \beta) = \sin 2\pi\cos\beta - \cos 2\pi\sin\beta = -\sin\beta$$
$$\cos(2\pi - \beta) = \cos 2\pi\cos\beta + \sin 2\pi\sin\beta = +\cos\beta$$
$$\sin(2\pi + \beta) = \sin 2\pi\cos\beta + \cos 2\pi\sin\beta = +\sin\beta$$
$$\cos(2\pi + \beta) = \cos 2\pi\cos\beta - \sin 2\pi\sin\beta = +\cos\beta$$

and for all additions of quadrants these values will be repeated in the same order. The same results might have been inferred by combining the four figures at p. 422 in one, and reasoning from known geometrical relations.

The tangents and co-tangents might also be inferred from the figure in the same way; or they might be obtained from these results by the equation $\tan\gamma = \frac{\sin\gamma}{\cos\gamma}$, and $\cot\gamma = \frac{\cos\gamma}{\sin\gamma}$. The secant and cosecant also being the reciprocals of the cosine and sine have the same signs as those functions.

Some interesting discussions of the signs connected with these functions may be found in Professor Young's *Mathematical Dissertations*, p. 8.

VI. THE VALUES OF THE TRIGONOMETRICAL FUNCTIONS OF CERTAIN ARCS.

1. To find cos 30°, sin 30°, and tan 30°. If $a = 30°$ we have $\cos 3a = \cos 90° = 0$. Whence taking the value of $\cos 3a$ from p. 428, we have,
$4\cos^3 30° - 3\cos 30° = \cos 90° = 0$. Whence $\cos^2 30° = \frac{3}{4}$, or $\cos 30 = \frac{1}{2}\sqrt{3}$, and $\sin^2 30° = 1 - \cos^2 30° = 1 - \frac{3}{4} = \frac{1}{4}$; or $\sin 30° = \frac{1}{2}$. Whence also $\tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$.

2. Find sin 45°, cos 45°, and tan 45°. As before, $\cos 2.45° = \cos 90° = 0$; and hence $2\cos^2 45° - 1 = 0$; or $\cos^2 45° = \frac{1}{2}$, and $\sin^2 45° = 1 - \cos^2 45° = \frac{1}{2}$; and we have $\cos 45° = \frac{1}{2}\sqrt{2}$, $\sin 45° = \frac{1}{2}\sqrt{2}$, and $\tan 45° = 1$.

3. Find sin 60°, cos 60°, tan 60°.

These may be inferred from (1), since the sine, cosine, and tangent of 30°, are the cosine, sine, and cotangent of 60°; but for illustration of the method of proceeding, the investigation is annexed independently of 30°.
$\sin 3.60° = \sin 180° = 0$; whence, as before, $3\sin 60° - 4\sin^3 60° = 0$, or $\sin^2 60° = \frac{3}{4}$, $\cos^2 60° = 1 - \frac{3}{4} = \frac{1}{4}$; and $\sin 60° = \frac{1}{2}\sqrt{3}$, $\cos 60° = \frac{1}{2}$, and $\tan 60° = \sqrt{3}$.

Having obtained the functions of these arcs, (the final expressions for which are the simplest that occur throughout the quadrant, for any arcs,) we can continually obtain their halves or doubles: their halves by the resolution of quadratic equations, and their doubles by squaring certain functions of the sine and cosines already obtained. We can also obtain *expressions* containing the functions of the third part of an arc by the resolution of cubic equations, and so on to any extent. Two examples are annexed, to find the functions of 15° and 10°, the half and the third parts of 30°.

$\cos 30° = \cos 2.15° = 2\cos^2 15° - 1 = \frac{1}{2}\sqrt{3}$; hence

$\cos^2 15° = \frac{1}{4}(2 + \sqrt{3})$, and $\sin^2 15° = 1 - \cos^2 15° = \frac{1}{4}(2 - \sqrt{3})$.

Hence, extracting the roots, we have the following expressions of value:
$\cos 15° = \frac{1}{2}\{\sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}}\}$, $\sin 15° = \frac{1}{2}\{\sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}}\}$, $\tan 15° = 2 - \sqrt{3}$.

Again, for sin 10° we have $\sin 3.10° = -4\sin^3 10° + 3\sin 10°$: hence $4\sin^3 10° - 3\sin 10° + \frac{1}{2} = 0$, and by Cardan's formula, we have,
$\sin 10° = \frac{1}{2}\sqrt[3]{-\frac{1}{2} + \frac{1}{2}\sqrt{-3}} + \frac{1}{2}\sqrt[3]{-\frac{1}{2} - \frac{1}{2}\sqrt{-3}}$.

This expression taking an imaginary form, indicates that all three roots are real, whilst neither of them can be exhibited in a real form by such a process. The same circumstance happens universally in obtaining the sine or cosine of an arc, by supposing it a third part of an arc whose sine or cosine are given, except when that given sine or cosine is 0. The method of trisection, therefore, is inapplicable to the finding of *useful expressions* * for these functions; but it gives an opportunity of making a remark which will be further expanded in the second volume.

We have $\sin 30° = \sin 390° = \sin 750° = \frac{1}{2}$; and hence the problem which is virtually put into equations, has in reality three different cases, according as we suppose these three angles to be trisected. Hence the roots are sin 10°, sin 130°, and sin 250°, all which are real, and answer to the real roots of the equation before found. We might, hence, have anticipated this result: and, indeed, the

* However, in all cases the values of the roots can be readily calculated by Horner's method; and as the same reasoning will apply to every section of a given arc, it is quite clear that we can always *actually compute* any function of any given part of an arc or its angle, when we are in possession of the value of any one of its trigonometrical functions.

double values of the radicals in the solution of the other problems indicate the same kind of circumstance, viz. two values of the sine, cosine, and tangent sought; which, on the same principle, were indications of the sine, cosine, and tangent of $\frac{1}{2}\,a$ and $\pi + \frac{1}{2}\,a$.

The surd values of the sines, and hence of the cosines, and the tangents which may be obtained from them, are given in the *Introduction to Hutton's Tables*, p. xxxix. for every third degree of the quadrant. The deduction of these will furnish sufficient exercise to the student.

VII. THE CALCULATION OF TRIGONOMETRICAL FUNCTIONS.

These functions can be expressed in a series of positive integer powers of the arc itself, and the coefficients of the series determined; and conversely, the arc can be expressed in a series of positive integer powers of any one of these functions. These series may be found either by indeterminate coefficients, or by the differential and integral calculus. The former method, however, is laborious; and the latter implies a degree of acquirement beyond our present progress. Hence, we shall adopt a more simple method of proceeding in this place, leaving the deduction of the series in question for its proper analytical position in the *Course*.

The method is founded on the principle, that in very small arcs the sine varies very nearly as the arc itself. For let a be a minute arc, and β one more minute, by which a is increased. Then $\sin(a+\beta) = \sin a \cos\beta + \cos a \sin\beta$; and since a and β are minute arcs, $\cos a$ and $\cos\beta$ are very nearly equal to unity. Hence, taking them actually as unity, we get $\sin(a+\beta) = \sin a + \sin\beta$, and the arc, therefore, increases nearly as the sine, when these arcs are very small.

Now, by IV. 15, we have $\sin a = \frac{1}{2}\left\{\sqrt{1+\sin 2a} - \sqrt{1-\sin 2a}\right\}$.

If, then, we put $2a$ successively equal to $30°, \frac{30°}{2}, \frac{30°}{2^2} \ldots \frac{30°}{2^{11}}$ and compute the several sines, we at last arrive at $\sin\frac{30°}{2^{11}} = \sin\frac{225}{256}.\,1' = \cdot 000255625$, and hence $\sin 1' = \frac{256}{225}\,.\,\sin\frac{30°}{2^{11}} = \cdot 0002908882$ nearly.

The cosine, tangent, or any other function of $1'$ can now be obtained, as $\cos 1' = \sqrt{1-\sin^2 1'} = \cdot 9999999577$, and $\tan 1' = \frac{\sin 1'}{\cos 1'} = \cdot 0002908882$, whence so far as the first ten decimals, there is no difference between the sine and tangent of $1'$.

Again, from (IV. 3,) we have $\sin(n+1)\beta = 2\sin n\beta \cos\beta - \sin(n-1)\beta$: and if we put $n = 1, 2, 3, \ldots\ 1799$, and $\beta = 1'$, we shall be able to calculate the sines of all angles from 0° to 30°, for every minute of a degree; and, concurrently, all the other trigonometrical functions of the same arcs.

To calculate those from 30° to 45°, we may use the formula thus obtained:—

$\sin(30°+\beta) + \sin(30°-\beta) = 2\sin 30°\cos\beta = \cos\beta$, from which,

$\sin(30°+\beta) = \cos\beta - \sin(30°-\beta)$

and making β successively equal to $1', 2', 3', \ldots\ 899'$, we shall obtain the sines, and thence the other functions of the arcs from 30° to 45° inclusively.

Also, the sine of any arc is the cosine of its complement; and hence, as we have computed all the complementary functions, we have the direct functions of all arcs from 45° to 90°, and the functions of the entire quadrant are computed.

The functions of arcs greater than 90° are at once obtained from the equations *chapter* V., p. 428.

VIII. THE CONSTRUCTION AND USE OF THE TABLES OF TRIGONOMETRICAL FUNCTIONS.

1. Since by the preceding method we can calculate the sines to radius 1, of all the angles from 1′ up to 90°, we may suppose them prepared for tabulation; and thence also by means of the relations deduced in *chapter* II. all the other functions. In Hutton's tables they are computed to seven decimal places. On each page are given the values of all the functions, *sin, cos, tan, cot, sec, cosec, versin,* and *coversin*, of all the minutes from p degrees to $(p + 1)$ degrees inclusive. The number p, if under 45°, is found at the head of the page to the left; and if 45° or upwards to 90°, at the bottom of the page to the right. The minutes, if p be less than 45°, are numbered from the top 0′, to 60′ at the bottom, the numbers being the left column of the page; but if 45° or upwards, they range from the bottom 0′ to the top 60′, and constitute the right column of the page. The name of each column of functions is placed at the top or bottom as p is less or greater than 45°.

It will also appear that the degrees at the top and the minutes at the left side, together with the degrees at the bottom and the minutes at the right side, of any horizontal column, together make 90°; or in other words, that any given function of a given arc is the complementary function of the complement of that arc in the structure of the tables. Thus sin 9° 10′ = ·1593069 = cos 80° 50′ (see page 286 of the Tables), and so of the other functions. This is an arrangement depending on the equation sin a = cos $(90 - a)$, and reduces the table to half the dimensions it would otherwise require to carry the functions up to 90°. These natural sines, natural cosines, etc. are always placed on the left page whenever we open the tables, and headed "NATURAL SINES, &c." The differences between the sines and between the cosines of each two consecutive arcs differing by 1′, are placed in columns and adjacent to them, marked "DIFFERENCES;" thus sin 9° 10′ — sin 9° 9′ = ·1593069 — ·1590197 = ·0002872, the effective figures 2872 of which is found on a line lying horizontally between sin 9° 9′ and 9° 10′; or again cos 9° 9′ — cos 9° 10′ = ·9871827 — ·9871363 = ·0000464, and the effective figures 464 are put down horizontally.

Again, let a be any number of degrees and minutes; then since covers $a = 1 - \sin a$, we have covers a — covers $(a + 1') = (1 - \sin a) - \{1 - \sin (a+1')\} = \sin (a + 1') - \sin a$. The differences between two consecutive coversines is equal to the difference between the sines of the same angles. The coversines of the angles are therefore put down on the opposite side of the column of differences from the sines, the same difference applying to each of the columns. For the same reason the versines are placed on the opposite side of the column of differences from the cosines. No other remark remains to be made on the table of natural functions.

The table of "LOG SINES, &c." on the right hand page is formed by taking the logarithms of the numbers on the opposite page. Thus log sin 9° 10′ = log ·1593069 = $\bar{1}$·2022345, and so of all the rest. However, to avoid the *negative indices* in the logarithms, which would create great difficulties in printing and much liability to mistakes in calculation, 10 is added to all the logarithms of the sines, etc. throughout the entire tables. Hence tabular sin a = 10 + log sin a, and hence tab. sin 9° 10′ = 10 + $\bar{1}$·2022345 = 9·2022345; and similarly with all the other functions and values of a.

The succession of columns in the two tables is different. In the table of

natural functions, the sines and coversines have the same differences; and for this reason, the sines and coversines are placed in succession with their difference-column between them; and the cosines and versines in the same manner. In the logarithms of these functions other relations exist, bringing together the sine and cosecant, the cosine and secant, and the tangent and cotangent.

For $\text{cosec } a = \frac{1}{\sin a}$, and hence $\log \text{cosec } a = - \log \sin a$; and in a similar manner $\log \text{cosec } (a + 1') = - \log \sin (a + 1)$; hence $\{10 + \log \text{cosec } a\} - \{10 + \log \text{cosec } (a + 1')\} = \{10 + \log \sin (a + 1')\} - \{10 + \log \sin a\}$ or tab cosec a — tab cosec $(a + 1')$ = tab sin $(a + 1')$ — tab sin a, whence the sines and cosecants are brought together, with the column of common differences intervening. The relations $\sec a = \frac{1}{\cos a}$ and $\cot a = \frac{1}{\tan a}$, give rise to a corresponding arrangement respecting these functions.

As the sines and tangents at the commencement of the table, and the cosines and cotangents at the end, vary very rapidly, the tabular functions for *every second* of the first two degrees are given at pp. 238—267. The structure will be evident on inspection.

2. Having explained the construction of the tables, their usage is next to be described.

To take out the sines, cosines, etc. or their tabular logarithms to degrees and minutes not greater than 90° is the immediate and first application of the tables, and the method is obvious from the construction of them already explained.

When the function is that of an angle greater than 90° we must have recourse to the results obtained at p. 428, *chapter V.*

Now $\sin a = \sin(\pi - a)$ and $\cos a = - \cos(\pi - a)$; hence

$$\tan a = \frac{\sin a}{\cos a} = - \frac{\sin(\pi - a)}{\cos(\pi - a)} = - \tan(\pi - a)$$

$$\cot a = \frac{\cos a}{\sin a} = - \frac{\cos(\pi - a)}{\sin(\pi - a)} = - \cot(\pi - a)$$

$$\sec a = \frac{1}{\cos a} = \frac{1}{\cos(\pi - a)} = - \sec(\pi - a)$$

$$\text{cosec } a = \frac{1}{\sin a} = + \frac{1}{\sin(\pi - a)} = + \text{cosec } (\pi - a)$$

$$\text{vers } a = 1 - \cos a = 1 + \cos(\pi - a)$$

$$\text{covers} a = 1 - \sin a = 1 - \sin(\pi - a)$$

Hence if a be greater than 90° and less than 180°, the trigonometrical functions of $\pi - a$ may be substituted for them, subjected to the changes of sign indicated by the above equations. Thus if $a = 96° 10'$ we have $\sin 96° 10' = \sin 83° 50'$, $\cos 96° 10' = - \cos 83° 50'$, and so on; all which values, signs excepted, fall amongst the tabulated numbers.

In the same manner, if the arc be greater than 180° and less than 270°, the corresponding values of the functions of a in terms of those of $a - \pi$ can be [illegible]; and so on to any number, p, of complete quadrants above or below the given value of a. Whence the functions of any arc, however large, can be expressed in functions of an arc not greater than 90°, and therefore the tables to 90° suffice for all the purposes of angular calculation.

The only thing remaining is the calculation of the interpolated values of the functions for any number of seconds intermediate between two consecutive minutes in the tables, and conversely the number of seconds corresponding to the excess of a given function over the less of those for two consecutive minutes

in the tables. This is effected on the principle, that (except in extreme cases) the difference of two arcs, being less than 1′, the difference of any of their trigonometrical functions is as the difference of the arcs themselves. This is sufficiently apparent as to fact, from the very slight changes of the differences between each tabular function and that of its consecutive minutes in successive steps of the increase or decrease of the arc. Thus,

nat sin θ	Δ nat sin θ	Δ^2 nat sin θ
sin 15° 0′ = ·2588190		
	·0002810	
sin 15 1 = ·2591000		·0000000
	·0002810	
sin 15 2 = ·2593810		—·0000001
	·0002809	
sin 15 3 = ·2596619		

And the same is true for any other part of the table, except at its extreme limits, and for all trigonometrical functions of the arc, both natural and logarithmic.

It will, however, conduce to clearness to show under what circumstances this law holds good, at the same time that a general mode of proving the truth of the assertion is pointed out. Now we have, nearly,

$$\frac{\sin(\theta+p'')-\sin\theta}{\sin(\theta+60'')-\sin\theta}=\frac{2\cos(\theta+\frac{1}{2}p'')\sin\frac{1}{2}p''}{2\cos(\theta+\frac{1}{2}60'')\sin\frac{1}{2}60''}=\frac{\sin\frac{1}{2}p''}{\sin\frac{1}{2}60''}=\frac{\sin p''}{\sin 60''}$$

except θ be *very small*. In this case, therefore, the difference of the *sines* is sensibly as the difference of the *arcs* within the assigned limits; and in a similar manner it may be shown for the other natural functions.

Again, for the logarithmic functions, it has already been shown (*p.* 255) that for small differences of the numbers, the differences of the logarithms are as the differences of the numbers. Hence in all the functions, except near the beginning and end of the quadrant, dif for 60″ : dif for p'' : : 60″ : p''. Hence for corrections *generally* we have, according as the difference of the arcs or the difference of the functions is given, respectively,

$$\text{dif for } p''=\frac{p''\text{ dif for }60''}{60''}\text{ and } p''=\frac{60''\text{ dif for }p''}{\text{dif for }60''}.$$

The differences for 60″ are those before spoken of, and are found by actual subtraction; but generally tabulated to save the trouble of the subtraction in each case. When arcs between 0° and 2°, or between 88° and 92°, or 178° and 182°, *etc.* are concerned, it will be necessary to use the corresponding tables, and work for decimals of seconds to one or two places at least.*

It is necessary to bear in mind that the sine, tangent, and secant (under 90° for which the tables are constructed) *increase* as the arc increases; whilst the

---

* It often happens that we have to use one function of an angle where another has been previously calculated, as, for instance, given tan θ to find sec θ, or given sin χ to find cos χ. The direct mode of proceeding is, of course, to find θ or χ to seconds from the given values of tan θ or sin χ, and thence to find the values of sec θ or cos χ as above explained. This, however, is not necessary; for the correction of the angle from the tangent, and the correction of the secant from the angle, involve six proportional terms, of which the middle ones are superfluous. Thus, if d be the difference between tan θ and the next less tabular tangent, and D the difference for 60″; and if D_1 be the tabular difference of the secant on the same horizontal line, and d_1 the corresponding correction of the secant; and finally, if p'' be the seconds corresponding to the correction of the tangent:—

$D : d_1 : : 60'' : p''$, and $60'' : p'' : : D_1 : d_1$; and, hence, $D : d : : D_1 : d_1 = \frac{dD_1}{D}$ = correction of the secant. The same holds with respect to any other functions, and in fact the correction is given rather more accurately, as well as with less work, than by finding the intermediate quantity, p''.

cosine, cotangent, and cosecant *decrease* as the arc increases. This will require the corrections connected with $\sin a$, $\tan a$, $\sec a$, to be *added*, and those connected with $\cos a$, $\cot a$, $\operatorname{cosec} a$, are to be *subtracted*, whether arcs or their functions be sought from the tables.*

For instance, given $\sin \chi = 9{\cdot}8265832$, to find $\cos \chi$. *(See note, p. 433.)*

Given $\sin \chi = 9{\cdot}8265832$; and *(p. 353 of the tables, seventh edition).*

next less sine $= 9{\cdot}8264910$, and the corresponding cosine $= 9{\cdot}8702756$

Hence $d =$	922	correction for $\cos \chi$ $=$	$-\ 754$
But $D =$	1397	and corrected $\cos \chi$ $=$	$9{\cdot}8702002$
and $D_1 =$	1143		

whence correction for $\cos \chi = \dfrac{dD_1}{D} = \dfrac{922.1143}{1397} = 754$, which is to be taken $-$.

EXAMPLES.

1. Find the tabular cosine of $28° \ 10' \ 15''$.

Here we have *(from the tables, p. 325,)* $\cos 28° \ 10' = 9{\cdot}9452609$, and

$$\text{pp } 15'' = -\frac{15 \text{ dif for } 60''}{60} = -\frac{677}{4}; \text{ or pp } 15'' = \quad -169$$

$$\cos 28° \ 10' \ 15'' = 9{\cdot}9452440$$

2. Find nat tan $212° \ 15' \ 18''$ and tab sin $169° \ 18' \ 45''$.

$$\text{Here } \tan 212° \ 15' \ 18'' = \frac{\sin 212° \ 15' \ 18''}{\cos 212 \ \ 15 \ \ 18} = \frac{\sin (180° + 32° \ 15' \ 18'')}{\cos (180 + 32° \ 15' \ 18'')}$$

$$= \frac{-\sin 32° \ 15' \ 18''}{-\cos 32° \ 15' \ 18''} = \tan 32° \ 15' \ 18''.$$

Hence nat $\tan 32° \ 15' = {\cdot}6309530$, and $\text{pp } 18'' = \dfrac{18 \text{ dif. } 60''}{60} = \dfrac{18.4068}{60}$

$$\text{or pp } 18'' = \quad +1220$$

$${\cdot}6310750 = \text{nat tan } 212° \ 15' \ 18''$$

Also $\sin 169° \ 18' \ 45'' = \sin (180° - 169° \ 18' \ 45'') = \sin 10° \ 41' \ 15''$.

Hence tab $\sin 10° \ 41' = 9{\cdot}2680647$, and $\text{pp } 15'' = \dfrac{15.6691}{60}$, or,

$$\text{pp } 15'' = \quad +1673$$

$$9{\cdot}2682320 = \text{tab sin } 169° \ 18' \ 45''$$

3. Find θ from n $\sin \theta = {\cdot}1625946$, and χ from tab $\operatorname{cosec} \chi = 10{\cdot}1653829$.

Looking in the tables for the next less nat sin and next less tab cosec we have

given nat $\sin \theta$	$= {\cdot}1625946$	Also $p'' = \dfrac{60'' \,.\, 1296}{\text{dif. } 60''} = \dfrac{60''.1296}{2870} = 27''$
nat $\sin 9° \ 21'$	$= {\cdot}1624650$	
	1296	

and hence $\theta = 9° \ 21' \ 27''$.

Again, given tab $\operatorname{cosec} \chi$	$= 10{\cdot}1653829$	Also $p'' = -\dfrac{60''.1126}{1349} = -50''$;
tab $\operatorname{cosec} 43° \ 7'$	$= 10{\cdot}1652703$	
	1126	

and hence $\theta = 43° \ 7' \ 0'' - 50'' = 43° \ 6' \ 10''$.

4. Find tab cos $50° \ 30' \ 35''$, tab cos $157° \ 10' \ 18''$, tab cot $196° \ 10' \ 18''$, and tab cosec $325° \ 10' \ 15''$.

5. Find nat cos $57° \ 18' \ 15''$, nat cot $59° \ 59' \ 59''$, and nat sec $525° \ 15' \ 58''$.

---

* It is usual to write n sin, n tan, n sec, *etc.* instead of natural sine, *etc.*; and for tab sin, tab tan, *etc.*, simply sin, tan, *etc.* Also to place the work of finding the parts in any vacant space apart from the general working formulæ.

6. (II, *p.* 34.) Find the values, natural and tabular, of the following expressions:—

(1.) sin 1° 5′ 10″ sin 91° 4′ 15″ sin 196° 10′ 18″ sin 300° 10′ 15″.

(2.) tan 18° tan 108° tan 196° tan 271° tan 305° tan 375° tan 400°.

(3.) $\dfrac{\sin(-18^\circ)\sin 367^\circ\cos 95^\circ\cos(-195^\circ)\tan 300^\circ\,10'\,16''}{\cos 18^\circ\cos(-367^\circ)\sin(-95^\circ)\cos 195^\circ\tan(-300^\circ\,10'\,16)}$.

(4.) $\dfrac{\sin 270^\circ\,10'\,16''\sin 175^\circ\,0'\,16''-\sin 536^\circ\,10'\,15''\cos 100^\circ}{\cos 17^\circ\,18'\,16''-\sin(-100^\circ\,15'\,16'')}$.

(5.) $\dfrac{\sin 100^\circ\,15'\,18''\sin 375^\circ\,18'\,16''-\cos 92^\circ\,0'\,16''\cos 325^\circ}{\cos 100^\circ\,15'\,16''-\cos 460^\circ\,15'\,52''}$.

IX. TO EXPAND SIN x AND COS x IN TERMS OF x.

ASSUME $\sin x = a + bx^\beta + cx^\gamma + dx^\delta + \ldots.$

and $\cos x = a_1 + b_1x^{\beta_1} + c_1x^{\gamma_1} + d_1x^{\delta_1} + \ldots.$

the several indices and coefficients being yet unknown.

1. Since for any given value of x there can be only a single determinate value of $\sin x$ or $\cos x$, these sines can contain *no fractional indices*. For if any one of them do contain a fractional index, it indicates multiple values of the term in which it appears, and therefore also multiple values of the sine and cosine themselves; that is, a determinate quantity has several different values, which is impossible. Hence all the indices of both series are integers.

2. These series can contain *no negative indices*. For if they can, let them be δ and δ_1. Now as the expansion is independent of the value of x, it is true for every value, and hence for $x = 0$. Now we have seen (p. 424) that $\sin 0 = 0$, and $\cos 0 = 1$; whence we shall have in this case,

$$0 = a + b0^\beta + c0^\gamma + \frac{d}{0^\delta} + e0^\epsilon + \ldots.$$

$$= a + 0 + 0 + \textit{infinity} + 0 + \ldots.$$

which is impossible. Hence the series for $\sin x$ contains no negative indices.

Nor are there any in the series for $\cos x$, since in this case, also, we should have

$$1 = a_1 + 0 + 0 + \textit{infinity} + 0 + \ldots.$$

which is, again, impossible.

The indices, therefore, of both series are positive integers.

3. Since $\sin x = \quad a + bx^\beta + cx^\gamma + dx^\delta + \ldots$

$-\sin x = -a - bx^\beta - cx^\gamma - dx^\delta - \ldots$ and (*p.* 424)

$\sin(-x) = \quad a + b(-x)^\beta + c(-x)^\gamma + d(-x)^\delta + \ldots$

But (424), $\sin(-x) = -\sin x$: hence the two series for these must be identical: and this can, obviously, only be the case when $a = 0$ and $\beta, \gamma, \delta, \ldots$ are *odd numbers*. Hence we may represent, with all possible generality, the series by $\sin x = h_1 x + h_3 x^3 + h_5 x^5 + \ldots.$ .... (a)

Again, $\cos x = a_1 + bx^{\beta_1} + c_1x^{\gamma_1} + d_1x^{\delta_1} + \ldots.$

and $\cos(-x) = a_1 + b_1(-x)^{\beta_1} + c_1(-x)^{\gamma_1} + d_1(-x)^{\delta_1} + \ldots.$

But (424), $\cos(-x) = \cos x$; and hence the two series just given are identical. Now this gives $a_1 = 1$, and requires that all the indices $\beta_1, \gamma_1, \delta_1, \ldots$ shall be *even numbers*. Hence representing a_1 by h_0x^0, we may generally write the series as follows:

$$\cos x = h_0x^0 + h_2x^2 + h_4x^4 + \ldots. \qquad \ldots\ldots\ldots (b)$$

4. Put $x = y + z$: then by the theorems (III, 1. 2), p. 425, we have

$$\sin x = \sin y \cos z + \cos y \sin z, \text{ and } \cos (y + z) = \cos y \cos z - \sin y \sin z \,..(c)$$

But $\sin x = \sin (y + z) = h_1(y + z) + h_3(y + z)^3 + h_5(y + z)^5 + \ldots.$

and $\cos x = \cos (y + z) = h_0(y + z)^0 + h_2(y + z)^2 + h_4(y + z)^4 + \ldots.$

Next insert the expansions (a), (b), in the equations (c): then we shall have

$$\sin x = (h_1 y + h_3 y^3 + \ldots)(h_0 z^0 + h_2 z^2 + \ldots) + (h_1 z + h_3 z^3 + \ldots)(h_0 y^0 + h_2 y^2 + \ldots) + \ldots$$
$$\cos x = (h_0 y^0 + h_2 y^2 + \ldots)(h_0 z^0 + h_2 z^2 + \ldots) - (h_1 y + h_3 y^3 + \ldots)(h_1 z + h_3 z^3 + \ldots) + \ldots$$

Expand the first pair of series by the binomial theorem, and arrange the results according to powers of y; and multiply out the factors of the latter pair, and arrange them in the same manner. Then since the two series for $\sin x$ are but different forms for the same function of y, the coefficients of the like powers of y must be equal, each to each; and, in the same manner, the coefficients of the like powers of y in the series for $\cos x$ must be equal each to each. It will be sufficient for our present purpose, to consider the coefficients of the first power of y only in the values of $\sin x$ and $\cos x$; and these give the equations

$$h_1 + 3h_3 z^2 + 5h_5 z^4 + 7h_7 z^6 + \ldots = h_1 + h_1 h_2 z^2 + h_1 h_4 z^4 + h_1 h_6 z^6 + \ldots$$
$$2h_2 z + 4h_4 z^3 + 6h_6 z^5 + 8h_8 z^7 + \ldots = -h_1^2 z - h_1 h_3 z^3 - h_1 h_5 z^5 - h_1 h_7 z^7 - \ldots$$

Now we have already seen that $h_0 = 1$, and we must find h_1. In the expression for the sine we have $\dfrac{\sin x}{x} = h_1 + h_3 x^2 + h_5 x^4 + \ldots\ldots$; and as x decreases, $\dfrac{\sin x}{x}$ approximates continually towards 1 as its limit, and hence when

$$x = 0, \text{ we have } 1 = h_1 + h_3 0^2 + h_5 0^4 + \ldots, \text{ or } h_1 = 1.$$

Again, these being *expansions*, the coefficients of the like powers of z are equal in each of them. Taking those of z, z^2, z^3 ...... in succession, we have

$$2h_2 = -h_1^2, \text{ or } h_2 = -\frac{1}{2} \qquad 5h_5 = +h_1 h_4, \text{ or } h_5 = \frac{1}{2.3.4.5}$$
$$3h_3 = +h_1 h_2, \text{ or } h_3 = -\frac{1}{2.3} \qquad 6h_6 = -h_1 h_5, \text{ or } h_6 = -\frac{1}{2.3.4.5.6}$$
$$4h_4 = -h_1 h_3, \text{ or } h_4 = \frac{1}{2.3.4}$$

and so on to any required extent.

Hence, finally, we have the series for the sine and cosine converted into *

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \ldots.$$
$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \ldots.$$

---

* In the same way we may obtain series for the tangent and cotangent:

$$\text{Let } \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{1.2.3} + \dfrac{x^5}{1.2.3.4.5} - \ldots}{1 - \dfrac{x^2}{1.2} + \dfrac{x^4}{1.2.3.4} - \ldots} = x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

and multiplying out and equating the coefficients of the like powers of x, we obtain,

$$\frac{\sin x}{\cos x} = \tan x = x + \frac{2x^3}{1.2.3} + \frac{2^4 x^5}{1.2.3.4.5} + \ldots.$$

$$\text{Similarly, } \frac{\cos x}{\sin x} = \cot x = \frac{1}{x} - \frac{2x}{1.2.3} - \frac{2^4 x^3}{1.2.3.4.5} + \ldots.$$

By reverting these several series (see p. 272), the arc itself may be found in terms of $\sin x$, $\cos x$, $\tan x$, or $\cot x$: but as these expressions are of little use in our present stage, and they can be obtained more simply by other processes, they will not be further discussed till the second volume. The entire investigation of all series of this nature, is best effected by means of the *integral calculus*; but as the series in the text were essential to elementary trigonometry, it was deemed advisable to give investigations of them here by the method of *indeterminate coefficients*.

X. Euler's and Demoivre's theorems.

1. *To prove Euler's theorem, viz. that*

$$\cos\theta = \frac{1}{2}\left\{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}\right\}, \text{ and } \sin\theta = \frac{1}{2\sqrt{-1}}\left\{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}\right\}$$

By (*p.* 251) $e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \ldots$

In this, substitute successively $\theta\sqrt{-1}$, and $-\theta\sqrt{-1}$ for x: then we shall have

$$e^{\theta\sqrt{-1}} = 1 + \frac{\theta\sqrt{-1}}{1} - \frac{\theta^2}{1.2} - \frac{\theta^3\sqrt{-1}}{1.2.3} + \frac{\theta^4}{1.2.3.4} + \ldots.$$

$$e^{-\theta\sqrt{-1}} = 1 - \frac{\theta\sqrt{-1}}{1} - \frac{\theta^2}{1.2} + \frac{\theta^3\sqrt{-1}}{1.2.3} + \frac{\theta^4}{1.2.3.4} - \ldots.$$

Whence, by addition, subtraction, and division by 2 and $2\sqrt{-1}$, we shall have

$$\frac{1}{2}\left\{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}\right\} = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \ldots. = \cos\theta;$$

$$\frac{1}{2\sqrt{-1}}\left\{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}\right\} = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \ldots. = \sin\theta.$$

2. *To prove Demoivre's theorem, viz. that for all values of* n,

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^n = \cos n\theta \pm \sqrt{-1}\sin n\theta,$$

By addition or subtraction of the two preceding results, we have at once $e^{\pm\theta\sqrt{-1}} = \cos\theta \pm \sqrt{-1}\sin\theta$; and as this is independent of the value of θ, we also have $e^{\pm n\theta\sqrt{-1}} = \cos n\theta \pm \sqrt{-1}\sin n\theta$, whatever n may be. Hence we have universally

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^n = \left\{e^{\pm\theta\sqrt{-1}}\right\}^n = e^{\pm n\theta\sqrt{-1}} = \cos n\theta \pm \sqrt{-1}\sin n\theta.$$

XI. Applications of the theorems of Euler and Demoivre.

1. *To expand $\cos n\theta$ and $\sin n\theta$ in terms of $\sin\theta$ and $\cos\theta$.*

For $\cos n\theta + \sqrt{-1}\sin n\theta = (\cos\theta + \sqrt{-1}\sin\theta)^n$ and

$$\cos n\theta - \sqrt{-1}\sin n\theta = (\cos\theta - \sqrt{-1}\sin\theta)^n;$$

and by addition and subtraction we have

$$2\cos n\theta = (\cos\theta + \sqrt{-1}\sin\theta)^n + (\cos\theta - \sqrt{-1}\sin\theta)^n$$

$$2\sqrt{-1}\sin n\theta = (\cos\theta + \sqrt{-1}\sin\theta)^n - (\cos\theta - \sqrt{-1}\sin\theta)^n.$$

Expand the second sides of these by the binomial theorem: then in the former all the odd powers of $\sqrt{-1}\sin\theta$ will mutually cancel, and in the latter all the even powers. This being done, and the equations divided by 2 and $2\sqrt{-1}$ respectively, we get

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{1.2}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cos^{n-4}\theta\sin^4\theta - \ldots$$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}\theta\sin^3\theta + \ldots$$

These may, obviously, be reduced to expressions containing only cosines and sines respectively, by means of II. 1, *p.* 423.

2. *We may also obtain, in a similar manner, the tangent of a multiple arc.* Thus,

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{1}{\sqrt{-1}} \frac{\left\{\cos\theta + \sqrt{-1}\sin\theta\right\}^n - \left\{\cos\theta - \sqrt{-1}\sin\theta\right\}^n}{\left\{\cos\theta + \sqrt{-1}\sin\theta\right\}^n + \left\{\cos\theta - \sqrt{-1}\sin\theta\right\}^n}$$

$$= \frac{1}{\sqrt{-1}} \frac{\left\{1 + \sqrt{-1}\tan\theta\right\}^n - \left\{1 - \sqrt{-1}\tan\theta\right\}^n}{\left\{1 + \sqrt{-1}\tan\theta\right\}^n + \left\{1 - \sqrt{-1}\tan\theta\right\}^n}$$

$$= \frac{n\tan\theta - \frac{n(n-1)(n-2)}{1.2.3}\tan^3\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\tan^5\theta}{1 - \frac{n(n-1)}{1.2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\tan^4\theta - \ldots}$$

3. *To find* $\sin^n\theta$ *and* $\cos^n\theta$ *in terms of the sines and cosines of multiples of* θ.

By *Euler's theorem* we have $\cos^n\theta = \frac{1}{2^n}\left\{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}\right\}^n$, or, expanding and bringing together the terms equidistant from the extremes of the series, we get

$$\cos^n\theta = \frac{1}{2^n}\left\{\left\{e^{n\theta\sqrt{-1}} + e^{-n\theta\sqrt{-1}}\right\} + \frac{n}{1}\left\{e^{(n-2)\theta\sqrt{-1}} + e^{-(n-2)\theta\sqrt{-1}}\right\} + \ldots\right.$$

$$= \frac{1}{2^n}\left\{\cos n\theta + \frac{n}{1}\cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta + \ldots\right\}$$

$$\sin^n\theta = \frac{1}{(2\sqrt{-1})^n}\left\{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}\right\}^n,$$ and proceeding as before, we have

$$= \frac{1}{(2\sqrt{-1})^n}\left\{\sin n\theta - \frac{n}{1}\sin(n-2)\theta + \frac{n(n-1)}{1.2}\sin(n-4)\theta + \ldots\right\}$$

where the signs depend on $(\sqrt{-1})^n$, and are easily assigned, and need no remark. For a complete discussion of multiple and submultiple arcs, the reader may, however, be referred to *Poinsot, Recherches sur l'Analyse des Sections Angulaires.*

4. *To express the value of the arc* θ *in terms of* $\tan\theta$.

Here, $e^{2\theta\sqrt{-1}} = \frac{e^{\theta\sqrt{-1}}}{e^{-\theta\sqrt{-1}}} = \frac{\cos\theta + \sqrt{-1}\sin\theta}{\cos\theta - \sqrt{-1}\sin\theta} = \frac{1 + \sqrt{-1}\tan\theta}{1 - \sqrt{-1}\tan\theta}$;

and taking $\log_e$ of both sides, we have, as at p. 250, after dividing by $2\sqrt{-1}$, the following series: $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots$

This theorem, which is by foreign writers attributed to *Leibnitz* was discovered and published originally by *James Gregory, Professor of Mathematics in the College of St. Andrew's.* Applications and modifications of it will be found in the *Mensuration* in this volume.

XII. SUBSIDIARY ANGLES.

This term is applied to those angles which it is either necessary or convenient to calculate, as intermediary steps between the data and the final solution of a problem. Their use is not, however, confined to cases where angles naturally enter into the inquiry: though their chief advantage occurs in the solution of trigonometrical problems.

It is not usual to designate by this term those angles which inevitably arise as intermediate subjects of calculation between the data and the solution: but only such as facilitate the calculation of those which are necessary, whilst that calculation might have been effected, though more laboriously, without the introduc-

tion of these subsidiary angles. For instance, suppose we have the equation $c = \sqrt{a^2 - 2ab\cos C + b^2}$: we may compute c by natural numbers, without any contrivance whatever beyond the operations indicated by the symbols: but as this is very laborious, an intermediate angle may be found, which, by the aid of logarithms, will render the operation much more concise. The angle so employed is, then, according to this description, *a subsidiary angle.* A few examples of the contrivances to be employed, are annexed; and others will occur in the places where they can be advantageously used.

1. Let $x = \sqrt{a^2 - b^2}$ be given for solution by a subsidiary angle. It may be written $x = a\sqrt{1 - \frac{b^2}{a^2}}$; and finding $\sin\theta = \frac{b}{a}$, we have $x = a\sqrt{1 - \sin^2\theta}$ $= a\cos\theta$. The value of $\sin\theta$ may be computed by logs, and hence we have $\log x = \log a + \log\sin\theta = \log a + \text{tab}\sin\theta - 10$.

Suppose, for instance, the expression had been $x = \sqrt{(32{\cdot}965)^2 - (2{\cdot}7682)^2}$: then $\sin\theta = \frac{2{\cdot}7682}{32{\cdot}965}$; and we have

$$\begin{aligned} 10 + \log 2{\cdot}7682 &= 10{\cdot}4421975 \\ \log 32{\cdot}965 &= 1{\cdot}5180531 \\ \hline & 8{\cdot}9241444 = \sin 4^\circ\, 49' \text{ nearly.} \end{aligned}$$

$$\begin{aligned} \text{and } \cos 4^\circ\, 49' &= 9{\cdot}9984636 \\ \log 32{\cdot}965 &= 1{\cdot}5180531 \\ \hline & 11{\cdot}5165167 = 10 + \log 32{\cdot}8486;\ \text{or} \\ & x = 32{\cdot}8486 \text{ nearly.} \end{aligned}$$

2. If $x = \sqrt{a^2 + b^2}$, we may write it $x = a\sqrt{1 + \frac{b^2}{a^2}}$; and taking $\frac{b}{a} =$ $\tan\theta$, we have $x = a\sqrt{1 + \tan^2\theta} = a\sec\theta$. This may be calculated similarly to the last *.

3. Suppose $x = \sqrt{a^2 - 2ab\cos C + b^2}$: then it admits of the following transformations:

$$x = \sqrt{a^2 - 2ab\cos C + b^2} = (a+b)\sqrt{1 - \frac{2ab\,(1 + \cos C)}{(a+b)^2}}$$

$$= (a+b)\sqrt{1 - \frac{4ab\cos^2 \tfrac{1}{2}C}{(a+b)^2}}$$

$$x = \sqrt{a^2 - 2ab\cos C + b^2} = (a-b)\sqrt{1 + \frac{2ab\,(1 - \cos C)}{(a-b)^2}}$$

$$= (a-b)\sqrt{1 + \frac{4ab\sin^2 \tfrac{1}{2}C}{(a-b)^2}}.$$

In the former case take $\cos\theta = \frac{2\cos\frac{1}{2}C}{a+b}\cdot\sqrt{ab}$, in the latter $\tan\chi = \frac{2\sin\frac{1}{2}C}{a-b}\cdot\sqrt{ab}$: then, $x = (a+b)\sin\theta$ in the former, and $x = (a-b)\sec\chi$ in the latter case: and both final solutions will give the correct values of x to the extent that the logarithmic tables enable us to carry the computations.

---

* It will be obvious that we might have taken $\cos\theta = \frac{b}{a}$ in the first example, and $\cot\theta = \frac{b}{a}$ in this. The same numerical results would have been finally obtained, as the student will see on working out any given numerical examples.

4. The coefficient $\frac{a-b}{a+b}$ often occurs in trigonometrical calculations, whilst only log a and log b are given. To avoid going to the tables to find a and b, and thence the above quotient, a subsidiary angle is generally used. The process is as follows:—

$$\frac{a-b}{a+b} = \frac{1-\frac{b}{a}}{1+\frac{b}{a}}, \text{ where if } \frac{b}{a} = \cos 2\theta, \text{ we have } \frac{a-b}{a+b} = \frac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta,$$

and as tab cos $2\theta = \log b - \log a + 10$; whence tan θ can be found from the logarithms without first finding the numbers a and b themselves.

Or had we taken $\frac{b}{a} = \tan \chi$, we should have had $\frac{a-b}{a+b} = \frac{1-\tan\chi}{1+\tan\chi} = \frac{\cos\chi - \sin\chi}{\cos\chi + \sin\chi} = \tan(45^\circ - \chi)$.

This latter is, perhaps, the better mode, when the fraction itself is the quantity sought, and the former when its square root is required.

5. The expression $x = a \sin A \pm b \cos A$ is one of frequent occurrence. Put $\frac{b}{a} = \tan\theta$; then $x = a\left(\sin A \pm \frac{b}{a}\cos A\right) = a(\sin A \pm \cos A \tan\theta)$,

$$= \frac{a}{\cos\theta}\left\{\sin A \cos\theta \pm \cos A \sin\theta\right\} = a \sec\theta \sin(A \pm \theta),$$

which is in an entirely logarithmic form.

6. Let the roots of $x^2 - px - q = 0$ be calculated by a subsidiary angle. Here $x = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q} = \sqrt{q}\left\{\frac{p}{2\sqrt{q}} \pm \sqrt{\frac{p^2}{4q} + 1}\right\}$. Take $\frac{p}{2\sqrt{q}} = \cot\theta$; then $x = \sqrt{q}\{\cot\theta \pm \operatorname{cosec}\theta\} = \sqrt{q}.\frac{\cos\theta \pm 1}{\sin\theta} = -\sqrt{q}.\tan\frac{1}{2}\theta$, or $\sqrt{q}.\cot\frac{1}{2}\theta$.

Had the equation been $x^2 - px + q = 0$, assume $\sin^2\theta = \frac{4q}{p^2}$; then the roots would have been $x = p\cos^2\frac{1}{2}\theta$, or $p\sin^2\frac{1}{2}\theta$.

7. The real root of the cubic equation $x^3 - qx - r = 0$ is capable of being put under the form

$$x = \sqrt{\frac{q}{3}}\left\{\sqrt[3]{\sqrt{\frac{27r^2}{4q^3}} + \sqrt{\frac{27r^2}{4q^3} - 1}} + \sqrt[3]{\sqrt{\frac{27r^2}{4q^3}} - \sqrt{\frac{27r^2}{4q^3} - 1}}\right\}.$$

Let $\operatorname{cosec}^2\theta = \frac{27r^2}{4q^3}$: then substituting, we obtain

$$x = \sqrt{\frac{q}{3}}\left\{\sqrt[3]{\frac{1+\cos\theta}{\sin\theta}} + \sqrt[3]{\frac{1-\cos\theta}{\sin\theta}}\right\} = \sqrt{\frac{q}{3}}\left\{\sqrt[3]{\cot\tfrac{1}{2}\theta} + \sqrt[3]{\tan\tfrac{1}{2}\theta}\right\}$$

Again, find $\tan\phi = \sqrt[3]{\tan\frac{1}{2}\theta}$; and the final value will be expressed by

$$x = \sqrt{\frac{q}{3}}\left\{\cot\phi + \tan\phi\right\} = \sqrt{\frac{4q}{3}}.\operatorname{cosec} 2\phi.$$

If the equation belong to the irreducible case, or $4q^3$ be greater than $27r^2$, let $x = a\cos\theta$: then $\cos 3\theta = \frac{4x^3}{a^3} - \frac{3x}{a}$, or $x^3 - \frac{3a^2}{4}x - \frac{a^3}{4}\cos 3\theta = 0$. To make this coincide with the given equation, we shall have $q = \frac{3a^2}{4}$, $r = \frac{a^3\cos 3\theta}{4}$, from which a and θ are determinable; and thence $x = a\cos\theta$ may be found.

It might, also, have been solved by making $x = \cos\left\{\frac{2\pi}{3} + \theta\right\}$, or $x = \cos\left\{\frac{4\pi}{3} + \theta\right\}$; and hence the three roots of the given equations may be found: which in this case are all real.

8. The following expression, not involving angles at all, occurs in physical astronomy, where e is always less than 1, viz.:

$$P = (1 + e_1)(1 + e_2)(1 + e_3)(1 + e_4)\ldots.(1 + e_n), \text{ where}$$

$$e_1 = \frac{1 - \sqrt{1 - e^2}}{1 + \sqrt{1 - e^2}},\ e_2 = \frac{1 - \sqrt{1 - e_1^2}}{1 + \sqrt{1 - e_1^2}},\ \ldots\ e_n = \frac{1 - \sqrt{1 - e^2_{n-1}}}{1 + \sqrt{1 - e^2_{n-1}}}$$

Put $e = \sin\theta$, then $\sqrt{1 - e_2} = \cos\theta$, and we get successively,

$$\frac{1 - \sqrt{1 - e^2}}{1 + \sqrt{1 - e^2}} = \frac{1 - \cos\theta}{1 + \cos\theta} = \tan^2\tfrac{1}{2}\theta;\ \text{and}\ 1 + e_1 = 1 + \tan^2\tfrac{1}{2}\theta = \sec^2\tfrac{1}{2}\theta.$$

Also, since the numerators are in all cases less than the denominators, $\tan^2\frac{1}{2}\theta$, and therefore $\tan\frac{1}{2}\theta$ is less than 1: the process may, therefore, be continued without limit, by making $\tan\frac{1}{2}\theta = \sin\theta_1$, and thence finding $\sec^2\frac{1}{2}\theta_1$, and then again $\sec^2\frac{1}{2}\theta_2$, and so on. The expression is, therefore, reducible to $P = \sec^2\frac{1}{2}\theta\ \sec^2\frac{1}{2}\theta_1\ \sec^2\frac{1}{2}\theta_2 \ldots\ldots$, which is adapted to the application of logs at once: and this, perhaps, is the only way in which the calculation could be practically effected without extreme labour.

XIII. CHANGING THE RADIUS IN TRIGONOMETRICAL EQUATIONS.

In all the preceding investigations, the radius has been assumed as unity: but inquiries sometimes occur in which it becomes necessary to transform trigonometrical equations formed on this hypothesis into others where the radius is some different quantity r; and, conversely, from equations to radius r into others to radius 1.

This is effected at once, in equations involving only *direct* functions of the arcs, by rendering all the terms *homogeneous;* and since, as a general principle, the terms can only cease to be homogeneous by some of the linear factors becoming unity. The rule, therefore, will be, to restore the general value r of the radius in all the terms, so as to render them homogeneous. Thus to radius 1, we have $\sin 3\theta = 3\sin\theta - 4\ \sin^3\theta$; and to render it homogeneous, we must have $r^2 \sin 3\theta = 3r^2 \sin\theta - 4\ \sin^3\theta$, there being now three linear factors in each term.

In the case of *inverse functions*, where the occasion for this change most frequently occurs, and where the mode of proceeding is less obvious, the change is still easily made.

Since arcs subtending the same angle are to one another as the radii to which they are described (*th.* 94, p. 337), if we denote by θ_1 and θ_r the arcs subtending an angle to the radii 1 and r, we have $\theta_1 : 1 :: \theta_r : r$; whence $\theta_r = r\theta_1$ and $\theta_1 = \frac{\theta_r}{r}$. Making these substitutions according as the expression in radius r or radius 1 is given, the transformation will be complete. The reduction from radius r to radius 1, is that which almost always occurs; and it is perpetually required in calculating the numerical values of integrals, as they are thus reduced to the use of the common tables.

XIV. ON INVERSE NOTATION AND OPERATIONS.

TAKING 1 as the base of all number, x^1 or x signifies 1 multiplied by x, and x^{-1} or $\frac{1}{x}$ signifies 1 divided by x: that is, x^1 and x^{-1} indicate *inverse operations*, or operations such that they mutually destroy each other. In analogy to this idea, Sir John Herschel proposed a notation for the general expression of inverse operations, which, from its great convenience, has been generally adopted in this country, and of which a very brief explanation is annexed.

Whatever function any one quantity be of another, the inverse function is expressed by the index, $^{-1}$, written after the symbol indicating the function. Thus, in the case above, x^1 becomes x^{-1}; in $\log h = b$, we have $\log^{-1} \log^1 h = \log^{-1} b$; and since $\log^{-1}$ neutralises $\log^1$ or log, it becomes $h = \log^{-1} b$. The same applies to trigonometrical and all other functions; as, for instance, if $\sin \theta = \kappa$, then $\sin^{-1} \sin^1 \theta = \theta = \sin^{-1} \kappa$; or if $\theta = \tan^{-1} \kappa$, we have $\tan \theta = \kappa$; and so on *.

Though the value of this notation is most obvious in the integral calculus, it is not destitute of utility even in elementary trigonometry, as the following examples will show.

1. Let $\tan a = t$, $\tan a_1 = t_1$, $\tan a_2 = t_2$, $\ldots \tan a_n = t_n$: then

$$\text{since } \tan (a - a_1) = \frac{\tan a - \tan a_1}{1 + \tan a \tan a_1} = \frac{t - t_1}{1 + tt_1}; \text{ hence we have}$$

$$\tan^{-1} \frac{t - t_1}{1 + tt_1} = a - a_1 = \tan^{-1} t - \tan^{-1} t_1.$$

Employing the same notation, we have the following equations in succession:

$$\tan^{-1} t - \tan^{-1} t_1 = \tan^{-1} \frac{t - t_1}{1 + tt_1},$$

$$\tan^{-1} t_1 - \tan^{-1} t_2 = \tan^{-1} \frac{t_1 - t_2}{1 + t_1 t_2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tan^{-1} t_{n-1} - \tan^{-1} t_n = \tan^{-1} \frac{t_{n-1} - t_n}{1 + t_{n-1} t_n}.$$

and by addition of all these results, the remarkable formula

$$\tan^{-1} t - \tan^{-1} t_n = \tan^{-1} \frac{t - t_1}{1 + tt_1} + \tan^{-1} \frac{t_1 - t_2}{1 + t_1 t_2} + \ldots \tan^{-1} \frac{t_{n-1} - t_n}{1 + t_{n-1} t_n}.$$

2. Given the equations $\theta + \phi = a$ and $\tan \theta = m \tan \phi$ to find ϕ and θ. From the second $\theta = \tan^{-1} (m \tan \phi)$, which inserted in the first, gives

$\tan^{-1} (m \tan \phi) + \phi = a$, or taking the tangents it becomes $\dfrac{m \tan \phi + \tan \phi}{1 - m \tan \phi \tan \phi} = \tan a$,

or, $m \tan a \tan^2 \phi + (m + 1) \tan \phi = \tan a$.

From this, $\tan \phi$, and hence $\tan \theta = m \tan \phi$, will be found by the solution of a quadratic equation: thus giving,

$$\tan \phi = \frac{-(m+1) \pm \sqrt{4m \tan^2 a + (m+1)^2}}{2m \tan a}, \quad \tan \theta = \frac{-(m+1) \pm \sqrt{4m \tan^2 a + (m+1)^2}}{2 \tan a}$$

* Before this notation was invented, the circumlocutions, θ = arc whose sine is κ, θ = arc whose tangent is κ, *etc.* were obliged to be used to express the equations $\theta = \sin^{-1} \kappa$, $\theta = \tan^{-1} \kappa$, *etc.* Many foreign writers still adhere to the old notation, which gives their books an extremely awkward appearance, and renders them much more difficult to read.

EXAMPLES FOR PRACTICE.

1. Show that $\operatorname{cosec}^{-1}\sqrt{50} + \operatorname{cosec}^{-1}\sqrt{65} = \operatorname{cosec}^{-1}\frac{\sqrt{130}}{3}$,

and $\operatorname{cosec}^{-1}\sqrt{10} + \operatorname{cosec}^{-1}\sqrt{26} = \operatorname{cosec}^{-1}\frac{\sqrt{65}}{4}$.

2. Show that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = 45^\circ$.

3. Solve the equations $\phi + \theta = a$, and $\cos\phi : \cos\theta :: m : n$.

4. Show that if $2\tan^{-1}t = \sin^{-1}2s$, then $s = \frac{t}{1+t^2}$.

5. Resolve the equation $\operatorname{vers}^{-1}\frac{x}{a} - \operatorname{vers}^{-1}\frac{bx}{a} = \operatorname{vers}^{-1}(1-b)$.

6. Find ϕ and θ from $\sin\phi = \sqrt{2}\sin\theta$ and $\tan\phi = \sqrt{3}\tan\theta$.

7. Establish the following equalities amongst arcs:—

(1.) $\sin^{-1}\frac{\sqrt{2+\sqrt{2}}}{2} + \sin^{-1}\frac{\sqrt{2-\sqrt{2}}}{2} = \frac{\pi}{2}$.

(2.) $\cos^{-1}\frac{3}{4} + \cos^{-1}\frac{9}{16} + \cos^{-1}\frac{1}{8} = \pi$.

(3.) $\cot^{-1}\frac{1}{2} + \cot^{-1}\frac{1}{3} - \cot^{-1}\frac{1}{2+\sqrt{3}} = \frac{\pi}{3}$.

(4.) $\operatorname{cosec}^{-1}\sqrt{10} + \operatorname{cosec}^{-1}\sqrt{26} + \operatorname{cosec}^{-1}\sqrt{50} + \operatorname{cosec}^{-1}\sqrt{65} = \frac{\pi}{4}$.

(5.) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = 2\tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{5} = \frac{\pi}{4}$.

(6.) $\tan^{-1}\frac{1}{99} - \tan^{-1}\frac{1}{70} + 4\tan^{-1}\frac{1}{5} = \frac{\pi}{4}$.

8. When $\tan^{-1}(x+1) = 3\tan^{-1}(x-1)$, show that $x = \pm\sqrt{2}$.

9. In $\cos^3\theta = 2\sin^2\theta$, $\theta = \sec^{-1}\left\{\sqrt[3]{\frac{1}{4} + \frac{1}{12}\sqrt{\frac{11}{3}}} + \sqrt[3]{\frac{1}{4} - \frac{1}{12}\sqrt{\frac{11}{3}}}\right\}$

10. Given $\cos^{-1}\frac{a^2+x^2-c^2}{2ax} \pm \cos^{-1}\frac{a^2+x^2-b^2}{2ax} = \pm\frac{\pi}{3}$ to find x*.

MISCELLANEOUS EXAMPLES FOR EXERCISES ON ARCS.

1. Investigate the equation $\tan a + \sec a = \tan(45^\circ + \frac{1}{2}a)$.

2. Show that $\sec 2\theta = \frac{1+\tan^2\theta}{1-\tan^2\theta}$, and $\operatorname{cosec} 2\theta = \frac{1+\tan^2\theta}{2\tan\theta} = \frac{\sec^2\theta}{2\tan\theta}$.

3. If $x = \cos\theta$ and $y = \sin\theta$, find θ from $Axy = By^2 + Cx^2 + D$.

4. Prove that $\tan^2\chi - \sin^2\chi = \tan^2\chi\sin^2\chi$.

5. If $\tan\phi = \sqrt{n}$, show that $\sin\phi = \sqrt{\frac{n}{1+n}}$ and $\cos\phi = \sqrt{\frac{1}{1+n}}$: and assign θ when n is 1, 2, 3, 4, and $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, in succession.

---

* This equation expresses trigonometrically the problem:—given the three radii a, b, c, of concentric circles to find the side x, of an equilateral triangle which shall have its angular points in the three circumferences.

From the same equation is also deduced the remarkable theorem: if on the sides of any triangle, equilateral triangles be described *exteriorly*, the lines drawn from the vertex of each to the opposite angles of the first triangle will be equal to each other, and intersect in the same point.

6. Find χ and θ from $\sin\chi = n\cos\chi$ and $\sec\theta = m\tan\theta$, both generally, and when m and n are 5·6 and 10 respectively *.

7. (II. 25.) Demonstrate the assigned values of sin A, cos A, and tan A given in *Hutton's Tables, pages* 362, 363.

8. (II. 10, 11.) Show that $\sin^2\theta + \text{vers}^2\theta = 2\,\text{vers}\,\theta$; and $\sin^2\theta = \frac{1}{2}\,\text{vers}\,2\theta$.

9. (II. 12.) Prove that $\sec\chi = \tan\chi + \tan\frac{1}{2}(90 - \chi)$, for all values of χ.

10. Establish the surd values of the sines and cosines, *p.* xxxix of *Hutton's Tables.*

11. (II. 18.) Show that $\sec 60^\circ = 2\tan 45^\circ$, and $\tan 45^\circ \sec 60^\circ = \sec^2 45^\circ$.

12. (II. 21, 22.) Solve the equations $\tan\phi + \cot\phi = 4$, $\cot\phi + \tan\phi = 2n$, and $\sin\phi + \cos\phi = a$; and show the limits of possibility.

13. (II. 9.) If $\phi + \theta = 60$ and $\chi + \omega = 90$, show that we shall have

$$\frac{\sin\phi - \sin\theta}{\sin\frac{1}{2}(\phi - \theta)} = \sqrt{3}, \text{ and } \frac{\sin\chi - \sin\omega}{\sin\frac{1}{2}(\chi - \omega)} = \sqrt{2}.$$

14. (II. 25, 3.) Show that $\cos^4 m - \sin^4 m = \cos 2m$, and that

$$\sin^2 m\sin^2 n + \cos^2 m\cos^2 n + \cos^2 m\sin^2 n + \cos^2 n\sin^2 m = 1.$$

15. (II. 36.) When $\tan^3 x = \cos b\sec a$, then $\dfrac{\cos a}{\cos x} + \dfrac{\cos b}{\sin x} = \left\{\cos^{\frac{2}{3}} a + \cos^{\frac{2}{3}} b\right\}^{\frac{3}{2}}$,

16. (II. 42.) Determine the arc which is a third of the arc whose sine is s, or find θ from $\sin 3\theta = s$; and show the meaning of the three answers.

17. (II. 43.) Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = 45^\circ$, and that

$$\tan^{-1}a \pm \tan^{-1}b = \tan^{-1}\frac{a \pm b}{1 \mp ab}.$$

18. (II. 47.) If a, b, c, be any three angles, prove the equality, $\tan a\tan b\tan c$

$$= \frac{-\sin(a+b+c) + \sin(-a+b+c) + \sin(a-b+c) + \sin(a+b-c)}{\cos(a+b+c) + \cos(-a+b+c) + \cos(a-b+c) + \cos(a+b-c)}.$$

19. (II. 26. 1.) If $\dfrac{\cos\phi}{\cos\theta} = \dfrac{m+n}{m-n}$ and $\dfrac{\sin\phi}{\sin\theta} = \dfrac{m+n}{\sqrt{2mn}}$ find the expressions for ϕ and θ, and ascertain whether they be real or not.

20. If α, β denote two given arcs, show that in the equation $\sec\alpha\sec\phi + \tan\alpha\tan\phi = \sec\beta$, we shall have

$$\sec\phi = \sec\alpha\sec\beta - \tan\alpha\tan\beta, \text{ and } \tan\phi = \sec\alpha\tan\beta - \tan\alpha\sec\beta.$$

21. Prove that $\tan 9^\circ = 1 + \sqrt{5} - \sqrt{5 + 2\sqrt{5}}$, and assign the arc ϕ from the equation $\tan\phi = 1 + \sqrt{5} + \sqrt{5 + 2\sqrt{5}}$.

22. If $\sin\phi + \cos\theta = a$, and $\cos\phi + \sin\theta = b$, show that

$$\tan\phi = \frac{a + bn}{b - an}, \text{ and } \tan\theta = \frac{b + an}{a - bn}, \text{ where } n = \sqrt{\frac{4}{a^2 + b^2} - 1}$$

23. Find the value of θ in the equation $x^2 - x = 2$, where

$$x = \sqrt{\sec^2\theta + \sqrt{\sec^2\theta + \sqrt{\sec^2\theta + \ldots\ ad\ inf.}}}$$

24. Resolve the equation, $\cot\theta\tan 2\theta - \tan\theta\cot 2\theta = 2$.

* The questions, as far as (6), stand in the same order as in the last edition of vol. i, and occupied p. 405. Those which follow are mainly from vol. ii. p. 34—42 of the last edition, and though subjected to a new arrangement here, the old numbers are still retained in parenthesis for more ready reference to another work dependent on the former arrangement.

25. In $\sin n\theta = \dfrac{x^n - x^{-n}}{2\sqrt{-1}}$ and $\cos m\theta = \dfrac{x^{3n} + x^{-3n}}{2}$, what is the relation between m and n?

26. Show that the four roots of $x^4 - 2x^2 \cos 2a + 1 = 0$, are $\pm\, \alpha^{\pm 1}$, and assign the value of α.

27. If $\dfrac{\cot^2\theta + 1}{\cot^2\theta - 1} = \operatorname{cosec}^2\phi + \cot^2\phi$, find the relation between ϕ and θ.

28. Show that $\tan\phi = \dfrac{3 \sin\phi - \sin 3\phi + 3 \sin\phi \cos 2\phi - \sin 3\phi \cos 2\phi}{3 \cos\phi + \cos 3\phi - 3 \cos\phi \cos 2\phi - \cos 3\phi \cos 2\phi}$.

29. If $\alpha + \beta + \gamma + \delta = \frac{1}{2}\pi$, prove that $1 + \tan\alpha \tan\beta \tan\gamma \tan\delta =$ $\tan\alpha \tan\beta + \tan\alpha \tan\gamma + \tan\alpha \tan\delta + \tan\beta \tan\gamma + \tan\beta \tan\delta + \tan\gamma \tan\delta$.

30. If $c = \mathrm{ch}\, a$, $c_1 = \mathrm{ch}\, \frac{1}{2}a$, $c_2 = \mathrm{ch}\, \frac{1}{4}a$, ...; then

$$\sin a = c - c\,c_1\,c_2 + c\,c_1\,c_2\,c_3\,c_4 - c\,c_1\,c_2\,c_3\,c_4\,c_5\,c_6 + \ldots.$$
$$\cos a = 1 - c\,c_1 + c\,c_1\,c_2\,c_3 - c\,c_1\,c_2\,c_3\,c_4\,c_5 + \ldots.$$

31. If $\sqrt[3]{\tan(45 - \frac{1}{2}\theta)} = \tan\phi$, then it is required to prove that

$$\sqrt[3]{\tan\theta + \sec\theta} + \sqrt[3]{\tan\theta - \sec\theta} = 2 \cot 2\phi.$$

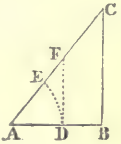
32. Prove *Euler's* series, $a = \sin a \sec \frac{1}{2}a \sec \frac{1}{4}a \sec \frac{1}{8}a$ .... *ad inf.*

33. Given $\cos\phi + \cos\chi = a_1$, and $\cos 5\phi + \cos 5\chi = b$, to find ϕ and χ.

34. Given $\phi + \chi = a$ and $\sin\chi \cos\phi = \sin\phi \cos\chi$, to find ϕ and χ.

XV. THE PROPERTIES OF PLANE TRIANGLES.

I. *The right-angled triangle.*



Let ABC be a triangle right-angled at B; with centre A and the unit-radius describe the arc DE, and draw DF perpendicular to AB. Then DF is the tangent of the arc DE or angle A; and AF is the secant.

Also, since the triangles ADF, ABC are similar, we have

AB : BC : : AD : DF; that is, AB . DF = BC.AD,
AB : AC : : AD : AF; that is, AB . AF = AC.AD.

But these in trigonometrical symbols become, since AD = 1,

BC = AB tan A .... (1)	AB = BC cot A .... (3)
AC = AB sec A .... (2)	AB = AC cos A .... (4)

Also from (3, 4) we get

$$\text{AC} \cos \text{A} = \text{BC} \cot \text{A} = \frac{\text{BC} \cos \text{A}}{\sin \text{A}}, \text{ or BC} = \text{AC} \sin \text{A} \quad \ldots. (5)$$

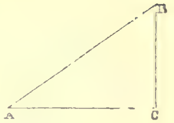
But C + A = $\frac{1}{2}\pi$, or C is the complement of A; and substituting this value in (1....5) we have the remaining equations

BC = AB cot C......(6)	AB = BC tan C .... (8)
AC = AB cosec C....(7)	AB = AC sin C .... (9)

BC = AC cos ACB...... .....(10)

When, therefore, of three parts of a right-angled triangle (except all three be sides) any two are given, the third can be found from these equations. When the three sides are concerned, the most generally commodious rule is that furnished by *Euc.* i. 47, or *Theor.* 34 *Geom.* The solution, however, may, even in this case, be effected by the equations given above.

EXAMPLES.



1. In the right-angled triangle ABC right-angled at C, there are given the side BC = 379·628 and the angle BAC = 39° 26′ 15″: to find the other parts.

Here A + B = 90°, or B = 90° − A = 50° 33′ 45″. Also AC = BC tan B, and AB = BC sec B. The work will be

log 379·62 =	2·5793491	log 379·62 =	2·5793491
pp 8 =	92	8 =	92
tan 50° 33′ =	10·0846678	sec 50° 33′ =	10·1969496
pp 45″ =	1931	pp 45″ =	1152
log AC =	2·6642192	log AB =	2·7764231

Hence AC = 461·5504 and AB = 597·6171 nearly.

The advantage of drawing out the forms for the entire operation cannot be too much insisted on. In the present very simple example, log BC occurs twice, and therefore may be written down at the same stage of the work: and tan B, sec B, occur at the same opening of the tables; and therefore may be taken out in another stage with one single reference to the book. In taking the logs of numbers, the corrections are set down by inspection; and the differences for one minute may be registered at the time of taking out the log functions to the nearest minute, and the calculation of the correction made and entered in its place afterwards. The method of making these corrections has already been explained (p. 434).

2. Given BA = 402·015 and B = 56° 7′ 18″ to find the other parts.

Here A = 90° − B = 33° 52′ 42″, BC = AB cos B, and AC = AB sin B.

log 402·01 =	2·6042369	log 402·01 =	2·6042369
pp 5 =	54	pp 5 =	54
cos 56° 7′ =	9·7462477	sin 56° 7′ =	9·9191694
pp 18″ =	−565	pp 18″ =	254
	2·3504900	log AC =	2·5234371
log BC =	2·3504335		

Hence BC = 224·0957 and AC = 333·7621 nearly.

In this example as the correction for cos B is negative, the positive terms are added, giving 2·3504900, and 565 subtracted *downwards*, leaving the corrected log BC as the remainder. When there are several subtractive corrections, it will be more convenient to add the positive terms together and then the negative together, and take the result of the two sums, as in the example above.

3. Given AB = 501·625 and AC = 437·128 to find the other parts.

Here $\sin B = \dfrac{CA}{AB}$, $A = 90° - B$, $BC = \sqrt{AB^2 - AC^2}$, or $BC = AB \cos B$.

log 437·12 =	2·6406007	log 501·62 =	2·7003748
pp 8 =	79	pp 5 =	44
Adding 10,	12·6406086		2·7003792
	2·7003792		
sin B =	9·9402294		

or B = 60° 37′ 28″, and hence, A = 29° 22′ 32″.

Then to find BC we have $BC = \sqrt{(AB + AC)(AB - AC)}$, and hence

$$\begin{aligned}
\text{By the question, } AB &= 501{\cdot}625 \\
\text{and } AC &= 437{\cdot}128 \\
\log (AB + AC) &= \log 938{\cdot}753 = 2{\cdot}9725514 \\
\log (AB - AC) &= \log\ 64{\cdot}497 = 1{\cdot}8095395 \\
& \qquad 2 \,|\, 4{\cdot}7820909
\end{aligned}$$

$$\log BC = 2{\cdot}3910455, \text{ or } BC = 246{\cdot}0626$$

Or again, by the equation $BC = AB \cos B = AB \sin A$ we have

$$\begin{array}{rcr|rcr}
\log 501{\cdot}625 & = & 2{\cdot}7003792 & \text{or } \log 501{\cdot}625 & = & 2{\cdot}7003792 \\
\cos 60^\circ 37' & = & 9{\cdot}6907721 & \sin 29^\circ 22' & = & 9{\cdot}6905476 \\
pp\ 28'' & = & -1048 & pp\ 32'' & = & 1197 \\
\hline
 & & 2{\cdot}3911513 & & & 2{\cdot}3910465 \\
\log BC & = & 2{\cdot}3910465 & \text{or } BC & = & 246{\cdot}0631.
\end{array}$$

The former process is generally the more accurate, but the latter is the less laborious method, as log AC had already been found in the preceding part of the solution, and the tables were open to cos B in finding its value from $\sin B = \frac{CA}{AB}$. Also, it conduces to convenience, as the work shows, to use the direct functions sin, tan, sec, in preference to the complementary ones, cos, cot, cosec.

4. Given $AC = 299{\cdot}015$, $BC = 325{\cdot}162$, to find the other parts.

$$\tan B = \frac{AC}{BC},\ A = 90^\circ - B, \text{ and } AB = BC \sec B.$$

$$\begin{array}{rcr|rcr}
\log\ 299{\cdot}01 & = & 2{\cdot}4756857 & \log\ 325{\cdot}16 & = & 2{\cdot}5120971 \\
pp\ 5 & = & 73 & pp\ 2 & = & 27 \\
\hline
\log AC + 10 & = & 12{\cdot}4756930 & \log\ BC & = & 2{\cdot}5120998 \\
 & & 2{\cdot}5120998 & & & \\
\hline
 & & 9{\cdot}9635932 & = \log \tan 42^\circ 36' 5'', & &
\end{array}$$

Hence $B = 42^\circ 36' 5''$ and $A = 47^\circ 23' 55''$.

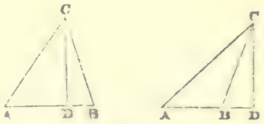
Then $AB = BC \sec B = 325{\cdot}162 \sec 42^\circ 36' 5''$; and hence

$$\begin{aligned}
\log 325{\cdot}162 &= 2{\cdot}5120998 \\
\sec 42^\circ 36' &= 10{\cdot}1330649 \\
pp\ 5'' &= 97 \\
\log AB &= 2{\cdot}6451744, \text{ or } AB = 441{\cdot}7478.
\end{aligned}$$

5. Given $AB = {\cdot}62985$, and $ABC = 50^\circ 10' 33''$, to find the other parts.
6. Given $BC = 358{\cdot}26$ and $CA = 286{\cdot}325$, to find the rest.
7. Given $\log BC = 3{\cdot}1296578$, and $\log BA = 3{\cdot}2965782$, to find the rest.
8. Given $AC = 5$, $BC = 6$, to find AB and the angles.
9. Given $AC = 162$, $BAC = 53^\circ 7' 48''$, to find the rest.
10. Given $AB = 25$ and $BC = 24$, to find the angles.

II. *Oblique-angled triangles.*

1. Let ABC be a triangle, and from the angle C draw the perpendicular CD to the base AB. Denote the angles by A, B, C, and the sides respectively opposite them by a, b, c.



By the right-angled triangles ACD, BCD we have $AB = AD + DB$, or $c = a \cos B + b \cos A$. Forming similar equations with respect to the other sides, we get

$$a = b\cos C + c\cos B;\ \text{from which}\ a^2 = ab\cos C + ac\cos B,$$
$$b = a\cos C + c\cos A;\ \ldots\ldots\ \ b^2 = ab\cos C + bc\cos A,$$
$$c = a\cos B + b\cos A;\ \ldots\ldots\ \ c^2 = ac\cos B + bc\cos A.$$

Subtract each of these equations from the sum of the other two; then we get

$$\left.\begin{array}{ll} 2bc\cos A = -a^2 + b^2 + c^2; & \text{or } \cos A = \dfrac{-a^2+b^2+c^2}{2bc} \\ 2ac\cos B = \quad a^2 - b^2 + c^2; & \text{or } \cos B = \dfrac{a^2-b^2+c^2}{2ac} \\ 2ab\cos C = \quad a^2 + b^2 - c^2; & \text{or } \cos C = \dfrac{a^2+b^2-c^2}{2ab} \end{array}\right\} \ldots\ldots (1)$$

These expressions are convenient for calculating the cosines of the angles of a triangle when the sides are given in small numbers, or in large numbers which have the same ratio with three small numbers. They are essential when a, b, c, are themselves the square roots of numbers not exact squares.

2. Since $\sin A = \sqrt{1 - \cos^2 A}$, if we substitute for $\cos A$ its value just found, and proceed similarly for $\cos B$, $\cos C$, we shall have

$$\left.\begin{array}{l} 2bc\sin A = \sqrt{\{2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4\}} \\ 2ac\sin B = \sqrt{\{2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4\}} \\ 2ab\sin C = \sqrt{\{2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4\}} \end{array}\right\} \ldots\ldots (2)$$

From the equality of the right-hand sides of these equations we get

$$2bc\sin A = 2ac\sin B = 2ab\sin C;$$

and from these three, taken two and two, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

which is usually expressed in words by saying that *the sides of a triangle are proportional to the sines of their opposite angles;* and often written

$$\sin A : \sin B : \sin C :: a : b : c \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The quantity under the radical in (2) is rarely adapted to convenient calculation, and never, except when also the equations in (1) are also adapted; and as (1) is less laborious than (2), the latter is never used for that purpose, but merely a step in general investigation. It is, in fact, the expression for the product of one side of a triangle and perpendicular upon it from the opposite angle. We shall presently find another form for it.

3. Resume the equations (1): then if $a + b + c = 2s$

$$\sin^2\tfrac{1}{2}A = \tfrac{1}{2}(1-\cos A) = \tfrac{1}{2}\left\{1 - \frac{-a^2+b^2+c^2}{2bc}\right\} = \frac{a^2-(b-c)^2}{4bc}$$
$$= \frac{(a-b+c)(a+b-c)}{4bc} = \frac{(s-b)(s-c)}{bc}$$

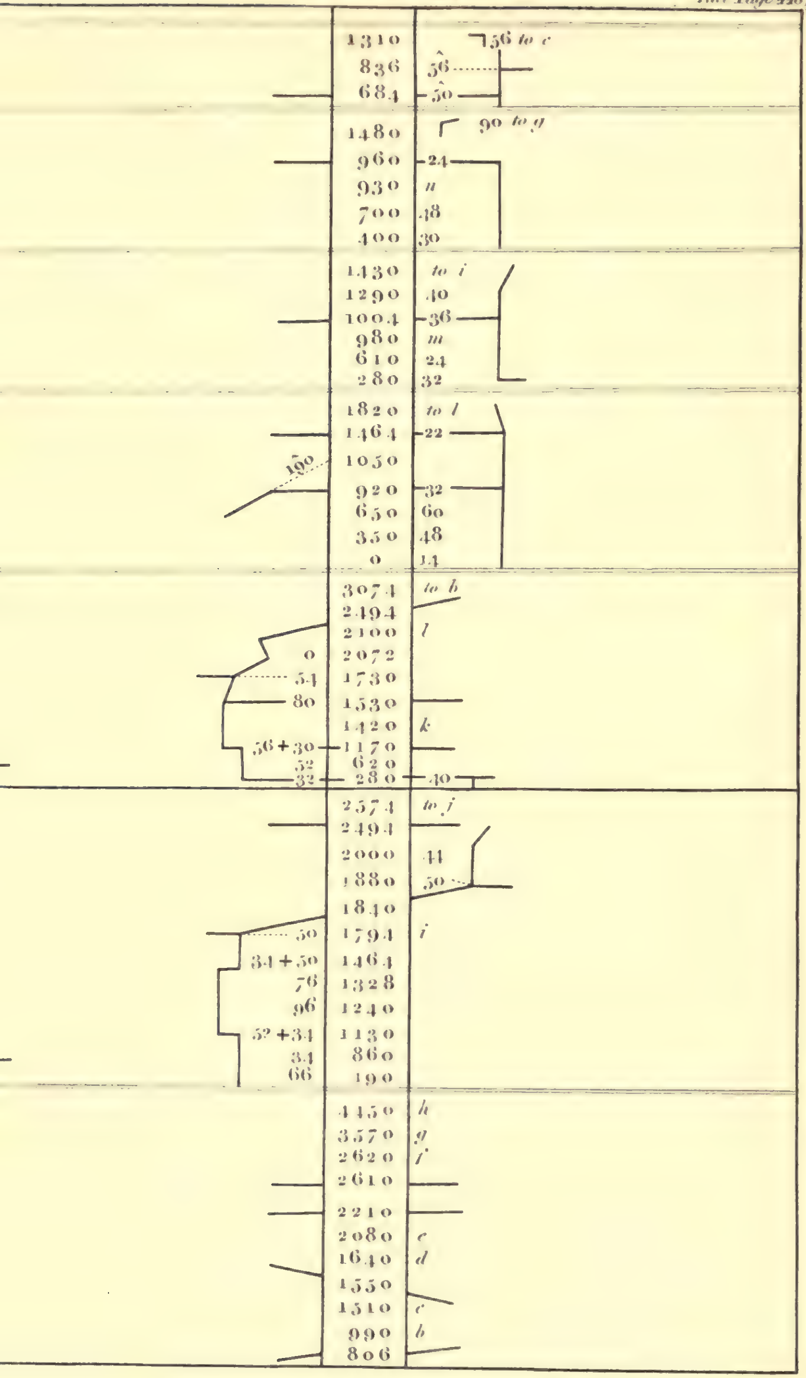
$$\cos^2\tfrac{1}{2}A = \tfrac{1}{2}(1+\cos A) = \tfrac{1}{2}\left\{1 + \frac{-a^2+b^2+c^2}{2bc}\right\} = \frac{-a^2+(b+c)^2}{4bc}$$
$$= \frac{(-a+b+c)(a+b+c)}{4bc} = \frac{s(s-a)}{bc}$$

From these two last results $\tan^2\tfrac{1}{2}A = \dfrac{(s-b)(s-c)}{s(s-a)}$,

Performing similar operations for B and C, we get the following equations:—

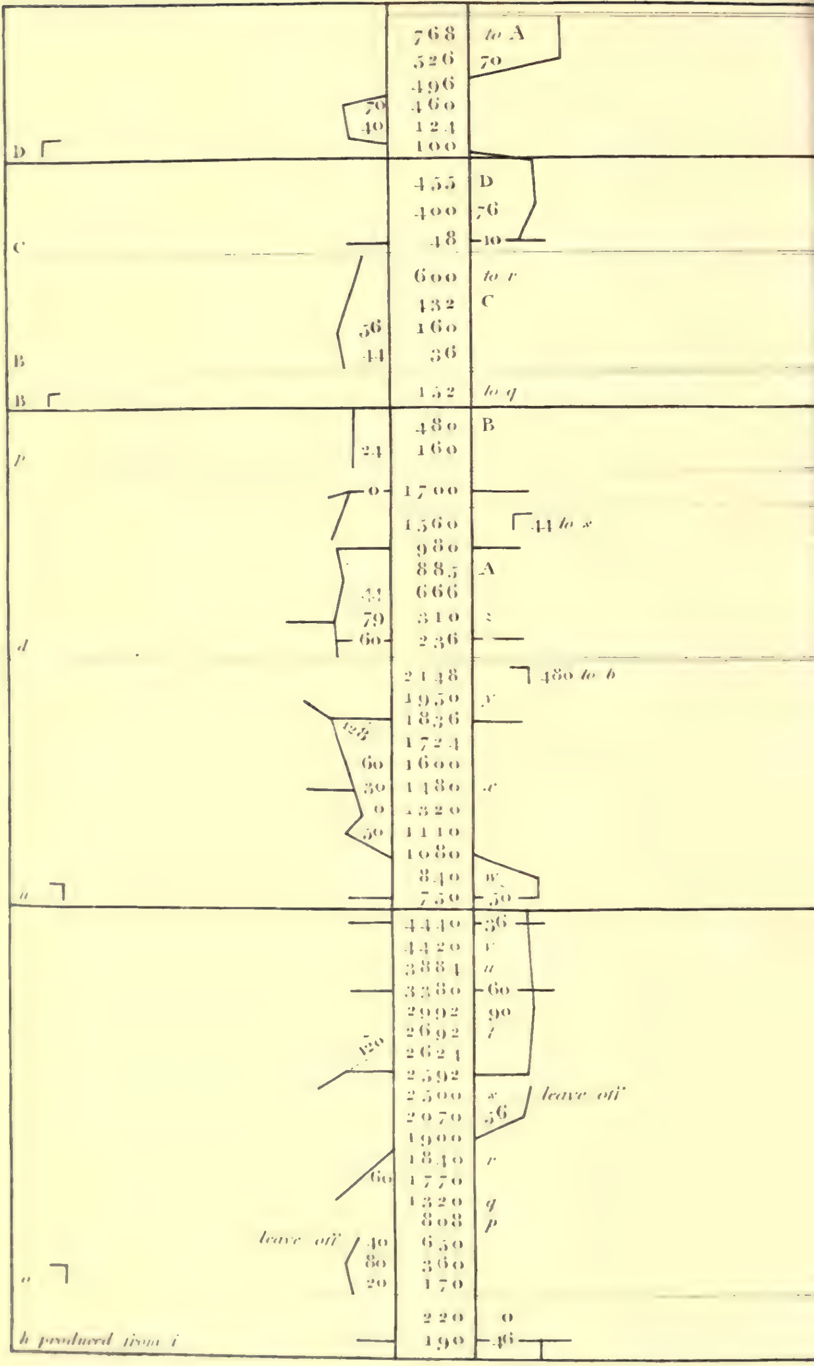
$$\left.\begin{array}{lll} \sin\tfrac{1}{2}A = \sqrt{\dfrac{(s-b)(s-c)}{bc}} & \cos\tfrac{1}{2}A = \sqrt{\dfrac{s(s-a)}{bc}} & \tan\tfrac{1}{2}A = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} \\ \sin\tfrac{1}{2}B = \sqrt{\dfrac{(s-a)(s-c)}{ac}} & \cos\tfrac{1}{2}B = \sqrt{\dfrac{s(s-b)}{ac}} & \tan\tfrac{1}{2}B = \sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}} \\ \sin\tfrac{1}{2}C = \sqrt{\dfrac{(s-a)(s-b)}{ab}} & \cos\tfrac{1}{2}C = \sqrt{\dfrac{s(s-c)}{ab}} & \tan\tfrac{1}{2}C = \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}} \end{array}\right\} \ldots (5)$$

Field Book.


1310
56 to c
836
56
684
50
1480
90 to g
960
24
930
n
700
48
400
30
1430
to i
1290
40
1004
36
980
m
610
24
280
32
1820
to l
1464
22
100
1050
920
32
650
60
350
48
0
14
3074
to h
2494
2100
l
0
2072
54
1730
80
1530
1420
k
56 + 30
1170
52
620
32
280
40
2574
to j
2494
2000
44
1880
50
1840
50
1794
i
34 + 50
1464
76
1328
96
1240
52 + 34
1130
34
860
66
190
4450
h
3570
g
2620
f
2610
2210
2080
e
1640
d
1550
1510
c
990
b
806

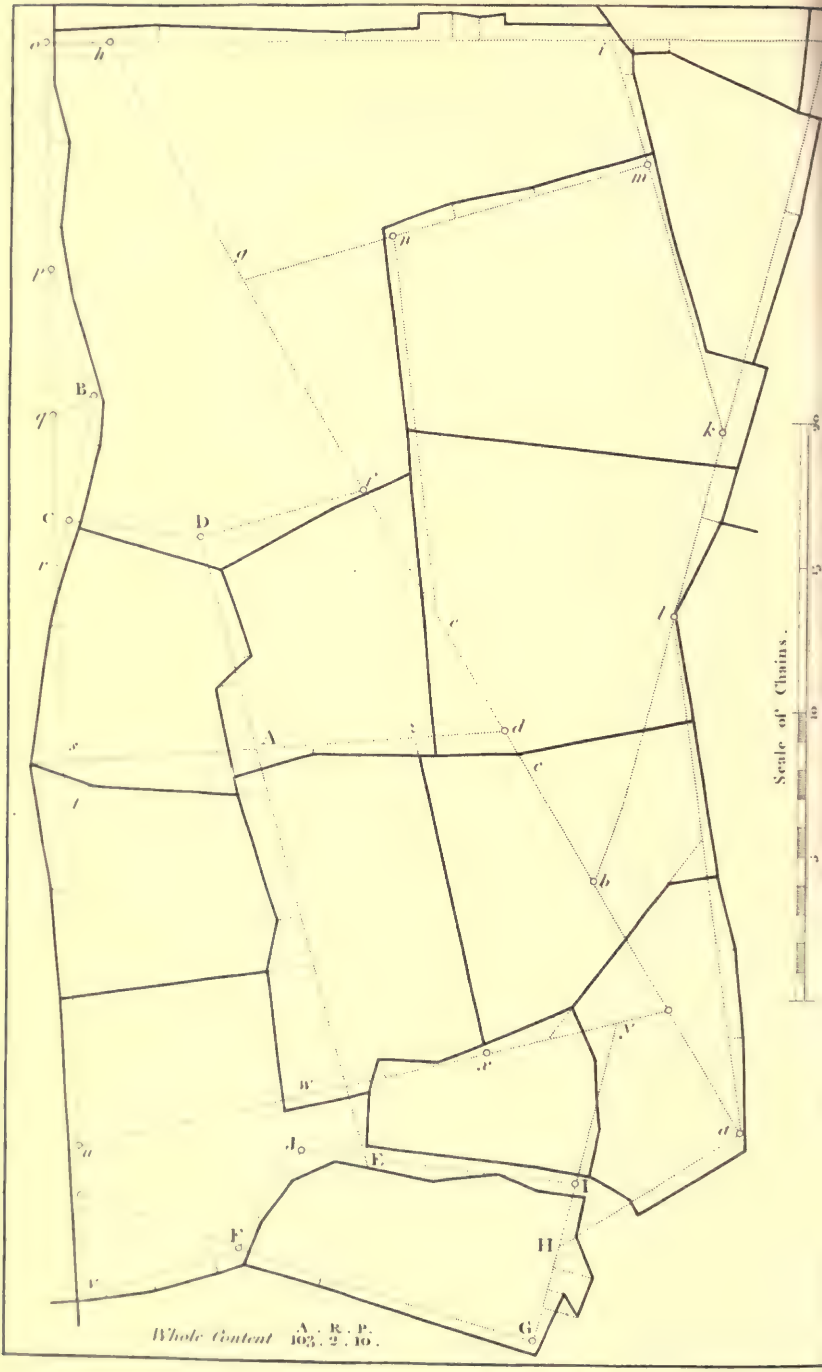
1.

Field Book.


768 to A
526 70
496
70 460
40 124
D 100
455 D
400 76
C 48 40
600 to r
132 C
56 160
B 44 36
B 152 to q
480 B
p 24 160
0 1700
1560 44 to s
980
885 A
44 666
79 310 z
d 60 236
2148 480 to b
1950 y
1836
128 1724
60 1600
30 1480 x
0 1320
50 1110
1080
840 w
a 750 50
4440 36
4420 v
3884 u
3380 60
2992 90
2692 t
120 2624
2592
2500 s leave off
2070 56
1900
1840 r
60 1770
1320 q
808 p
leave off 40 650
80 360
a 20 170
220 o
h produced from i 190 46

	580	*to* v
40	500	
76	300	
76	100	
	420	*to* F
20	150	
	954	J
15	850	
	740	*to* E
30	490	
	340	60
0	280	
20	170	50
	725	*to* H
	672	0
70	450	0
50	15	
	1160	*to* y
32	1000	
	890	
	780	32
	590	40
	570	I
	530	40
	376	H
	256	150
	190	64
	144	130
	1676	G
	1676	30
	896	24
	632	
	620	50
from u towards x	588	F
	620	*to* t
	488	32
	2260	
	2250	E
20	2210	
56	2050	
	2030	
	1990	130 *to* w
	1552	180
	1380	96
	950	110
	860	

Plan from the foregoing Field Book.



When it is required to compute all the angles of the triangle, the third set of formulæ is the most convenient, as there are only four logarithms and three arithmetical complements to be taken out. In each of the others there are seven required to obtain the three angles. A still more convenient form, however, is the following, as it requires only four logarithms:—

Put $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$: then we shall have for the tangents,

$$\tan \tfrac{1}{2}\text{A} = \frac{r}{s-a},\ \tan \tfrac{1}{2}\text{B} = \frac{r}{s-b},\ \text{and } \tan \tfrac{1}{2}\text{C} = \frac{r}{s-c} \quad \ldots\ldots (6)$$

4. Resume the equations marked (3): then, since $\frac{a}{b} = \frac{\sin \text{A}}{\sin \text{B}}$, we shall have

$$\frac{a-b}{a+b} = \frac{\sin \text{A} - \sin \text{B}}{\sin \text{A} + \sin \text{B}} = \frac{\tan \frac{1}{2}(\text{A} - \text{B})}{\tan \frac{1}{2}(\text{A} + \text{B})} = \frac{\tan \frac{1}{2}(\text{A} - \text{B})}{\cot \frac{1}{2}\text{C}} \ldots\ldots\ldots (7)$$

and similar expressions for each of the other pairs of sides in succession.

Or again, from the same equations

$$\left.\begin{aligned} \frac{a+b}{c} &= \frac{\sin \text{A} + \sin \text{B}}{\sin \text{C}} = \frac{\sin \text{A} + \sin \text{B}}{\sin(\text{A}+\text{B})} = \frac{\cos \frac{1}{2}(\text{A}-\text{B})}{\cos \frac{1}{2}(\text{A}+\text{B})} \\ \frac{a-b}{c} &= \frac{\sin \text{A} - \sin \text{B}}{\sin \text{C}} = \frac{\sin \text{A} - \sin \text{B}}{\sin(\text{A}+\text{B})} = \frac{\sin \frac{1}{2}(\text{A}-\text{B})}{\sin \frac{1}{2}(\text{A}+\text{B})} \end{aligned}\right\} \ldots\ldots (8)$$

These equations may be put in a convenient form for the calculation of the side c when the sides a, b, and angle C are given: viz.

$c = (a - b) \cos \frac{1}{2}\text{C} \operatorname{cosec} \frac{1}{2}(\text{A} - \text{B})$, and $c = (a + b) \sin \frac{1}{2}\text{C} \sec \frac{1}{2}(\text{A} - \text{B})$ and the two values of c thus computed will be a mutual check upon the accuracy of the work, by which each was obtained.

5. Returning to equation (1) we have in succession,

$$c^2 = a^2 - 2ab \cos \text{C} + b^2 = (a+b)^2 - 2ab\,(1 + \cos \text{C})$$
$$= (a+b)^2 \left\{ 1 - \frac{4ab \cos^2 \frac{1}{2}\text{C}}{(a+b)^2} \right\}$$

Put $\cos \theta = \frac{2\sqrt{ab} \cos \frac{1}{2}\text{C}}{a+b}$; then $\sin^2\theta = 1 - \frac{4ab \cos^2 \frac{1}{2}\text{C}}{(a+b)^2}$ and we have

$$c^2 = (a+b)^2 \sin^2\theta, \text{ or } c = (a+b) \sin \theta \ \ldots\ldots\ldots\ldots\ldots (9)$$

Or, again, we may write the equation in the following successive forms,

$$c^2 = (a-b)^2 + 2ab\,(1 - \cos \text{C}) = (a-b)^2 \left\{ 1 + \frac{4ab \sin^2 \frac{1}{2}\text{C}}{(a-b)^2} \right\}$$

Put $\tan \chi = \frac{2\sqrt{ab} \sin \frac{1}{2}\text{C}}{a-b}$, then we have as before

$$c^2 = (a-b)^2 \sec^2\chi, \text{ or } c = (a-b) \sec \chi \ \ldots\ldots\ldots\ldots (10)$$

This method is much used by the continental mathematicians: but it is not so convenient for use as those in (7, 8).

6. Multiplying together the values of $\sin \frac{1}{2}\text{A}$, $\cos \frac{1}{2}\text{A}$ found in (5), we have

$$2 \sin \tfrac{1}{2}\text{A} \cos \tfrac{1}{2}\text{A} = \sin \text{A} = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc}, \text{ or}$$

$$bc \sin \text{A} = 2\sqrt{s(s-a)(s-b)(s-c)} \ \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

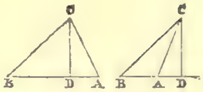
and similar expressions for sin B, and sin C.

Hence, comparing this with equation (2) we have the expressions there given in another form, and adapted to logarithmic use.

7. Since, in the triangle ABC we have $\text{CD} = b \sin \text{A}$, we get from (11)

$$\text{CD} = p_3 = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{c} \ \ldots\ldots\ldots\ldots\ldots (12)$$

and similar values for the perpendiculars p_2, p_1, from B and A.



8. Again, let CD be the perpendicular from C upon c as before. Then,

$$\mathrm{CD}\cot \mathrm{B} \pm \mathrm{CD}\cot \mathrm{A} = c\,; \text{ or}$$

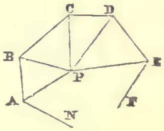
$$\mathrm{CD} = \frac{c}{\cot \mathrm{B} \pm \cot \mathrm{A}} = \frac{c \sin \mathrm{A} \sin \mathrm{B}}{\sin(\mathrm{A} \pm \mathrm{B})} = \frac{c \sin \mathrm{A} \sin \mathrm{B}}{\sin \mathrm{C}} \quad \ldots\ldots (12)$$

This might have been derived, but not quite so briefly, from equation (3): and then

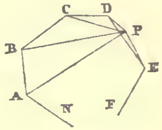
$$\mathrm{BD} = \mathrm{CD} \cot \mathrm{B} = \frac{c \sin \mathrm{A} \cos \mathrm{B}}{\sin \mathrm{C}}, \text{ and } \mathrm{DA} = \mathrm{CD} \cot \mathrm{A} = \frac{c \cos \mathrm{A} \sin \mathrm{B}}{\sin \mathrm{C}} \quad \ldots (13)$$

These formulæ are often useful, both in the determination of the heights or distances of objects, and in mensuration.

9. Let ABCDE.... be any polygon, and P a point in the same plane, or not, and situated either within or without the polygon: draw lines from P to all the angular points of the figure.



Put PA $= a$	PAN $= \alpha_1$	PAB $= \alpha$
PB $= b$	PBA $= \beta_1$	PBC $= \beta$
PC $= c$	PCB $= \gamma_1$	PCD $= \gamma$
....	....	....



Then,

$$\begin{aligned} a \sin \alpha_1 &= n \sin \nu \\ b \sin \beta_1 &= a \sin \alpha \\ c \sin \gamma_1 &= b \sin \beta \\ d \sin \delta_1 &= c \sin \gamma \\ &\ldots\ldots\ldots \\ m \sin \mu_1 &= l \sin \lambda \\ n \sin \nu_1 &= m \sin \mu \end{aligned}$$

and multiplying these together we get, since the linear factors mutually cancel

$$\sin \alpha_1 \sin \beta_1 \sin \gamma_1 \ldots \sin \mu_1 \sin \nu_1 = \sin \alpha \sin \beta \sin \gamma \ldots \sin \mu \sin \nu \quad \ldots. (14)$$

XVII. THE NUMERICAL SOLUTION OF PLANE TRIANGLES.

When only the parts of the triangle itself are concerned in the inquiry, either as given or required, there are three general cases, the formulæ necessary for the solution of which have already been given.

1. *When a side and its opposite angle are amongst the data.*
2. *When two sides and their included angle form the data.*
3. *When the three sides form the data.*

CASE I.

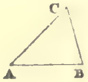
When a side and its opposite angle are amongst the data.

This divides itself into two subordinate cases, according as the third datum is an angle or a side.

1. *Let the third datum be an angle.* Then (*eq.* 3, p. 448.) if ABC be the triangle, and we have given a, A, B; then $\mathrm{C} = \pi - (\mathrm{A} + \mathrm{B})$

$$b = \frac{a \sin \mathrm{B}}{\sin \mathrm{A}} = a \sin \mathrm{B} \operatorname{cosec} \mathrm{A}.$$

$$c = \frac{a \sin \mathrm{C}}{\sin \mathrm{A}} = a \sin (\mathrm{A} + \mathrm{B}) \operatorname{cosec} \mathrm{A}.$$



Ex. Let BC = 305·296, B = 51° 15′ 35″, and C = 37° 21′ 25″ be given. Here A = 180° — B — C = 91° 23′, and sin A = sin 91° 23′ = sin 88° 37′.

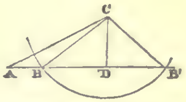
Also, $b = \dfrac{a \sin B}{\sin A}$ and $c = \dfrac{a \sin C}{\sin A}$: whence the work will stand thus,

log 305·29	=	2·4847126	log 305·29	=	2·4847126
pp 6	=	85	pp 6	=	85
sin 51° 15′	=	9·8920303	sin 37° 21′	=	9·7829614
pp 35″	=	591	pp 25″	=	690
cosec 88° 37′	=	10·0001266	cosec 88° 37′	=	10·0001266
log b	=	2·3769371	log c	=	2·2678781
or, b	=	238·1974.	or, c	=	185·3011.

2. *Let the third datum be a side.* Then if we have given a, b, A to find the other parts, we shall have $\sin B = \dfrac{b \sin A}{a}$; in which case two of the angles will become known, and we can proceed as in the former sub-case.

It is necessary, however, to observe that as sin B = sin (π — B) we have no reason to prefer one value of B to the other: or as it is usually expressed, the solution is double. There are, in fact, two different triangles which will fulfil the given conditions, and yet two of whose quæsita are different from each other. This is altogether analogous to the double solutions of questions in algebra which give rise to quadratic equations; and, indeed, if this example were solved algebraically for the third side, we should find (as may be easily shown) that its value was given by a quadratic equation.

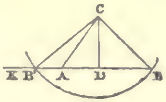
To explain this more clearly, let us consider the geometrical problem, in which are given two sides AC, CB, and the angle CAB opposite to one of them to construct the triangle.

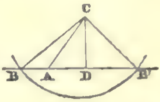


Constr. With centre C and distance CB describe a circle, cutting AB in B, and B′. Join CB,CB′: then obviously *either* of the triangles ABC or AB′C fulfils all the conditions, and may be taken as that required.

Also, since BC = B′C, the angle CBB′ = CB′B; but CBA is the supplement of CBB′ and hence also of CB′B. This accords entirely with the inference drawn above, that the two values of the angle B are supplementary, the one of the other. It will also follow that the angle C will be double according to the values of B, and likewise that the side C will be double from the same cause, viz. AB and AB′.

In the preceding investigation of the ambiguous case, our reasonings have turned upon the assumption that *the less of the given sides is opposite to the given angle.* Suppose now that the greater of the given sides is opposite to the given angle, and construct as before: then since CB is greater than CA, CB′ will also be greater than CA, and the line CB′ will lie on the opposite side of CA from CB, and the triangle formed by the lines CA, AB, CB′ (produced where necessary for the formation of the triangle) will be ACB′, having the angle CAB′ the supplement of CAB instead of equal to it. Two triangles, therefore, fulfilling all the conditions can only be formed when the less of the given sides is opposite to the given angle. This, therefore, is the only ambiguous case.





Ex. 1. Given $c = 195\cdot265$, $b = 203\cdot162$, and $B = 45^\circ\ 0'\ 55''$ to find the rest. Here since the greater side is opposite to the given angle, the problem is not ambiguous. The formulæ of solution are

$$\sin C = \frac{c \sin B}{b}, \text{ and } a = b \sin A \operatorname{cosec} B.$$

$$\begin{aligned} \log 195\cdot265 &= 2\cdot2906244 \\ \sin 45^\circ\ 0'\ 55'' &= 9\cdot8496008 \\ \text{ac} \log 203\cdot162 &= 7\cdot6921575 \\ \hline \sin C &= 9\cdot8323827 \\ \text{or, } C &= 42^\circ\ 49'\ 42''; \\ \text{and hence, } A &= 92^\circ\ \ 9'\ 23''. \end{aligned}$$

$$\begin{aligned} \log 203\cdot162 &= 2\cdot3078425 \\ \sin 92^\circ\ 9'\ 23'' &= 9\cdot9996923 \\ \operatorname{cosec} 45^\circ\ 0'\ 55'' &= 10\cdot1503992 \\ \hline \log a &= 2\cdot4579340 \\ \text{whence } a &= 287\cdot0350 \end{aligned}$$

Ex. 2. Given $a = 350\cdot169$, $b = 236\cdot291$, and $B = 38^\circ\ 39'\ 15''$ to find A, B, c. In this, since the less side is opposite to the given angle, the solution is double, or belongs to the ambiguous case. The formulæ are

$$\sin A = \frac{a \sin B}{b}, \ C = 180^\circ - (A + B), \text{ and } c = a \sin C \operatorname{cosec} A.$$

$$\begin{aligned} \log 350\cdot169 &= 2\cdot5442777 \\ \text{ac} \log 236\cdot291 &= 7\cdot6265529 \\ \sin 38^\circ\ 39'\ 15'' &= 9\cdot7956146 \\ \hline \sin A &= 9\cdot9664452. \end{aligned}$$

$$\begin{aligned} \text{Hence } A &= 67^\circ\ 45'\ 58'' \\ B &= 38\ \ 39\ \ 15 \\ \hline A + B &= 106\ \ 25\ \ 13 \\ C &= 73\ \ 34\ \ 47 \end{aligned}$$

$$\begin{aligned} \text{and } A' &= 112^\circ\ 14'\ \ 2'' \\ B &= 38\ \ 39\ \ 15 \\ \hline A' + B &= 150\ \ 53\ \ 17 \\ C' &= 29\ \ \ 6\ \ 43 \end{aligned}$$

$$\begin{aligned} \log 350\cdot169 &= 2\cdot5442777 \\ \sin 73^\circ\ 34'\ 47'' &= 9\cdot9819155 \\ \operatorname{cosec} 67\ \ 45\ \ 58 &= 10\cdot0335548 \\ \hline \log c &= 2\cdot5597480 \end{aligned}$$

$$\begin{aligned} \log 350\cdot169 &= 2\cdot5442777 \\ \sin 29^\circ\ 6'\ 43'' &= 9\cdot6870985 \\ \operatorname{cosec} 112\ \ 14\ \ 2 &= 10\cdot0335548 \\ \hline \log c' &= 2\cdot2649310 \end{aligned}$$

Hence, we have the side AB = 362·8674, and the side A′B = 184·048 0

EXAMPLES FOR PRACTICE.

1. Given $A = 37^\circ\ 20'$, $a = 232$ and $c = 345$ yards, to find b, B, C.
2. Given $c = 365$, $A = 57^\circ\ 12'$, $B = 24^\circ\ 45'$, to find a, b, C.
3. Given the following parts of the triangles, to find the remaining ones.

	a	b	c	A	B	C
(1)	197	....	237	....	....	90° 0′ 0″
(2)	310	....	....	....	62° 9′ 0″	41 13 10
(3)	....	305	217	....	45 0 0	....
(4)	516	329	....	....	37 10 15	....
(5)	232·198	345·261	....	142° 29′ 10″	....	....
	....	203·162	195·265	92° 9′ 23″	....	....

4. Given $a = 10$, $b = 12$, and $A = 37^\circ\ 15'$ to find the remaining parts without the employment of logarithms.

5. Given $a + b : a - b :: 15 : 2$ and $b + c : b - c :: 14 : 3$ to find the angles of the triangle.

6. In the same triangle, if $3a + 4b - 6c = 17\cdot86$, find the sides of the triangle.

CASE II.

When two sides, a, b, and their included angle C are given.

First method. By *th.* 7. p. 449, find $\tan \frac{1}{2}(A - B) = \frac{a-b}{a+b} \cot \frac{1}{2}C$: then we shall have $\frac{1}{2}(A + B) + \frac{1}{2}(A - B) = A$, and $\frac{1}{2}(A + B) - \frac{1}{2}(A - B) = B$; and the third side by *Case* I.

When the logs of a and b are given, as is generally the case when the sides have been found by a previous investigation, employ the method explained at p. 440 for finding $\tan \frac{1}{2}(A - B)$ and complete the solution as above.

Second method. When only the third side is required, employ equations (8) p. 449, and the subsequent formula there given. This is rather shorter than the other, though it has not been much used from not being so generally known.

Third method. Under the same circumstances employ equations (9, 10), p. 449; which also furnishes a good mode of solution.

Fourth method. Employ the equations $c = \sqrt{a^2 - 2ab \cos C + b^2}$,
$\sin B = \frac{b \sin C}{\sqrt{a^2 - 2ab \cos C + b^2}}$, and $\sin A = \frac{a \sin C}{\sqrt{a^2 - 2ab \cos C + b^2}}$.

The first of these methods was invented by *Vieta:* the second by *Thacker;* and it was published in 1743, but no attention appears to have been given to it till Dr. Gregory, in a former edition of this work, gave it a practical form. Dr. Wallace has also discussed it in the *Edinb. Trans. vol.* x. The author of the third I am not able to assign, but it is that most commonly employed by the continental mathematicians. The fourth is not adapted to calculation, and its only use is, in the transformation of trigonometrical impressions.

Ex. 1. Given $a = 16{\cdot}9584$, $b = 11{\cdot}9613$, $C = 60^\circ\ 43'\ 36''$.

First method.

Here, $\tan \frac{1}{2}(A - B) = \frac{a-b}{a+b} \cot \frac{1}{2}C$; and $c = a \sin C \operatorname{cosec} A = b \sin C \operatorname{cosec} B$.

For the coef of cot $\frac{1}{2}C$	*For the value of* tan $\frac{1}{2}(A-B)$	*For A, B, separately.*
a = 16·9584	log 4·9971 = 0·6987180	$\frac{1}{2}$(A+B)=59°38′12″
b = 11·9613	ac log 28·9197 = 8·5388062	$\frac{1}{2}$(A−B)=16 26 0·23
	cot 30° 21′ 48″ = 10·2322235	
$a + b$ = 28·9197		A=76 4 12·23
$a - b$ = 4·9971	tan $\frac{1}{2}$(A−B) = 9·4697477	B=43 12 11·77

For $c = \frac{a \sin C}{\sin A}$		*For* $c = \frac{b \sin C}{\sin B}$	
log 16·9584 =	1·2293848	log 11·9613 =	1·0777784
sin 60° 43′ 36″ =	9·9406644	sin 60° 43′ 36″ =	9·9406644
cosec 76 4 12·23 =	10·0129638	cosec 43 12 11·77 =	10·1645703
	1·1830130		1·1830131

From both of which we have $c = 15{\cdot}24098$ nearly.

Second method.

In this, having found $\tan \frac{1}{2}(A-B)$, as before, we have the following process:

$c = (a + b) \cos \frac{1}{2}(A + B) \sec \frac{1}{2}(A - B)$.		$c = (a-b) \sin \frac{1}{2}(A+B) \operatorname{cosec} \frac{1}{2}(A-B)$.	
log 28·9197 =	1·4611938	log 4·9971 =	0·6987180
cos 59° 38′ 12″ =	9·7037054	sin 59° 38′ 12″ =	9·9359289
sec 16 26 0·23 =	10·0181139	cosec 16 26 0·23 =	10·5483662
	1·1830131		1·1830131

which give the same values of c as were obtained in the former solution.

Third method.

We have $\cos\theta = \dfrac{\sqrt{4ab}\cos\frac{1}{2}C}{a+b}$, and $c = (a+b)\sin\theta$, for one solution: and

$$\tan\chi = \frac{\sqrt{4ab}\sin\frac{1}{2}C}{a-b}, \text{ and } c = (a-b)\sec\chi, \text{ for the other.}$$

For log $2\sqrt{ab}$ or *log* $\sqrt{4ab}$.

log 16·9584	=	1·2293848
log 11·9613	=	1·0777784
log 4.	=	0·6020600
	2	2·9092232
log $2\sqrt{ab}$	=	1·4546116

$$\cos\theta = \frac{2\sqrt{ab}\cos\frac{1}{2}C}{a+b}.$$

log $2\sqrt{ab}$	=	1·4546116
cos 30° 21′ 48″	=	9·9359289
ac log 28·9197	=	8·5388062
cos θ	=	9·9293467

$c = (a+b)\sin\theta$

log 28 9197	=	1·4611938
sin 31° 48′ 13″·32	=	9·7218194
log c	=	1·1830132

$$\tan\chi = \frac{2\sqrt{ab}\sin\frac{1}{2}C}{a-b}.$$

log $2\sqrt{ab}$	=	1·4546116
sin 30° 21′ 48″	=	9·7037055
ac log 4·9971	=	9·3012820
tan χ	=	10·4595991

$c = (a-b)\sec\chi$.

log 4·9971	=	0 6987180
sec 70° 51′ 37″·11	=	10·4842951
log c	=	1·1830131

which are very nearly the same logarithms as those found by the other modes of solution.

Scholium.

When it can be done, it is desirable to obtain check-solutions. In this case it can be done in each of the methods; but where the operation is one of consequence, it will always be better to employ the first and third, or the second and third methods, in preference to the checks by the same method. If, however, the same method be employed for the solution and its check, the third is the best; and perhaps shorter altogether than the separate application of the first and second, as checks upon each other.

The first solution requires ten openings of the tables, the second five, and the third eleven, for the solution and its check.

Ex. 2. Let the logs of the given sides be 2·2293848 and 2·0777784, and C = 60° 43′ 36″ to resolve the triangle.

$$\text{Here } \frac{a-b}{a+b} = \frac{1-\dfrac{b}{a}}{1+\dfrac{b}{a}} = \frac{1-\tan\chi}{1+\tan\chi} = \cot(45° + \chi), \text{ where } \tan\chi = \frac{b}{a}.$$

---

* It may surprise the student to be informed that it is always much easier to observe angles to any given degree of accuracy than to measure lines: but such, owing to a variety of causes hereafter to be explained, is the case. The consequence is, that after the careful measurement of a single base, all the subsequent distances are found by calculation from that base and observed angles. This renders it *essential* to check every calculation, either by a different process, or by having the computations performed by two persons independently of each other. Few persons are capable of such steadiness of attention to numerical operation, as to be entirely certain of avoiding mistakes; and no one would trust to a single computation where a slight error at any one of perhaps ten thousand steps would vitiate the entire result, and the entire labour bestowed upon the work be thrown away. The practice of checking solutions should, therefore, be commenced early in every course of study of this nature.

$$\tan \chi = \frac{b}{a}.$$

$$\log b = 12{\cdot}0777784$$
$$\log a = 2{\cdot}2293848$$
$$\tan \chi = 9{\cdot}8483936$$
$$\text{or, } \chi = 35^\circ\ 11'\ 47''{\cdot}75$$

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \cot \tfrac{1}{2}C.$$

$$\cot 80^\circ\ 11'\ 47''{\cdot}75 = 9{\cdot}2375241$$
$$\cot 30\ \ 21\ \ 48 = 10{\cdot}2322235$$
$$\tan \tfrac{1}{2}(A - B) = 9{\cdot}4697476$$

very nearly as before found.

Then having the logs of the sides given, and all the angles, the solution is completed as in the first method.

EXAMPLES FOR EXERCISE.

1. Given $a = 345{\cdot}02$, $b = 174{\cdot}07$, $C = 37^\circ\ 20'\ 30''$, to resolve the triangle.
2. Given $c = 365$, $b = 154{\cdot}33$, $A = 57^\circ\ 12'\ 10''$ to resolve the triangle.
3. Given $a = 112$, $b = 120$, $C = 57^\circ\ 58'\ 39''$, to resolve the triangle.
4. $\log a + \log b = 5{\cdot}1693765$, $\log a - \log b = {\cdot}7629876$, together with $n \tan C = 1{\cdot}8656729$, to find the remaining parts of the triangle.
5. If $a : b :: 3 : 4$, $\cos C = 0$, and the perpendicular from C to c be 100, what are the sides, and other angles of the triangle?

CASE III.

When the sides a, b, c, are given, to find the angles A, B, C.

First method. By equation 1, p. 448, we have the solution from

$$\cos A = \frac{-a^2 + b^2 + c^2}{2bc},\ \cos B = \frac{a^2 - b^2 + c^2}{2ca},\ \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Second method. Find $s = \frac{1}{2}(a + b + c)$, and form $s—a$, $s—b$, $s—c$: then

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s\,(s-a)}},\ \tan \tfrac{1}{2}B = \sqrt{\frac{(s-c)\,(s-a)}{s\,(s-b)}},\ \tan \tfrac{1}{2}C = \sqrt{\frac{(s-a)\,(s-b)}{s\,(s-c)}}.$$

Third method. Find $r = \sqrt{\frac{(s-a)\,(s-b)\,(s-c)}{s}}$; then (eq. 6, p. 449.)

$$\tan \tfrac{1}{2}A = \frac{r}{s-a},\ \tan \tfrac{1}{2}B = \frac{r}{s-b},\ \tan \tfrac{1}{2}C = \frac{r}{s-c}.$$

The first of these methods is adapted to the case where a, b, c are small numbers, or are in the ratio of small numbers. The second (which was discovered about 200 years ago by William Purser of Dublin) is well adapted, like the first, to the case where either one or all the angles are required; the use of the tangents is, however, preferable to that of the sines and cosines of $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$, given in the same place, as it requires the use of fewer logarithms. The third is an elegant modification of the second, and was first proposed by Dr. Wallace.

In every case, to secure accuracy, compute the three angles or semiangles as the case may require: then if the work be correct, their sum will be 180° in the first, and 90° in the second and third methods.

Ex. 1. Let $a = 3$, $b = 5$, $c = 7$: find A, B, C.

Here the first method applies, and gives successively,

$$\cos A = \frac{-3^2 + 5^2 + 7^2}{2\,.\,5\,.\,7} = \frac{13}{14} = {\cdot}9285714 = \cos\ 21^\circ 47'\ 12''\tfrac{1}{2}$$

$$\cos B = \frac{3^2 - 5^2 + 7^2}{2\,.\,3\,.\,7} = \frac{11}{14} = {\cdot}7857142 = \cos\ 38\ \ 12\ \ 47\tfrac{1}{2}$$

$$\cos C = \frac{3^2 + 5^2 - 7^2}{2\,.\,3\,.\,5} = -\frac{1}{2} = -{\cdot}5000000 = \cos 120\ \ 0\ \ 0$$

$$\text{Proof of the work, } A + B + C = 180\ \ 0\ \ 0$$

Ex. 2. Given $a = 32986$, $b = 43628$, $c = 62984$ to find the angles A, B, C.

Here the *second method* may be used.

For the factors.

$$\begin{array}{rcr} a & = & 32986 \\ b & = & 43628 \\ c & = & 62984 \\ \hline & 2 & 139598 \\ \hline s & = & 69799 \\ s-a & = & 36813 \\ s-b & = & 26171 \\ s-c & = & 6815 \end{array}$$

For $\tan \frac{1}{2}A$.

$$\begin{array}{rcr} \log 26171 & = & 4{\cdot}4178203 \\ \log 6815 & = & 3{\cdot}8334659 \\ \text{ac} \log 69799 & = & 5{\cdot}1561508 \\ \text{ac} \log 36813 & = & 5{\cdot}4339988 \\ \hline & 2 & 18{\cdot}8414358 \\ \hline \tan 14^\circ\, 45'\, 35''\tfrac{1}{2} & = & 9{\cdot}4207179 \end{array}$$

Hence $A = 29^\circ\, 31'\, 11''$, nearly.

For $\tan \frac{1}{2}B$.

$$\begin{array}{rcr} \log 36813 & = & 4{\cdot}5660012 \\ \log 6815 & = & 3{\cdot}8334659 \\ \text{ac} \log 69799 & = & 5{\cdot}1561508 \\ \text{ac} \log 26171 & = & 5{\cdot}5821797 \\ \hline & 2 & 19{\cdot}1377976 \\ \hline \tan 20^\circ\, 20'\, 4'' & = & 9{\cdot}5688988 \end{array}$$

or $B = 40^\circ\, 40'\, 8''$, nearly.

For $\tan \frac{1}{2}C$.

$$\begin{array}{rcr} \log 26171 & = & 4{\cdot}4178203 \\ \log 36813 & = & 4\ 5660012 \\ \text{ac} \log 69799 & = & 5{\cdot}1561508 \\ \text{ac} \log 6815 & = & 6{\cdot}1665341 \\ \hline & 2 & 20{\cdot}3065064 \\ \hline \tan 54^\circ\, 54'\, 20''\tfrac{1}{2} & = & 10{\cdot}1532532 \end{array}$$

or $C = 109^\circ\, 48'\, 41''$, nearly.

By the *third method*, $r = \sqrt{\dfrac{(s-a)\ (s-b)\ (s-c)}{s}}$, and then $\tan \frac{1}{2}A = \dfrac{r}{s-a}$, and so on. The work will, then, stand as follows:—

$$\begin{array}{rcl} \log 36813 & = & 4{\cdot}5660012 \\ \log 26171 & = & 4{\cdot}4178203 \\ \log 6815 & = & 3{\cdot}8334659 \\ \text{ac} \log 69799 & = & 5{\cdot}1561508 \\ \hline & 2 & 7{\cdot}9734382 \\ \hline 10 + \log r & = & 13{\cdot}9867191 \\ \log r - \log (s-a) & = & 9{\cdot}4207179 = \tan 14^\circ\, 45'\, 35''\tfrac{1}{2} \\ \log r - \log (s-b) & = & 9{\cdot}5688988 = \tan 20\ \ 20\ \ \ 4 \\ \log r - \log (s-c) & = & 10{\cdot}1532532 = \tan 54\ \ 54\ \ 20\tfrac{1}{2} \end{array}$$

Whence A, B, C are found by doubling these results respectively.

EXAMPLES FOR EXERCISE.

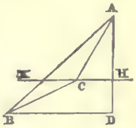
1. Given $a = 174{\cdot}07$, $b = 232$, $c = 345$ to find A, B, C.
2. Given $a = 309{\cdot}86$, $b = 154{\cdot}33$, $c = 365$, to resolve the triangle.
3. Given $a = 112$, $b = 112{\cdot}65$, $c = 120$, to find the angles.
4. Given $a = 3$, $b = 4$, $c = 5$, to find the angles.
5. Given $a + b = 48{\cdot}2106$, $b + c = 60{\cdot}1250$, and $c + a = 44{\cdot}4114$, to find the sides and angles.
6. Given 24804, 57876, 74412 to find the angles by the three methods.
7. Two sides of four triangles are the same, viz. 35 and 36; but the third sides are 70·99, 70·98, 70·97, and 70·96 respectively: find the angles of these four triangles.
8. Two sides of three triangles are 315965 and 315966, and the third sides are respectively 20, 21, and 22: required the angles of each triangle.

XVIII. THE APPLICATION OF TRIGONOMETRY TO THE DETERMINATION OF THE HEIGHTS AND DISTANCES OF OBJECTS.

A line is said, in technical language, to be *measured,* and an angle to be *observed.* It is not, however, intended here to enter into a description either of the instruments employed or the methods of using them: but a succinct account of both will be given in the GEODESY in the second volume. A few specimens of the class of calculations by which the solutions are obtained are given at the commencement of this chapter, and a series of unsolved questions for the exercise of the student is annexed.

EXAMPLES.

1. *From the foot B of a hill, the lower part of whose inclination was different from that of the upper, the elevation of A the summit was observed to be* 32° 10′ 15″, *and the inclination of the slope of the lower part to be* 15° 10′ 45″; *and at the upper end C of the slope the elevation of the summit was* 57° 10′ 30″: *what was the height of the hill, supposing the distance BC to be* 952·57 *yards?*



Let $BC = a$, $ABD = \beta$, $CBD = \alpha$, and $ACH = \gamma$. Then $KCB = CBD = \alpha$, $ACK = \pi - \gamma$, or $ACB = \pi - \gamma + \alpha$; also $BAC = \pi - ABC - BCA = \pi - (\beta - \alpha) - (\pi - \gamma + \alpha)$ $= \gamma - \beta$. We have, therefore,

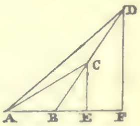
$$BA = \frac{BC \sin BCA}{\sin BAC} = \frac{a \sin(\gamma - \alpha)}{\sin(\gamma - \beta)}\text{; and hence we have}$$

$$AD = AB \sin ABD = \frac{a \sin\beta \sin(\gamma-\alpha)}{\sin(\gamma-\beta)}.$$

In the example we have $a = 952{\cdot}57$, $\beta = 32°\ 10'\ 15''$, $\alpha = 15°\ 10'\ 45''$, and $\gamma = 57°\ 10'\ 30''$; hence the calculation will be as follows:—

log 952·57	=	2·9788969	$\gamma-\alpha = 41°\ 59'\ 45''$
sin 32° 10′ 15″	=	9·7262751	$\gamma-\beta = 25\ \ 0\ \ 15$
sin 41 59 45	=	9·8254758	
cosec 25 0 15	=	10·3739840	
log 802·8451	=	2·9046318; or AD = 802·8451 yds.	

2. *The elevations of two mountains in the same line with the observer are* 10° *and* 20°; *but upon approaching four miles nearer they have both an elevation of* 40°: *it is required to find their heights, their distance apart, and the distances of the two points of observation from each.*



Let CE, DF be the heights of the two mountains, and A, B the places of observation. Put, generally, $AB = a$, $CAF = \alpha$, $DAF = \alpha_1$, $DBF = \beta$: then, by *Case* 1. *pl. tr.*

$$BD = \frac{a \sin\alpha_1}{\sin(\beta-\alpha_1)} \text{ and } BC = \frac{a \sin\alpha}{\sin(\beta-\alpha)}\text{: hence we have}$$

$$CE = BC \sin\beta = \frac{a \sin\alpha \sin\beta}{\sin(\beta-\alpha)} \qquad DF = DB \sin\beta = \frac{a \sin\alpha_1 \sin\beta}{\sin(\beta-\alpha_1)}$$

$$BE = BC \cos\beta = \frac{a \sin\alpha \cos\beta}{\sin(\beta-\alpha)} \qquad BF = BD \cos\beta = \frac{a \sin\alpha_1 \cos\beta}{\sin(\beta-\alpha_1)}$$

$$EF = BF - BE = a \cos\beta \left\{\frac{\sin\alpha_1}{\sin(\beta-\alpha_1)} - \frac{\sin\alpha}{\sin(\beta-\alpha)}\right\}$$

$$= \frac{a \cos\beta \left\{\sin\alpha_1 \sin(\beta-\alpha) - \sin\alpha \sin(\beta-\alpha_1)\right\}}{\sin(\beta-\alpha)\sin(\beta-\alpha_1)}$$

$$= \frac{a \sin\beta \cos\beta \sin(\alpha_1-\alpha)}{\sin(\beta-\alpha)\sin(\beta-\alpha_1)} = \frac{a \sin 2\beta \sin(\alpha_1-\alpha)}{2\sin(\beta-\alpha)\sin(\beta-\alpha_1)}$$

For CE.

log 4 =	0·6020600
sin 10° =	9·2396702
sin 40° =	9·8080675
cosec 30° =	10·3010300
log CE =	$\bar{1}$·9508277
and CE =	·892951.

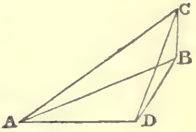
For DF.

log 4 =	0·6020600
sin 20° =	9·5340517
sin 40° =	9·8080675
cosec 20° =	10·4659483
log DF =	0·4101275
and DF =	1·57115.

For EF.

log 4 =	0·6020600
sin 80° =	9·9933515
sin 10° =	9·2396702
cosec 30° =	10·3010300
cosec 20° =	10·4659483
log 2EF =	0·6020600
and EF = 2.	

3. *The side of a hill forms an inclined plane whose angle of inclination is known; required the direction in which a rail or other road must run along the side of the hill, in order that it may have an ascent of* 1 *in every* n *feet.*



In the annexed figure, let ADB be in the horizontal plane, CD a line on the side or slope of the hill perpendicular to AD, and DB a horizontal line perpendicular to AD, to which, as well as to AB, BC is perpendicular, and AC the rail-road; its projection on the horizontal plane will be AB, and the angle BAC, therefore, will determine the angle. Now

$$\sin BAD = \frac{BD}{BA},\ \cot CDB = \frac{BD}{BC},\ \tan CAB = \frac{CB}{BA} = \frac{BD}{BA} \div \frac{DB}{BC} = \frac{\sin BAD}{\cot CDB};$$

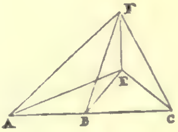
$$\text{or } \sin BAD = \tan CAB \cot CDB.$$

For example let CDB = 10° and $n = 100$, or the ascent 1 foot in 100: then

$$\tan^2 CAB = \frac{\sin^2 CAB}{1-\sin^2 CAB} = \frac{(\cdot01)^2}{1-(\cdot01)^2},$$

or $\tan CAB = \cdot01\sqrt{1+\cdot0001} = \cdot0100005$ very nearly. Hence, finally, $\sin BAD = \cdot0100005 \cot 10° = \sin 3° 15' 5''$, the horizontal angle required.

4. *Going along a horizontal straight road, I wished to find the height of a tower on a hill, and for this purpose measured two distances of* 75 *and* 80 *yds; and from each of the three points found the elevations of the tower above the horizon to be respectively* 52° 18′, 68° 15′, *and* 67° 10′: *find from these observations the height of the tower above the horizon.*



Let A, B, C, be the three stations at which the elevations were observed, and FE be perpendicular to the horizontal plane. Put $AB = a$, $BC = c$, and denote the elevations FAE, FBE, FCE, at A, B, C, by α, β, γ, respectively. Then since $EBA + EBC = \pi$, $\cos EBA + \cos EBC = 0$.

But $\cos EBC = \dfrac{EB^2 + BC^2 - CE^2}{2EB.BC}$, and

$\cos EBA = \dfrac{EB^2 + BA^2 - AE^2}{2EB.BA}$; which being inserted in the preceding equation, we shall have $BC.AE^2 + AB.CE^2 - AC.BE^2 = AC.CB.BA$.

Put $FE = x$, then $AE = x \cot \alpha$, $BE = x \cot \beta$, $CE = x \cot \gamma$: hence

$$cx^2 \cot^2\alpha - (a + c)\, x^2 \cot^2\beta + ax^2 \cot^2\gamma = ac\,(a + c), \text{ or again,}$$

$$x^2\{c\,(\cot^2\alpha - \cot^2\beta) + a\,(\cot^2\gamma - \cot^2\beta)\} = ac\,(a + c). \text{ But}$$

$$\cot^2\alpha - \cot^2\beta = (\cot\alpha + \cot\beta)(\cot\alpha - \cot\beta) = \frac{\sin(\beta+\alpha)\sin(\beta-\alpha)}{\sin^2\alpha \sin^2\beta}$$

$$\cot^2\gamma - \cot^2\beta = (\cot\gamma + \cot\beta)(\cot\gamma - \cot\beta) = \frac{\sin(\beta+\gamma)\sin(\beta-\gamma)}{\sin^2\gamma \sin^2\beta}$$

Hence the coefficient of x^2 may be written in the following form.

$$\frac{a\sin(\beta+\gamma)\sin(\beta-\gamma)}{\sin^2\beta\sin^2\gamma}\left\{1+\frac{c\sin(\beta+a)\sin(\beta-a)\sin^2\gamma}{a\sin(\beta+\gamma)\sin(\beta-\gamma)\sin^2 a}\right\}.$$

To calculate this, find $\tan^2\theta = \dfrac{c\sin(\beta+a)\sin(\beta-a)\sin^2\gamma}{a\sin(\beta+\gamma)\sin(\beta-\gamma)\sin^2 a}$: then

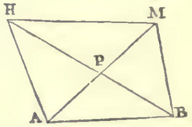
$$x^2 = \frac{c(a+c)\sin^2\beta\sin^2\gamma\cos^2\theta}{\sin(\beta+\gamma)\sin(\beta-\gamma)}.$$

In the particular example we have the elementary data as follows:—

$\frac{c}{a}=\frac{15}{16}=\cdot 9375$	$a = 67^\circ\ 10'$	$\beta + a = 135^\circ\ 25'$	$\beta+\gamma = 120^\circ\ 33'$
	$\beta = 68\ \ 15$	$\beta - a = \ \ 1\ \ 5$	$\beta-\gamma = \ 15\ \ 57$
$(c+a)c = 11625$	$\gamma = 52\ \ 18$		

log ·9375 =	$\bar{1}$·9719713	log 11625 =	4·0653930
sin 135° 25′ =	9·8463036	cosec 120° 33′ =	10·0649031
sin 1 5 =	8·2766136	cosec 15 57 =	10·5609858
cosec 120 33 =	10·0649031	2	4·6912819
cosec 15 57 =	10·5609858		2·3456410
2	18·7207774	sin 68 15 =	9·9679267
	9·3603887	sin 52 18 =	9·8982992
sin 52 18 =	9·8982992	cos θ =	9·9917448
cosec 67 10 =	10·0354398	log FE =	2·2036117
tan θ =	9·2941277	or, FE =	159·8129.

5. *From the following observations made at the extremities of a known base AB, it is required to find the distance between two inaccessible objects H, M: viz. AB = 900·625 yds, and the angles HAB, MAB, MBA, HBA, respectively* 93° 10′ 15″, 61° 15′ 25″, 84° 15′ 30″, *and* 52° 10′ 50″.



Denote the given quantities as follows :

AB = a	HAB $=\alpha$	MAB $=\alpha_1$	AMH $=\theta$	HAM $=\alpha-\alpha_1$	$\theta=\alpha_1+\beta-\omega$.
	HBA $=\beta$	MBA $=\beta_1$	BHM $=\omega$	HBM $=\beta_1-\beta$	

Then, *th.* 14, p. 450, $\dfrac{\sin\theta}{\sin\omega}=\dfrac{\sin\beta\sin(\alpha-\alpha_1)\sin(\alpha_1+\beta_1)}{\sin\alpha_1\sin(\beta_1-\beta)\sin(\alpha+\beta)}=h\sin(\alpha_1+\beta)$,

or $\sin\theta = h\sin(\alpha_1+\beta)\sin\omega$.

But $\sin\theta=\sin(\alpha_1+\beta-\omega)=\sin(\alpha_1+\beta)\cos\omega-\cos(\alpha_1+\beta)\sin\omega$. Equate these values of sines: then $\cot\omega = h+\cot(\alpha_1+\beta)$, from which ω becomes known, and thence also $\theta=\alpha_1+\beta-\omega$ can be found.

Finally, $\mathrm{HM}=\dfrac{\mathrm{MB}\sin(\beta_1-\beta)}{\sin\omega}=\dfrac{a\sin\alpha_1\sin(\beta_1-\beta)}{\sin(\alpha_1+\beta_1)\sin\omega}$, or

$$\mathrm{HM}=\frac{\mathrm{HA}\sin(\alpha-\alpha_1)}{\sin\theta}=\frac{a\sin\beta\sin(\alpha-\alpha_1)}{\sin(\alpha+\beta)\sin\theta}:$$

from either of which equations HM may be computed.

Scholium.

Had the distance HM been known, and the ground intervening between A and B been inconvenient to measure, this might have been found with equal ease, since the formulæ of relation would in both cases have been the same, and we should only be required to resolve the final formulæ for AB instead of HM,

giving $AB = \dfrac{HM \sin(\alpha_1 + \beta_1) \sin \omega}{\sin \alpha_1 \sin(\beta_1 - \beta)} = \dfrac{HM \sin(\alpha + \beta) \sin \theta}{\sin \beta \sin(\alpha - \alpha_1)}$. The problem does, in fact, appear more frequently under this form in actual surveys, than under that given in the question: whilst the ordinary mode of solution becomes exceedingly inconvenient under these circumstances.

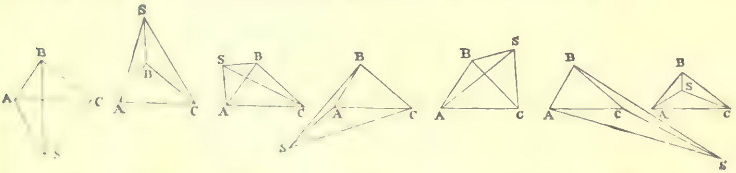
$\alpha = 93^\circ\ 10'\ 15''$	$\alpha - \alpha_1 = 31^\circ\ 54'\ 50''$	$\alpha_1 + \beta = 113^\circ\ 26'\ 15''$
$\beta = 52\ \ 10\ \ 50$	$\beta_1 - \beta = 32\ \ 4\ \ 40$	$\cot(\alpha_1 + \beta) = -\tan 23^\circ\ 26'\ 15''$
$\alpha_1 = 61\ \ 15\ \ 25$	$\alpha + \beta = 145\ \ 21\ \ 5$	$a = 900{\cdot}625.$
$\beta_1 = 84\ \ 15\ \ 30$	$\alpha_1 + \beta_1 = 145\ \ 30\ \ 55$	

sin $52^\circ\ 10'\ 50''$ =	9·8975980	h =	·9734420
sin 31 54 50 =	9·7231634	$-\tan 23^\circ\ 26'\ 15''$ =	$-$ ·4335159
sin 145 30 55 =	9·7529595		
cosec 61 15 25 =	10·0571069	$\cot \omega$ =	·5399261
cosec 32 4 40 =	10·2748483	or ω =	$61^\circ\ 38'\ 3''$
cosec 145 21 5 =	10·2452375	cosec $61^\circ\ 38'\ 3''$ =	10·0555509
cosec 113 26 15 =	10·0373965	cosec 145 30 55 =	10·2470405
		sin 61 15 25 =	9·9428931
$\log h$ =	$\bar{1}$·9883101	sin 32 4 40 =	9·7251517
		log 900·625 =	2·9545441
		log HM =	2·9251803

Hence the distance HM is 841·7446 yds.

6. *Three objects A, B, C, whose relative positions to each other are known, are observed from a point S to subtend the angles ASB, BSC, equal respectively to γ and α: and it is required to find the distance of S from each of the three objects A, B, C.*

Whichever three parts of the triangle ABC are given so as to render it determinate, the other three can be found, and hence we may consider all the parts as actually known. According to the position of the point S with respect to the triangle, several different cases will arise, as in the annexed figures: but the same general investigation serves for all of them, and only requiring a slight modification suggested by the several figures in the form of the results. The following investigation is immediately applicable to the second figure.



Put BAS $= \omega$; SCB $= \theta$; a, b, c, and A, B, C, the sides and angles of the given triangle; and $B - (\alpha + \gamma) = \delta$. Then, *th.* 14, *p.* 450, we have

$$\frac{\sin\theta}{\sin\omega} = \frac{\sin C \sin \alpha}{\sin A \sin \gamma} = \frac{c \sin \alpha}{a \sin \gamma}, \text{ or } \sin\theta = \frac{c \sin \alpha \sin \omega}{a \sin \gamma}.$$

Also, $\omega + \theta + \alpha + \gamma + A + C = \pi$, or $\theta = B - (\alpha + \gamma) - \omega = \delta - \omega$, or $\sin\theta = \sin(\delta - \omega) = \sin\delta \cos\omega - \cos\delta \sin\omega$. Then equating the two values of $\sin\theta$, and reducing, we readily obtain,

$$\cot\omega = \frac{c \sin\alpha \operatorname{cosec}\delta}{a \sin\gamma} + \cot\delta.$$

From this ω becomes known, and thence $\theta = \delta - \omega$, and thence again SBA $= \pi - \gamma - \omega$, and SBC $= \pi - a - \theta$; and the solution is completed by the first case of plane triangles.

The only alteration under different circumstances, will be in the value of δ.

However, for the purposes of calculation, this formula may be a little varied: for we have,

$$\cot\omega = \cot\delta + \frac{c \sin a \operatorname{cosec}\delta}{a \sin\gamma} = \cot\delta\left\{1 + \frac{c \sin a}{a \sin\gamma \cos\delta}\right\}.$$

If $\dfrac{c \sin a}{a \sin\gamma\cos\delta}$ be $+$, find $\tan^2\chi = \dfrac{c \sin a}{a \sin\gamma \cos\delta}$; and

if $\dfrac{c \sin a}{a \sin\gamma\cos\delta}$ be $-$, find $\cos^2\phi = -\dfrac{c \sin a}{a \sin\gamma \cos\delta}$.

Then in the former case we have $\cot\omega = \cot\delta \sec\chi$,

and in the latter .... $\cot\omega = \cot\delta \sin\phi$.

The forms for the work, with blank spaces for the numbers, are added here.

$$\cot\omega = \frac{c \sin a \operatorname{cosec}\delta}{a \sin\gamma} + \cot\delta$$

log c = ...

ac log a = ...

sin a = ...

cosec δ = ...

cosec γ = ...

log h = found by addition.

hence h = ...

n cot δ = ...

n cot ω = found by addition.

hence ω = ——° ——′ ——″

$$\cot\omega = \cot\delta\left\{1 + \frac{c \sin a}{a \sin\gamma\cos\delta}\right\}$$

log c = ...

ac log a = ...

sin a = ...

sec δ = ...

cosec γ = ...

2 found by addition.

tan χ or cos ϕ = ...

sec χ or sin ϕ = ...

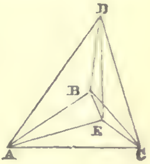
cot δ = ...

log cot ω = found by addition.

hence ω = ——° ——′ ——″

As numerical examples, take questions 26 and 27, of the following unsolved questions, the solutions being completed by *Case* 1, *Plane Trigonometry*.

7. *Given the angles of elevation of the summit of a hill or building, observed from three positions whose distances from each other are known, to find the height.*



Let A, B, C, be the three stations, and D the top of the hill or building; and draw DE perpendicular to the plane ABC, and join AE, BE, CE, AD, BD, CD. Denote the sides of the triangle ABC as usual by a, b, c; the angles of elevation of D from A, B, C, by α, β, γ; the distances AE, BE, CE, by x, y, z, and the perpendicular DE by u.

Then $x = u\cot\alpha = mu$, $y = u\cot\beta = nu$, $z = u\cot\gamma = pu$.

$$\cos \text{AEB} = \frac{x^2 + y^2 - c^2}{2xy} = \frac{(m^2+n^2)\,u^2 - c^2}{2mnu^2} \quad .... \quad (1)$$

$$\cos \text{BEC} = \frac{y^2 + z^2 - a^2}{2yz} = \frac{(n^2+p^2)\,u^2 - a^2}{2npu^2} \quad .... \quad (2)$$

$$\cos \text{CEA} = \frac{z^2 + x^2 - b^2}{2xz} = \frac{(p^2+m^2)u^2 - b^2}{2pmu^2} \quad .... \quad (3)$$

But AEB + BEC + CEA = 2π, and hence we obtain the relation *

$$1 - \cos^2\text{AEB} - \cos^2\text{BEC} - \cos^2\text{CEA} + 2\cos\text{AEB}\cos\text{BEC}\cos\text{CEA} = 0:$$

and, inserting in this, the preceding values of these cosines, we get the equation

$$\left\{\begin{array}{l} (m^2-n^2)\,(m^2-p^2)\,a^2 \\ +(n^2-p^2)\,(n^2-m^2)\,b^2 \\ +(p^2-m^2)\,(p^2-n^2)\,c^2 \end{array}\right\} u^4 - \left\{\begin{array}{l} (m^2+n^2)\,a^2b^2-p^2c^4 \\ +(m^2+p^2)\,a^2c^2-n^2b^4 \\ +(n^2+p^2)\,b^2c^2-m^2a^4 \end{array}\right\} u^2 + a^2b^2c^2 = 0 \;\ldots\; (4)$$

which is a quadratic equation in terms of u^2, and hence the solution is analytically effected.

The coefficients of this equation may be reduced in different ways, of which two are here annexed.

1. In the coefficient of u^4 write for c^2 its value, $a^2 - 2ab\cos C + b^2$, and in that of u^2, write for a^2, b^2, c^2, their values in corresponding forms. Then the equation is reduced to

$$\left\{\begin{array}{l} (m^2-p^2)^2\,a^2 + (n^2-p^2)^2\,b^2 \\ -\,2ab\,(m^2-p^2)\,(n^2-p^2)\cos C \end{array}\right\} u^4 - 2abc\left\{\begin{array}{l} m^2a\cos A \\ +\,n^2b\cos B \\ +\,p^2c\cos C \end{array}\right\} u^2 + a^2b^2c^2 = 0.$$

The former of these is the square of the side c_1 of the triangle, whose opposite angle is C, and its including sides are $(m^2 - p^2)\,a$, and $(n^2 - p^2)\,b$ respectively. This, therefore, can be easily computed by the second case of plane triangles. The terms of the second coefficient can also be computed by subsidiary angles.

2. The equation may be easily reduced to the form below:

$$\left\{\begin{array}{l} a^2(n^2-m^2)^2 + c^2\,(n^2-p^2)^2 \\ +(n^2-m^2)\,(n^2-p^2)\,(a^2-b^2-c^2) \end{array}\right\} u^4 + \left\{\begin{array}{l} a^2(a^2-b^2-c^2)m^2 \\ +b^2(b^2-c^2-a^2)\,n^2 \\ +c^2(c^2-b^2-a^2)\,p^2 \end{array}\right\} u^2 + a^2\,b^2\,c^2 = 0.$$

Put $nh = ma$, and $nk = pc$: then, since $y = nu = u\cot\beta$, the equation becomes

$$\left\{\begin{array}{l} a^2\,(c^2-k^2)^2 + c^2\,(a^2-h^2)^2 \\ +(b^2-c^2-a^2)\,(c^2-k^2)\,(a^2-h^2) \end{array}\right\} y^4 + \left\{\begin{array}{l} a^2\,b^2\,c^2\,(b^2-c^2-a^2) \\ +a^2\,c^2\,k^2\,(c^2-a^2-b^2) \\ +a^2\,c^2\,h^2\,(a^2-b^2-c^2) \end{array}\right\} y^2 + a^4\,b^2\,c^4 = 0.$$

Assume now the relation between h and k by means of the computable angle θ, from the equation $k^2 = h^2 - 2bh\cos\theta + b^2$, and for c^2 put its value $a^2 - 2ab\cos C + b^2$: then the coefficient of y^4 is transformed into

$$b^2\{(a^2-h^2)^2 - 4ah\,(a^2+h^2)\cos C\cos\theta + 4a^2h^2\,(\cos^2 C + \cos^2\theta)\}, \text{ or into}$$

$$b^2\{(a^2 - 2ah\cos C\cos\theta + h^2)^2 - (2ah\sin C\sin\theta)^2\}.$$

In the same manner the coefficient of y^2 is changed into

$$-\,2a^2b^2c^2\,\{a^2 - 2ah\cos C\cos\theta + h^2\}.$$

Hence, transposing $(2ah\sin C\sin\theta)^2$, the equation is transformed into

$$\left\{\begin{array}{l} a^2+h^2 \\ -2ah\cos C\cos\theta \end{array}\right\}^2 y^4 - 2a^2\,c^2\left\{\begin{array}{l} a^2+h^2 \\ -2ah\cos C\cos\theta \end{array}\right\} y^2 + a^4c^4 = \Big\{2ah\sin C\sin\theta\Big\}^2 y^4$$

both sides of which are complete squares: and extracting the root, we get

$$\{a^2 - 2ah\cos C\cos\theta + h^2\}\,y^2 - a^2\,c^2 = \pm\,2ah\sin C\sin\theta\,y^2, \text{ or}$$

$$\{a^2 - 2ah\,(\cos C\cos\theta \pm \sin C\sin\theta) + h^2\}\,y^2 = a^2c^2, \text{ or again,}$$

$$\{a^2 - 2ah\cos(C \mp \theta) + h^2\}\,y^2 = a^2\,c^2.$$

Hence $u = y\tan\beta = \dfrac{\pm\, ac\tan\beta}{\sqrt{a^2 - 2ah\cos(C \mp \theta) + h^2}}$; the double sign prefixed to the numerator indicating two inverted symmetrical positions of the point D,

* For, if $\phi + \theta + \chi = 2\pi$, we have $\cos(\phi+\theta) = \cos(2\pi - \chi)$, or again, $\cos\phi\cos\theta - \cos\chi = \sin\phi\sin\theta$; or squaring both sides, and writing $1 - \cos^2\phi$ and $1 - \cos^2\theta$ for $\sin^2\phi$ and $\sin^2\theta$, we obtain, after slight reduction, the formula in question. The same form of result also arises from $\cos(\phi+\theta) = \cos\chi$, as will be seen from performing the operation: and hence the theorem is true, whether the point E be within or without the triangle ABC.

one above and the other below the horizon of the places of observation: and the double sign in the denominator indicating two different positions of the point E. It may be shown, that one point will lie within the triangle ABC, and the other without it, and thus all ambiguity will be removed from the solution *.

Further examples for exercise.

8. From the edge of a ditch, of 36 feet wide, surrounding a fort, the angle of elevation of the top of the wall was found to be 62° 40′: required the height of the wall, and the length of a ladder to reach from my station to the top of it.
Ans. height of wall = 69·649272, ladder = 78·402942ft.

9. Required the length of a shoar, which strutting 11ft. from the upright of a building, will support a jamb 23ft 10in from the ground. Ans. 26·249ft.

10. A ladder, 40ft long, can be so placed, that it shall reach a window 33ft from the ground, on one side of the street; and by turning it over without moving the foot out of its place, it will do the same by a window 21ft high, on the other side: required the breadth of the street. Ans. 56·6493981ft.

11. A maypole, whose top was broken off by a blast of wind, struck the ground 15ft from the foot of the pole: what was the height of the whole maypole, the broken piece measuring 39ft in length? Ans. 75ft.

12. At 170ft distance from the bottom of a tower, the angle of its elevation was found to be 52° 30′: required the altitude of the tower. Ans. 221·55ft.

13. From the top of a tower, by the sea-side, of 143ft high, it was observed that the angle of depression of a ship's bottom, then at anchor, was 35°: what was the ship's distance from the bottom of the wall? Ans. 204·2271ft.

14. What is the perpendicular height of a hill; its angle of elevation, taken at the bottom of it, being 46°, and 200yds farther off, on a level with the bottom, the angle being 31°? Ans. 286.2906yds.

15. Wanting to know the height of an inaccessible tower: at the least distance from it, on the same horizontal plane, I took its angle of elevation equal to 58°; then going 300ft directly from it, found the angle there to be only 32°: required its height, and my distance from it at the first station.
Ans. height = 307·5456, distance = 192·162.

16. Being on a horizontal plane, and wanting to know the height of a tower placed on the top of an inaccessible hill, I took the angle of elevation of the top of the hill 40°, and of the top of the tower 51°; then measuring in a line directly from it to the distance of 200ft, I found the elevation to the top of the tower to be 33° 45′: what is the height of the tower? Ans. 93·33149ft.

---

* The fourth problem may be included in this, viz. when the three stations are in one line. For then $b = a + c$ and $C = \pi$; whence $b^2 - c^2 - a^2 = 2ac$, $c^2 - a^2 - b^2 = -2ab$, and $a^2 - b^2 - c^2 = -2bc$: and the fundamental equation becomes a complete square, and equivalent to $\left\{a \cot^2 \alpha - (a + c) \cot^2 \beta + c \cot^2 \gamma\right\} u^2 = ac\,(a + c)$, as found at p. 459.

Again, if the distances a, c, be equal, we get $\left\{\cot^2 \alpha - 2 \cot^2 \beta + \cot^2 \gamma\right\} u^2 = 2a^2$, either by substituting in the last, or in the fundamental, equation.

The following is the process indicated by the investigation for the solution of the problem.

1. Find h and k from $h = a \cot \alpha \tan \beta$, and $k = c \cot \gamma \tan \beta$.
2. Find C and θ from $c^2 = a^2 - 2ab \cos C + b^2$, and $k^2 = h^2 - 2ah \cos \theta + a^2$.
3. Find c_1 from $c_1^2 = a^2 - 2\,ah \cos (C \mp \theta) + h^2$; and finally
4. Find u from $u = \pm \dfrac{ac \tan \beta}{c_1}$.

Corresponding steps will be applicable to the case of the three stations in one line, though the solution will not be simpler than that already given, p. 459.

17. From a window near the bottom of a house, which was on a level with the bottom of a steeple, I observed the angle of elevation of the top of the steeple to be 40°; then from another window, 18ft above the former, the elevation was 37° 30′: required the height and distance of the steeple, and a general formula of solution. Ans. height = 210·436, distance = 250·792.

18. Wanting to know the height of, and my distance from, an object on the other side of a river, which was on a level with the place where I stood, close to the side of the river; and not having room to measure backward, in the same line, because of the immediate rise of the bank, I placed a mark where I stood, and measured, in a direction from the object, up the ascending ground, to the distance of 264ft, where it was evident that I was above the level of the top of the object; there the angles of depression were found to be, viz. of the mark left at the river's side 42°, of the bottom of the object 27°, and of its top 19°. Required the height of the object, and the distance of the mark from its bottom.

Ans. height = 57·2734, distance = 150·5058.

19. If the height of the Peak of Teneriffe be $2\frac{1}{3}$ miles, and the angle taken at the top of it, as formed between a plumb-line and a line conceived to touch the earth in the horizon, or farthest visible point, be 88° 2′; it is required from these measures to determine the magnitude of the whole earth, and the utmost distance that can be seen on its surface from the top of the mountain, supposing the form of the earth to be perfectly spherical.

Ans. greatest visible dist. = 135·943, diam. = 7917·85 miles.

20. Two ships of war, intending to cannonade a fort, are, by the shallowness of the water, kept so far from it, that they suspect their guns cannot reach it with effect. In order, therefore, to ascertain their distance, they separate from each other a quarter of a mile, or 440 yds; then each ship observes the angle which the other ship and the fort subtends, which angles are 83° 45′ and 85° 15′: what is the distance between each ship and the fort?

Ans. 2292·266 and 2298·051yds respectively.

21. Wanting to know the breadth of a river, I measured a base of 500yds in a straight line close by one side of it; and at each end of this line I found the angles subtended by the other end and a tree, close to the bank on the other side of the river, to be 53° and 79° 12′: what was the perpendicular breadth of the river? Ans. 529·4847yds.

22. Wanting to know the extent of a piece of water, or distance between two headlands, I measured from each of them to a certain point inland, and found the two distances to be respectively 735 and 840 yds; also the horizontal angle subtended between these two lines was 55° 40′: what was the distance of the two headlands? Ans. 741·2085yds.

23. A point of land was observed, by a ship at sea, to bear east-by-south; and after sailing north-east 12 miles, it was found to bear south-east-by-east: it is required to determine the position of that headland, and the ship's distance from it at the last observation. Ans. 26·07282 miles.

24. Wanting to know the distance between a house and a mill, which were seen at a distance on the other side of a river, I measured a base line along the side where I was, of 600 yds, and at each end of it took the angles subtended by the other end and the house and mill, which were as follow, viz. at one end the angles were 58° 20′ and 95° 20′, and at the other end the corresponding angles were 53° 30′ and 98° 45′: what was their distance? Ans. 959 604yds.

25. Wanting to know my distance from an inaccessible object O, on the other side of a river, and having only a chain for measuring distances, I chose two stations, A and B, 500yds asunder, and measured in the direction

from the object O, the lines AC and BD each equal to 100 yds; also the diagonals AD.BC equal to 550, 560 yds respectively: what was the distance of the object O from each station A and B? Ans. AO = 536·441, BO = 500·237.

26. In a besieged garrison are three remarkable objects, A, B, C, the distances of which from each other are discovered by means of a map of the place, to be as follow, AB = 266¼, AC = 530, and BC = 327½ yds. Now, having to erect a battery against it, at a certain spot without the place, and it being necessary to know whether my distances from the three objects be such, as that they may from thence be battered with effect, I observed the horizontal angles subtended by these objects from the station S, and found them to be ASB = 13° 30′, and BSC = 29° 50′. Required the three distances, SA, SB, SC; the object B being situated nearest to me, and between the two others A and C. Ans. SA = 757·1407, SB = 537·1028, SC = 654·0996.

27. Required the distances as in the last example, when the object B is the farthest from my station, but still seen between the two others as to angular position; and those angles being ASB = 33° 45′, and BSC = 22° 30′, also the three distances, AB = 600, AC = 800, BC = 400 yds respectively.

Ans. SA = 710·195, SB = 1042·545, SC = 934·29.

28. If CB (*fig.* 1. *p.* 422.) represent a portion of the earth's surface, and C the point where a levelling instrument is placed, then DG will be the difference between the true and the apparent level; and it is required to show that, for distances not exceeding 5 or 6 miles measured on the earth's surface, DG, estimated in feet, is nearly equal to $\frac{2}{3}$ CD^2, taken in miles.

29. On the opposite bank of a river to that on which I stood, is a tower, known to be 216 feet high, and with a pocket sextant I ascertained the vertical angle subtended by the tower's height to be 47° 56′. Required the distance, across the river, from the place where I stood, to the bottom of the tower; supposing my eye to be 5 feet above the horizontal plane which passes through it.

Ans. 200·22ft.

30. In the valley of Chamouni three positions, A, B, C, were selected, in a straight horizontal line, such that AB = 80, and BC = 75yds. Three remarkable points, A′, B′, C′, on the side of the Jura, were also chosen to be observed. The angles of elevation of A′, as seen from A, B, and C, were 67° 10′, 68° 15′, and 52° 18′; those of B′, as seen from the same points, were 72° 18′, 78° 15′, and 70° 10′; and finally, those of C′ were 60° 5′, 61° 10, and 58° 5′ respectively. It is required from these observations to find the heights of A′, B′, C′, from the horizon of the line of observation.

Ans. A′A″ = 159·8134, B′B″ = 286·3938, C′C″ = 323·3860.

31. (II. 1, p. 48.) An obstacle prevented my measuring the part BC of a line AD, and a point E was selected from which the angles subtended by the segments AB, BC, CD, were α, β, γ, respectively, and the two accessible segments AB and CD were found to be a and c respectively: from which data it is required to find the length of the line AD.

Ans. BC = x, is found from $(x + a)(x + c) = \dfrac{ac \sin(\alpha + \beta) \sin(\beta + \gamma)}{\sin \alpha \sin \gamma}$.

32. (II. 2.) Being on the opposite side of a river from two steeples O and W, which I knew from a previous survey to be at the distance of 6954 yds, and wishing to know the distance between two other objects on the side on which I stood, but which the irregularity of the ground prevented my measuring, I took the following horizontal angles, OAW = 85° 46′, BAW = 23° 56′, OBW = 31° 48′, and OBA = 68° 2′. What was the length of AB?

33. (II. 4.) A person walking from C to D on a straight horizontal road, can see a tower on the summit of the hill A at every point except E, where he can just see the top of the tower over the hill B. He then measures a base EC of 150 yds, and at C observes the elevation of A to be $59° 18' 15''$; and he also finds that $ACB = 10° 12' 20''$, $ACE = 69° 18' 30''$, and $AEC = 108° 12' 15''$. From these observations, the horizontal distance of the hills from each other, and from the places of observation, C and E are to be found.

34. (II. 5.) From three positions A, B, C, in the same horizontal plane whose distances were $AB = 150{\cdot}25$, $BC = 179{\cdot}69$, $AC = 205{\cdot}36$, the elevations of the top of a tower on a hill were observed to be $6° 10' 55''$, $7° 18' 3''$, and $6° 58' 58''$ respectively: whilst the elevation of the bottom of the tower from A was $6° 2' 58''$: and from these data the height of the tower is required.

35. (II. 6.) Four points A, B, C, D, are accessible, and three M, N, P, inaccessible, but are to be found from the following observations:

$AB = 815$	$ABC = 49° 54'$	$AMB = 80° 8'$	$CNP = 98° 44'$
$BC = 670$	$BCD = 73\ 57$	$BMN = 24\ 55$	$CPN = 29\ 13$
$CD = 660$		$CNM = 124\ 16$	$CPD = 51\ 19$

36. A tree growing on the side of a hill which rises due north at an angle of $30°$, had the upper part blown off 12 feet from the ground by a gale from W.S.W: now supposing the tree to stand perpendicularly to the horizon, and the top (before the other part was wholly separated from the tree) to strike the ground 40 feet from the bottom, what was its original height? Ans. 51·204ft.

37. Passing along a straight and level road, near a very lofty tower on the same horizontal plane with the road, I wished to know its height: but having no instrument for taking other than vertical angles, I proceeded thus: at a convenient point (A) on the road, I observed the angle of elevation of the top of the tower to be $30° 40'$, and 60 yds farther on the road the elevation was found to be $40° 33'$; at the end of another 60 yards, I was prevented, by a high wall, from taking the elevation, and therefore I measured 12 yds still farther, and found the angle of elevation to be $50° 23'$. From these data it is required to find the height of the tower, and its horizontal distance from each of the stations.

Ans. the height $= 94{\cdot}835$, and the distances 159·087, 110·8414, and 78·507, from the stations.

38. A person in a balloon observed the angle between two places A, B, bearing N. and S. of each other, and a miles apart to be $\alpha°$, and from B, which bore due east of him, to a point directly under him, to be $\beta°$: show that his altitude is expressed by $a \cot \alpha° \cos \beta°$.

39. A tower a feet high stands in the centre of a field whose form is an equilateral triangle, and each side subtends an angle of 2α: find the side of the field.

40. Walking along a horizontal road I observed the elevation of a tower to be $20°$, and the angular distance of the top of the tower and an object on the road to be $30°$; also the nearest distance of the tower from the road was 200ft: find its height. Ans. 187·57534.

41. From a station A, the angle subtended by two objects B and C was 2α, and at B and C the angles subtended by A and a fourth point D were right angles; also the distances AB and AC were b and c: show that if 2θ be the difference of the angles BAD, DAC, then $\tan \theta = \dfrac{b-c}{b+c} \tan \alpha$, and the distance $AD = b \sec (\alpha - \theta) = c \sec (\alpha + \theta)$.

42. AB is an obelisk on a hill BHG, and there is no horizontal ground in front of it: on the opposite hill, I therefore measured 160ft in the same vertical

plane with the castle from C upwards to E: the elevations of A and B from C were 47° 27′ and 45° 17′, and that of A from E was 46° 20′, whilst the inclination of CE to the horizon was 10° 10′. Find the height of the obelisk.

Ans. 367·851ft.

43. From a point on a level with the bottom of a flagstaff its elevation was 23° 8′ 15″, and from another point 18·2961ft higher, the angle subtended by the flag-staff was 23° 15′: required its height and distance.

44. A person on the mast-head at S, 105·6ft above the level of the sea, just sees over the earth's surface at P the top of a cliff T known to be 660 ft high: how far was the ship from the cliff, the earth's diameter being 7800 miles?

33.53 naut. m

Ans. 35·051miles. ?

45. A church O is to be built for the accommodation of three villages A, B, C, whose distances asunder are BC = 2·26, CA = 1·14, and AB = 1·58 miles: but as they contribute unequally to the expense, their distances are to be to one another in the ratio AO : BO : CO :: 5 : 12 : 9: what is its distance from each? Ans. If O be *within* ABC, AO = ·54699, BO = 1·31278, CO = ·98459,
.... *without* ..... AO = ·99486, BO = 2·38768, CO = 1·79076.

46. Three stations of the trigonometrical survey of Britain can be seen from the Eddystone lighthouse, viz. Kit's Hill, Carraton Hill, and Butterton Hill, (which denote by A, B, C respectively, and the lighthouse by D): and it appears from the survey that AB = 33427ft, BC = 131576ft, and CA = 100969ft; and likewise that from D, BC subtended an angle of 64° 1′ 48″, AC an angle of 48° 45′ 53″, and AB an angle of 15° 15′ 55″; from which it is required to determine the distance of the Eddystone from each of the stations.

Ans. AD = 123411ft, BD = 126896ft, and CD = 121123ft.

47. In the French trigonometrical survey, three stations, Villers Bretonneux (A), Vignacourt (B), and Sourdon (C), and a station (D) within the triangle, were taken, and the following data obtained: viz. log BC = 4·2734544, ABC = 49° 4′ 13″, ACB = 31° 49′ 57″·8, ADC = 130° 44′ 16″·5, and ADB = 60° 31′ 53″·8: to find the distances of D from each of the stations.

Ans. AD = 8064·61, BD = 11124·25, CD = 7733·49.

48. In the trigonometrical survey of Scotland the three stations High Pike (A), Cross-fell (B), and Crif-fell (C) were observed to subtend angles from Helvellin (D) as follows; BDC = 100° 17′ 45″·25, ADC = 32° 39′ 57″·25, BDA = 67° 37′ 48″; whilst the previously determined distances of the three stations were, BC = 255886·1ft, AC = 147733·5ft, and AB = 120904·9ft: it is required to find their respective distances from Helvellin.

Ans. AD = 65724·6, BD = 129531·8, and CD = 198738·6ft.

49. At the commencement of the trigonometrical survey, a base line BC of 27404·2 ft was measured on Hounslow Heath, between Hampton Poorhouse (C) and King's Arbour (B); and from both these stations, Hanger-Hill Tower (A) and St. Ann's Hill A′ (on opposite sides of BC) were visible, and the following angles were observed: ABC = 70° 1′ 47″, ACB = 67° 55′ 39″, A′CB = 51° 26′ 35″·5, and A′BC = 74° 14′ 35″, from which to find the distance between Hanger Hill Tower and St. Ann's Hill. Ans. 68896ft.

50. To find the distance of Inchkeith Lighthouse (A) and the spire of North Leith Church (A′), the following observations were made at the Edinburgh Observatory (C) and Beincleuch (B); viz. BC = 145314 ft, BCA = 73° 16′ 28″·5, BCA′ = 55° 38′ 41″·1, CBA = 11° 53′ 56″, CBA′ = 2° 34′ 2″·2; to find AA′, both points being on the same side of BC. Ans. 23045·53ft.

XIX. MISCELLANEOUS EXAMPLES FOR EXERCISES ON TRIANGLES.

1. (I. 7.) Let a, b, c denote the sides of a plane triangle, to find C, when $c^2 = a^2 \pm ab + b^2$, $c^2 = a^2 \pm \frac{1}{2}ab + b^2$, and when $c^2 = a^2 \pm \frac{1}{n}\, ab + b^2$.

2. (II. 6.) The three sides of a plane triangle are three consecutive terms in the series of integers, and the greatest angle is double the least: find the sides and angles.

3. (II. 14.) How must three trees A, B, C be planted so that the angle A may be double of the angle B, and the angle B double the angle C; and that a line of 400 yards may go round them?

4. (II. 15.) The sines of the three angles of a triangle are as the numbers 17, 15, 8, and the perimeter is 160; and it is required to find the sides, the angles, the perpendiculars, and the lines bisecting the sides.

5. (II. 16.) The logs of two sides are 2·2407293 and 2·5378191, and the included angle is 37° 20′ 1″. Determine the other side without finding the angles.

6. (II. 17.) The sides of a triangle are to one another as the fractions $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$: what are the angles? and under what angles do the lines from the angles to the middles of the sides intersect?

7. (II. 24.) If 2Δ denote the product of the two sides of a triangle about the right angle C, then it is required to show that

$$b = \sqrt{2\Delta \cot A},\ a = \sqrt{2\Delta \tan A}, \text{ and } c = 2\sqrt{\Delta \operatorname{cosec} 2A}.$$

8. (II. 26, 2.) If c be the hypothenuse of a right-angled triangle, then

$$\text{tab} \tan \tfrac{1}{2} A = 10 - \tfrac{1}{2}\left\{\log (c + b) - \log (c - b)\right\}.$$

9. (II. 27.) Two lines g and h are given, and a triangle, whose sides a, b, c are respectively the arithmetical, geometrical, and harmonical means between them, is to be constructed: find its angles, and exemplify it when $g = 4h$.

10. (II. 28.) From the vertex A of a triangle a line is drawn to cut the base a in segments which have the triplicate ratio of b to c: find the angles which it makes with its three sides, and show what ratio between b and c will cause the line so drawn to make a right angle with a.

11. (II. 29.) Given the radius of the inscribed circle and the angles at the base of a plane triangle, to find the sides and the radius of the circumscribing circle.

12. (II. 30.) If R, r, be the radii of the circumscribed and inscribed circles, and d the distance of their centres, then $d = \sqrt{R^2 - 2\,Rr}$; and, if the four circles be described, which touch the three sides of the triangle, and d_1, d_2, d_3, be the distances of their centres from the centre of the circumscribing circle, then

$$d^2 + d_1^2 + d_2^2 + d_3^2 = 12R^2.$$

13. (II. 31.) If a right pyramid SABCD on a square base ABCD be cut by any plane, and a, b, c, d be the distances from the vertex at which the edges taken in order are cut by the plane: then it is required to prove that

$$\frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{d}.$$

14. (II. 32.) Given the perimeter, $2s$, the difference, A − B, of the angles at the base, and the perpendicular p_3 from C to c, to find the angles of the triangle.

15. (II. 33.) The four sides of a quadrilateral, a, b, c, d, inscriptible in a circle, are given, to find the diagonals, the angles under which they intersect, and the radius of the circumscribing circle.

16. (II. 34.) Let A, B, C be the angles of a triangle, a, b, c its sides, and R, r the radii of its circumscribing and inscribed circles, and r_1, r_2, r_3 the radii of the described circles, and p_1, p_2, p_3 the perpendiculars from A, B, C on a, b, c; then,

(1) $$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

(2) $$\cot A \cot B + \cot B \cot C + \cot C \cot A = 1.$$

(3) $$\sin A \pm \sin B : \sin C :: a \pm b : c.$$

(4) $$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

(5) $$\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B + \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C + \tan \tfrac{1}{2}C \tan \tfrac{1}{2}A = 1.$$

(6) $$\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C = \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C.$$

(7) Find r, r_1, r_2, r_3 in terms of R and A, B, C.

(8) $$p_1 = \frac{b^2 \sin C + c^2 \sin B}{b+c}, \quad p_2 = \frac{a^2 \sin C + c^2 \sin A}{a+c}, \quad p_3 = \frac{a^2 \sin B + b^2 \sin A}{a+b}.$$

(9) $$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

17. (II. 35.) Given the angles A, B, C and circumscribing radius R, to find the three lines from the angles to the middles of the opposite sides, and likewise the three perpendiculars.

18. (II. 38.) Three of the angles of a quadrilateral figure circumscribing a circle whose radius is 10, are 29° 15′ 10″, 87° 15′ 12″, and 105° 15′ 18″: what are its sides?

19. (II. 39.) Find the remaining parts of the triangles where are given:

(1) b, c, $B \pm C$	(3) A, a, $b \pm c$	(5) A, a, bc
(2) B, C, $b \pm c$	(4) A, c, $a \pm b$	(6) A, B, $a + b + c$.

20. (II. 40.) Adopting the notation of *Ex.* 16, there are given,

(1) C, c, p_3 to find A, B, and a, b.

(2) $a + b + c$, C and $\frac{1}{2}ab \sin C$ to find c the remaining parts.

(3) a, b, c, to find the segments of c by a line bisecting C, the segments of C by a line bisecting c and p_1, p_2, p_3.

(4) p_1, p_2, p_3 to find a, b, c and A, B, C.

(5) a, b, C to find p_3, p_2, and p_1.

(6) $a + b + c$, B, and the equation $b^2 = ac$ to find A and C.

(7) $a + b + c$, $\frac{a}{b}$, $\frac{a}{c}$ to find the sides, angles, and perpendiculars.

21. (II. 41.) If a, b, c, be the chords of three arcs which together form a semicircle, what is the radius of it?

22. (II. 45.) Prove that in any triangle:

(1) $$\tan C = \frac{1 + \sec A \sqrt{\left(\frac{a}{b}\right)^2 - \sin^2 A}}{\tan A - \operatorname{cosec} A \sqrt{\left(\frac{a}{b}\right)^2 - \sin^2 A}}$$

(2) $$2a = \sqrt{c^2 + 2p_3 c \cot \tfrac{1}{2} C} \pm \sqrt{c^2 - 2p_3 c \tan \tfrac{1}{2} C}.$$

(3) $$\cos C = \frac{a - c \cos B}{\sqrt{a^2 - 2ac \cos B + c^2}}$$

(4) $$\cos C = \frac{a}{b} \sin^2 B \mp \cos B \sqrt{1 - \left(\frac{a}{b}\right)^2 \sin^2 B}.$$

23. (II. 46.) If α, β, and α_1, β_1 be any two angles of two triangles, then

$$\left\{\frac{\sin \beta_1}{\sin(\alpha_1 + \beta_1)} - \frac{\sin \beta}{\sin(\alpha+\beta)}\right\}^2 - \left\{\frac{\sin \alpha_1}{\sin(\alpha_1+\beta_1)} - \frac{\sin \alpha}{\sin(\alpha+\beta)}\right\}^2 = \frac{4\{\sin^2\frac{1}{2}(\alpha_1 - \alpha) - \sin^2\frac{1}{2}(\beta_1 - \beta)\}}{\sin(\alpha_1 + \beta_1)\sin(\alpha+\beta)}$$

24. From the angles A, B, C of a triangle draw the perpendiculars to the

opposite sides, meeting them in D, E, F respectively, and form the triangle DEF: then $\Delta DEF = 2\,\Delta ABC \,.\, \cos A \cos B \cos C$.

25. Prove the following properties of a right-angled triangle, C being the right-angle.

$$\sin^2\tfrac{1}{2}A = \frac{c-b}{2c} \quad \sin 2A = \frac{2ab}{b^2+a^2} \quad \sin(45^\circ \pm A) = \frac{b \pm a}{c\sqrt{2}} \quad \sin(A-B) = \frac{a^2-b^2}{c^2}$$

$$\cos^2\tfrac{1}{2}A = \frac{c+b}{2c} \quad \cos 2A = \frac{b^2-a^2}{b^2+a^2} \quad \cos(45^\circ \pm A) = \frac{b \mp a}{c\sqrt{2}} \quad \cos(A-B) = \frac{2ab}{c^2}$$

$$\tan^2\tfrac{1}{2}A = \frac{c-b}{c+b} \quad \tan 2A = \frac{2ab}{b^2-a^2} \quad \tan(45^\circ \pm A) = \frac{b \pm a}{b \mp a} \quad \tan(A-B) = \frac{a^2-b^2}{2ab}$$

26. In any isosceles triangle, C being the angle included by the equal sides:

$$\cos A = \frac{c}{2a},\ \text{vers}\ C = \frac{c^2}{2a^2},\ r = \frac{c}{2}\sqrt{\frac{2a-c}{2a+c}},\ \text{and}\ R = \frac{a^2}{\sqrt{4a^2-c^2}}.$$

27. Prove that in any plane triangle the following equations are true: *viz.*

$$\sin\tfrac{1}{2}(A-B) = \frac{a-b}{c}\cos\tfrac{1}{2}C \qquad \frac{\cot\frac{1}{2}A + \cot\frac{1}{2}C}{\cot\frac{1}{2}B + \cot\frac{1}{2}C} = \frac{b}{a}.$$

$$\cos\tfrac{1}{2}(A-B) = \frac{a+b}{c}\sin\tfrac{1}{2}C \qquad \frac{\tan\frac{1}{2}A - \tan\frac{1}{2}B}{\tan\frac{1}{2}A + \tan\frac{1}{2}B} = \frac{a-b}{c}.$$

$$\sin(A-B) = \frac{a^2-b^2}{c^2}\sin C \qquad \frac{\cot\frac{1}{2}B + \tan\frac{1}{2}A}{\cot\frac{1}{2}B - \tan\frac{1}{2}A} = \frac{a+b}{c}.$$

28. Also, that,

$$c^2 = (a+b)^2\sin^2\tfrac{1}{2}C + (a-b)^2\cos^2\tfrac{1}{2}C = \frac{(a+b)^2\sin^2\frac{1}{2}C - (a-b)^2\cos^2\frac{1}{2}C}{\cos(A-B)},$$

and that $c = \dfrac{(a+c)\tan\frac{1}{2}B + (a-c)\cot\frac{1}{2}B}{2\tan\frac{1}{2}(A+B-C)} = \dfrac{a}{\cos B + \sin B \cot C}.$

29. R and r being the radii of the circumscribed and inscribed circles, then,

$$R = \frac{s}{\sin A + \sin B + \sin C} = \frac{abc\tan\frac{1}{2}(A+B-C)}{a^2+b^2-c^2};\ \text{and}$$

$$r = s\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C = \frac{abc(\sin A + \sin B + \sin C)}{4s^2}.$$

30. If p_3 be the perpendicular from C upon c, it is required to prove that

$$p_3 = \frac{2s}{\cot\frac{1}{2}A + \cot\frac{1}{2}B} = \frac{a^2\sin B + b^2\sin A}{a+b} = \frac{c\cos(A-B) - c\cos(A+B)}{2\sin C}$$

and its distance from the middle of c will be expressed by the following:

$$\frac{c\sin(A-B)}{2\sin C},\ \frac{c}{2}\,\frac{\tan A - \tan B}{\tan A + \tan B},\ \text{and}\ \frac{(a^2-b^2)\sin(A+B)}{c\sin C}.$$

31. Given AB $= 100$, A $= 60^\circ$, B $= 45^\circ$, to find BC, CA without the aid of any tables. Ans. BC $= 50\sqrt{6}\{\sqrt{3}-1\}$, CA $= 100\{\sqrt{3}-1\}$.

32. The two sides of a triangle are as 3 to 5, and one of the angles at the base three times the other: what are the angles?

Ans. 35° 15′ 54″, 105° 47′ 42″, and 38° 56′ 24″.

33. In a triangle, ABC, there are given A $= 80^\circ$, $a = 400$, and $b + c = 600$; to find the other parts.

34. If $a = 3 + \sqrt{2}$, $b = 3 - \sqrt{2}$, and $c = 4$: then $\sin C = \frac{2}{7}\sqrt{10}$.

35. Show that in any plane triangle $\dfrac{a}{b} : \dfrac{s-a}{s-b} :: \sin^2\frac{1}{2}A : \sin^2\frac{1}{2}B$.

36. If lines be drawn from the angles of a triangle to a point within it so as to make equal angles with each other, their sum will be equal to

$$\sqrt{a^2 - 2ab\cos(A+60^\circ) + b^2}.$$

MENSURATION.

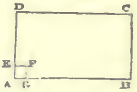
In the previous applications of algebraic methods to geometry, whether in their simple or trigometrical forms, our object has ultimately been to find either a line or an angle from certain data furnished by the problem. The object of that branch upon which we now enter is to find the lengths of curve lines, are the areas or volumes of given superficial or solid figures. To effect this purpose to any great extent requires the use of the differential and integral calculus: but in the few cases which will here be considered, it can be accomplished without that end.

All lines are transformed either entirely or approximately into straight lines expressed in terms of the linear unit; all surfaces into rectangles whose adjacent sides are expressed in terms of the same linear unit; and all solids into parallelopipedons, whose three adjacent edges are expressed also in the same terms. See *Application of Algebra to Geometry*, p. 413.

In this work the general investigations will be first given altogether for figures which belong to the same class; and then examples to each in precisely the same order as in the previous edition.

I. THE AREAS AND LENGTHS OF PLANE FIGURES.

1. *A rectangle whose sides are given.*



Let AB, AD be two adjacent sides of the rectangle, and AG, AE each equal to the unit of the scale by which the sides are measured. Let a and b be the number of times to which the sides AB, AD respectively contain AG or AE. Complete the square AEFG, and this will be the superficial unit by which AC is measured, or calculated.

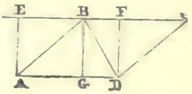
Now the parallelograms AF, AC being equiangular, they are to one another in a ratio compounded of the ratios of their sides. That is,

$$\text{par}^{m}\ AC : \text{par}^{m}\ AF :: AB.AD : AG.AE; \text{ or alternately}$$
$$\text{par}^{m}\ AC : AB.AD :: \text{par}^{m}\ AF : AG.AE.$$

But $\text{par}^{m}\ AF = AG.AE = 1$, and hence $\text{par}^{m}\ AC = AB.AD = ab$.

The area is, hence, expressed by the product of any two adjacent sides.

2. *When two adjacent sides of a parallelogram and their included angle are given.*



Let AB, AD, and the angle A be given. Draw the perpendiculars AE, BG, DF. Then the parallelogram AC is equal to the rectangle AF. Whence parallelogram $AC = AD.AE = AD.BG = AD.AB \sin A = ab \sin A$, where a and b denote AD and AB respectively.

Hence the area is expressed by the continued product of the two sides and the sine of their included angle.

Cor. 1. It is obvious from what is here shown, that when the one side and its distance from the opposite side are given, the area is their product, viz. AD.BG.

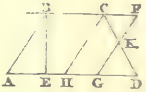
Cor. 2. When the base and perpendicular of a triangle are given, the area is half their product: for the triangle ABD is half the parallelogram AC.

Cor. 3. When two sides and the included angle of a triangle are given, as AB, AD, and A, we shall have, since trian ABD $= \frac{1}{2}$ parallelogram AC,

$$ABD = \tfrac{1}{2}AB.AD \sin A.$$

Cor. 4. When in the triangle ABC, the angle $C = \frac{\pi}{2}$, we have area $= \frac{1}{2}ab$*.

3. *When two of the opposite sides are parallel but not equal* (the trapezoïd), *and there are given those two sides with one of the other sides, and its inclination to either of the parallels.*



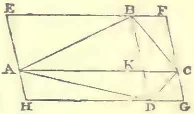
Let ABCD be the trapezoïd, the sides AD, BC which are parallel, being given, as likewise the side AB and angle A.

Draw CH parallel to AB, bisect HD in G and draw GF parallel to AB. Then GD = GH = CF. Hence the triangles GKD, CKF, having one side GD equal to one side CF, and the angles at the extremities of these sides equal each to each, the triangles are equal. Consequently adding ABCKG to each, the trapezoïd ABCD is equal to the parallelogram ABFG. Also BF + AG = 2AG = BC + AD, or AG $= \frac{1}{2}$(AD + BC), that is, to half the sum of the parallel sides.

Whence, ABCD = ABFG = AB.AG sin A $= \frac{1}{2}$AB (BC + AD) sin A.

Cor. 1. If the two opposite sides AD, BC and breadth BE be given, then the area is $\frac{1}{2}$BE (AD + BC).

4. *When the two diagonals of a trapezium and their angle of intersection are given.*



Let ABCD be the quadrilateral figure or trapezium, of which the diagonals AC, BD, and their angle K of intersection are given.

Through A, B, C, D draw lines parallel to the diagonals: then they will together form a parallelogram, EFGH whose angles are equal to the angles at K; and whose area is double that of the trapezium ABCD.

But EFGH = HE.HG sin H = BD.AC sin K; whence, finally, we have

$$ABCD = \tfrac{1}{2}BD.AC \sin K.$$

5. *When one diagonal and perpendiculars from the other angle are given.*

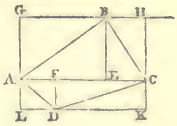
Let ABCD be the trapezium, AC the given diagonals, and BE, DF the given perpendiculars from the other angles B and D upon AC.

---

* The following, amongst many other expressions that might be given, for the area of a triangle in terms of different parts of the figure, possess an analytical interest to the inquiring mind. Their investigation will form good exercises for the student.

Expressions for the area of the triangle.

$\frac{1}{2}ab \sin C$	$\sqrt{\{s(s-a)bc\}} \sin \frac{1}{2}A$	$\dfrac{(a^2+b^2-c^2)\sin C}{4\tan\frac{1}{2}(A+B-C)}$
$\frac{1}{2}a^2 \dfrac{\sin B \sin C}{\sin A}$	$\sqrt{\{(s-b)(s-c)bc\}} \cos \frac{1}{2}A$	
$\frac{1}{2}a^2 \dfrac{\sin B \sin C}{\sin(B+C)}$	$\frac{1}{2}\sqrt[3]{\{a^2b^2c^2 \sin A \sin B \sin C\}}$	$\dfrac{a^2+b^2+c^2}{4(\cot A + \cot B + \cot C)}$
$s(s-a) \tan \frac{1}{2}A$	$\dfrac{2abc}{a+b+c} \cos\frac{1}{2}A \cos\frac{1}{2}B \cos\frac{1}{2}C$	$\dfrac{abc^2}{2(a^2-b^2)} \sin(A-B)$
$\frac{1}{4}(a^2-b^2)\dfrac{\sin A \sin B}{\sin(A-B)}$	$\frac{1}{4}\sqrt{\dfrac{4(a^4+b^4+c^4)-(a^2+b^2+c^2)^2}{\cot^2 A + \cot^2 B + \cot^2 C}}$	$\sqrt{\{s(s-a)(s-b)(s-c)\}}$



Through B, D, draw GH, LK parallel to AC, and through A, C draw GL, HK perpendicular to AC and therefore parallel to BE and DF. Then the rectangle LH is double the trapezium ABCD; and LG = BE + FD, and LK = AC. Whence

trapez ABCD $= \frac{1}{2}$par LH $= \frac{1}{2}$ LK.LG $= \frac{1}{2}$AC (BE + FD).

Scholium. If a, b, c, d be the four sides of a trapezoïd, which has the opposite sides a and c parallel; and if $c - a = \delta$: then the area of the figure is

$$\frac{a+c}{4\delta}\sqrt{(b+d+\delta)(-b+d+\delta)(b-d+\delta)(b+d-\delta)}$$

If the quadrilateral be inscriptible in a circle, $2s = a + b + c + d$, and

$$\text{area} = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

These theorems are left for the student's investigation by means of the trigonometrical calculus.

6. *When the three sides of a triangle are given to find the area.*

Let them be as usual denoted by a, b, c. Then, we have, from *p.* 448, *eq.* 11.

$$\sin C = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{ab};\ \text{and hence}$$

$$\text{area} = \tfrac{1}{2}ab\sin C = \sqrt{s(s-a)(s-b)(s-c)},$$

the same result as before found, *p.* 417, by a different method, but founded on the same principle.

When $b = c$, or the triangle isosceles, this becomes

$$\text{area} = \tfrac{1}{4}a\sqrt{(2b+a)(2b-a)}$$

and when $a = b = c$, or the triangle equilateral, area $= \frac{1}{4}a^2\sqrt{3}$.

Also, when the sides are expressed by the radicals of the second degree, $\sqrt{a}, \sqrt{b}, \sqrt{c}$, the form most convenient for computation will be

$$\text{area} = \tfrac{1}{4}\sqrt{2(ab+bc+ca)-(a^2+b^2+c^2)}$$

The area of a triangle may also be computed by a different formula, thus: put $b + c = h$, $b - c = k$, then area $= \frac{1}{4}\sqrt{(h^2-a^2)(a^2-k^2)}$, and these again may be separated into factors, giving the convenient equation

$$\text{area} = \tfrac{1}{4}\sqrt{(h+a)(h-a)(a+k)(a-k)}.$$

7. *Given two angles of a triangle and the included side.*

At p. 450, *eq.* 12, we found the expression for the perpendicular on c to be $p = c\sin A\sin B\operatorname{cosec} C$; and hence, area $= \frac{1}{2}pc = \frac{1}{2}c^2\sin A\sin B\operatorname{cosec} C$.

In all cases where sufficient data is given, the problem can be reduced to one or other of the forms laid down, and the area thence found: and in all cases of trapezia, the same remark applies. When we come to figures of more than four sides, it will generally be better to reduce them into triangles or trapezia, or into figures of both kinds, as the data may suggest, and take the sum of all the areas of these partial figures for that of the given one.

When the object to be measured is small, it is more easy to obtain the several lines and perpendiculars accurately than it is to obtain angles; as in boards, walls, and artificial objects generally. In larger ones, as fields or the assemblage of fields constituting a farm or estate, it is easier to obtain angles than lines; and in this case it will always be better to measure one side very carefully, and

obtain all the other requisite data, in angular measures. In this latter case, we reduce the figure to a series of triangles; and in the former to a combination of triangles and quadrilaterals, or often more conveniently to a series of triangles only. All the cases that are likely to occur have been already given; and any other that may incidentally present itself can easily be reduced to some of these by the principles of trigonometry; and, indeed, the difficulty can often be evaded altogether by a little difference of arrangement.

8. *In a circle of given radius, a regular polygon of* n *sides is inscribed, and it is required to find the perimeter and area of the polygon.*

Let AE or EP $=$ R be the given radius, and AP one of the equal sides. Then since all the sides are equal, the angles which they subtend at the centre are all equal, and each of them to the angle AEP. But they are all together equal to 2π, and hence AEP $= \frac{2\pi}{n}$. Draw the perpendicular ER to AP, and it will bisect the angle AEP. Hence AER $= \frac{\pi}{n}$; and by right-angled triangles AQ $=$ AE sin AER $=$ R $\sin\frac{\pi}{n}$, and AP $= 2\text{R}\sin\frac{\pi}{n}$. Whence the *perimeter* $= 2n\text{R}\sin\frac{\pi}{n}$.

Again for the area, we have n triangles each equal to AEP, and expressed by $n\text{AQ}.\text{QE} = n\text{R}\sin\frac{\pi}{n}.\ \text{R}\cos\frac{\pi}{n} = \frac{n}{2}\text{R}^2\sin\frac{2\pi}{n} =$ *area.*

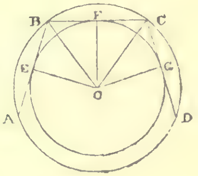
9. *About a given circle a polygon of* n *sides is described, whose perimeter and area are required.*

In the preceding figure, let AE $=$ EP $= r$, the radius of the given circle, and ST be one of the n sides of the polygon. Then, reasoning as in the preceding instance, we have

$$\text{ST} = 2r\tan\frac{\pi}{n},\ \text{and the } perimeter = 2nr\tan\frac{\pi}{n}.$$

And the triangle EST $=$ SR.RE $= r^2\tan\frac{\pi}{n}$, and that of the entire polygon is $nr^2\tan\frac{\pi}{n}$.

10. *Given one side of a regular polygon of* n *sides, to find its area, and the radii of the inscribed and circumscribed circles.*



Let AB, BC, CD ... be the sides of the polygon, each of which is denoted by $2a$.

Then BOF $= \frac{\pi}{n}$, and FO $=$ BF cot BOF, and BO $=$ BF cosec BOF. That is, FO $= a\cot\frac{\pi}{n}$, and BO $= a\ \text{cosec}\ \frac{\pi}{n}$, which are the radii respectively of the inscribed and circumscribing circles.

For the area, we have $n.\text{BF}.\text{FO} = na^2\cot\frac{\pi}{n}$.

11. *To calculate the perimeter of a circle of a given radius.*

(*See fig. in p.* 474, *No.* 8.)

Let ST, AP, be one of the sides of the regular circumscribed and inscribed polygons respectively of n sides. Then the perimeter of the circle is less than the perimeter of the former, but greater than that of the latter polygon. Also, as the value of n is increased, the perimeter of the circumscribed polygon is continually diminished, whilst that of the inscribed one is continually increased: and the perimeter of the circle is the limit towards which they both tend, the one by its continual diminution, and the other by its continual enlargement. For the one may be diminished so as to differ from that of the circle by a quantity less than the least assignable, whilst it can never become less than that of the circle: and on the other hand, the other perimeter may be so increased, by increasing n, as to differ from that of the circle by a quantity less than the least assignable, whilst it can never become greater than that of the circle. The perimeter of the circle, then, is a limit between the perimeters of the two polygons; and, therefore, to the same number of decimal places as the two polygons agree for any value of n, to that extent the perimeter of the circle is also obtained.

Let us suppose, then, that $n = 10800$, in which case the angle AEP $= 2'$; and AP $= 2r \sin 1'$, and ST $= 2r \tan 1'$; and the inscribed and circumscribed perimeters are, putting $2r = d$, $10800d \sin 1'$, and $10800d \tan 1'$ respectively.

We have already (p. 430) computed sin 1′ and tan 1′ to ten decimal places, and found them coincident to that extent: but in fact they only begin to differ in the fifteenth decimal place, and we have the following functions to radius 1; viz.

$$\sin 1' = \cdot 000290888208664, \text{ and } \tan 1' = \cdot 000290888208668.$$

Multiplying these by $21600r$ or $10800d$ respectively, and we shall obtain

$$\text{circum. perim.} = 3{\cdot}1415926536161 \ldots; \text{ inscr. perim.} = 3{\cdot}1415926535767 \ldots$$

These agree to nine places of figures, and hence the perimeter of the circle, whose value is intermediate to them, is accurately found to the same extent *.

---

* It may be desirable to point out other methods of computing the circumference of the circle: but as they mainly rest either upon the use of the imaginary symbol, or upon the integral calculus, the obvious principle above employed has been thought better adapted to the text.

1. By *Euler's* theorem (p. 437) we have $\tan \alpha = \dfrac{\sin \alpha}{\cos \alpha} \quad \dfrac{1}{\sqrt{-1}} \cdot \dfrac{e^{2\alpha\sqrt{-1}} - 1}{e^{2\alpha\sqrt{-1}} + 1}$,

or $e^{2\alpha\sqrt{-1}} = \dfrac{1 + \sqrt{-1}\tan\alpha}{1 - \sqrt{-1}\tan\alpha}$: whence, taking log. of both sides, we have the equation

$$2\alpha\sqrt{-1} = \log_e \left\{1 + \sqrt{-1}\tan\alpha\right\} - \log_e \left\{1 - \sqrt{-1}\tan\alpha\right\}$$

which being expanded, the equal terms with contrary signs cancelled, and the whole divided by $2\sqrt{-1}$, we have $\alpha = \tan\alpha - \frac{1}{3}\tan^3\alpha + \frac{1}{5}\tan^5\alpha - \frac{1}{7}\tan^7\alpha + \ldots$ *ad inf.*

This, from its discoverer, is called *Gregory's series*, and it is the foundation of almost every effective method usually employed. It is not, however, in its simple form, well adapted to use, on account of its slow convergency; as it requires a great number of terms to be computed for arriving at a moderate degree of approximation.

When $\alpha = 45°$, $\tan\alpha = 1$, and we have the following expression for the value of 45°, viz.

$$\frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \ldots.$$

When

This number might have been found correctly to a greater number of places, if we had taken AEP smaller, (that is, n greater,) as $2''$ for instance: and it has been actually computed to 128 decimal places by Delagny, and in a MS. at Oxford to 140 places. (*See Hutton's Dictionary, Art. Circle.*) It is generally represented by π, the initial letter of the Greek term *perimeter*. The same symbol has been used in the trigonometry to denote $180°$, or a semicircle to radius 1; and which, since the perimeters of circles are as their diameters, will properly represent the circle to diameter 1 or radius $\frac{1}{2}$.

12. *To find the area of a circle, whose diameter is given.*

Reasoning as in the last case, the area of the circle is the limit between the areas of the inscribed and circumscribed polygon of 2^n sides when n is continually increased. Now circumscribed polygon $= n\mathrm{R}^2 \tan\frac{\pi}{n}$, and inscribed polygon $= \frac{1}{2}\,nr^2 \sin\frac{2\pi}{n}$: and R, r, approach to equality as n is increased, whilst $\sin\frac{2\pi}{n}$ and $\tan\frac{\pi}{n}$ approach to $\frac{2\pi}{n}$ and $\frac{\pi}{n}$ respectively under the same circumstances; and at their limits these three equalities actually take place. Hence,

---

When $\alpha = 30° = \frac{\pi}{6}$ we have $\tan\alpha = \frac{1}{\sqrt{3}}$, and hence by actual substitution we have

$$\frac{1}{6}\pi = \frac{1}{\sqrt{3}}\left\{1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \dots\right\}$$

This latter is called *Halley's series*, but it, like the former, converges but very slowly.

2. Let $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$: then $\tan(\alpha+\beta) = \dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = 1$, or $\alpha+\beta = \frac{1}{4}\,\pi$.

But we have by Gregory's series,

$$\alpha = \frac{1}{2} - \frac{1}{3.2^3} + \frac{1}{5.2^5} - \frac{1}{7.2^7} + \dots \text{ and } \beta = \frac{1}{3} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \frac{1}{7.3^7} + \dots \text{ whence}$$

$$\alpha + \beta = \frac{\pi}{4} = \left\{\frac{1}{2} + \frac{1}{3}\right\} - \frac{1}{3}\left\{\frac{1}{2^3} + \frac{1}{3^3}\right\} + \frac{1}{5}\left\{\frac{1}{2^5} + \frac{1}{3^5}\right\} - \dots$$

This is ***Euler's series***, and converges with considerable rapidity.

3. Take $\tan\alpha = \frac{1}{5}$; then $\tan 2\alpha = 1\frac{5}{2}$, and $\tan 4\alpha = 1 + \frac{1}{119}$; and hence α is greater than $\frac{1}{4}\pi$.

Take $4\alpha = \theta + \frac{1}{4}\pi$, or $\theta = 4\alpha - \frac{\pi}{4}$: then $\tan\theta = \dfrac{\tan 4\alpha - 1}{\tan 4\alpha + 1} = \dfrac{1}{239}$.

Again $\frac{1}{4}\pi = 4\alpha - \theta$, and substituting these separately in Gregory's series, we have

$$4\alpha = 4\left\{\frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \dots\right\}, \text{ and } \theta = \frac{1}{239} - \frac{1}{3.239^3} + \frac{1}{5.239^5} - \dots$$

$$\text{or } \frac{\pi}{4} = 4\left\{\frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \dots\right\} - \left\{\frac{1}{239} - \frac{1}{3.239^3} + \frac{1}{5.239^5} - \dots\right\}$$

which converges with great rapidity. This series was discovered by *Machin*, and is called after him. The most convenient series, however, for the actual computation of the circumference, ever discovered, is the following, which was communicated to me some time ago by my colleague, *Mr. Rutherford*.—

$$\frac{\pi}{4} = 4\left\{\frac{1}{5} - \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} - \dots\right\} - \left\{\frac{1}{70} - \frac{1}{3}\cdot\frac{1}{70^2} + \dots\right\} + \left\{\frac{1}{99} - \frac{1}{3}\cdot\frac{1}{99^3} + \dots\right\}$$

The divisors 70 and 99 being easily employed from their reducibility into factors, would render the Oxford approximation an operation of by no means extraordinary labour. The series itself is derived from the equation xiv. 7. (6) *p.* 443,

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}.$$

supposing π to represent the accurate circumference, we have $d = 2r = 2R$, and $\frac{1}{2} nR^2 . \frac{2\pi}{n} = nr^2 \frac{\pi}{n} = r^2\pi = d^2 . \frac{\pi}{4}$ = area of the circle.

Cor. 1. Since the circumference of the circle is $2r\pi$, this value of the area is also represented by $\frac{1}{2} r.2r\pi$, or half the product of the radius and circumference.

Cor. 2. Since (*th.* 94, *Geom. p.* 337) the *sectors of a circle* are as the arcs or angles corresponding to them, these are found by the simple proportion,

$$360^\circ : a^\circ :: r^2\pi : \text{area of sector} = \frac{a^\circ}{360^\circ} . r^2\pi.$$

Cor. 3. The sector is also represented by $\frac{r^2}{2} . \beta$, where β is the length of the arc subtended by a°.

Cor. 4. The area of an *annulus*, or the space inclosed by the circumferences of two concentric circles, is represented by $r^2\pi - r_1^2 \pi = (r^2 - r_1^2)\pi = (r + r_1)(r - r_1)\pi$; where r and r_1 are the two radii.

13. *To find the area of the segment of a circle.*

(*See first fig. p.* 474.)

If a be the arc corresponding to the segment, we shall have the following values:

$$\text{segment ARP} = \text{sector AEP} \mp \text{triangle AEP}$$
$$= \frac{a^\circ}{360^\circ} r^2\pi \mp \frac{1}{2} r^2 \sin a^\circ = \frac{r^2}{2} \left\{ a \mp \sin a \right\}$$

where the upper or lower sign is used according as the segment taken is less or greater than a semicircle.

Scholia.

A few numerical particulars, which will greatly facilitate the actual solution of the following and similar problems, may be advantageously put together here.

1. The ratio of the diameter to the circumference of a circle may be with different degrees of approximation calculated from

diam. : circ. ::	7	: 22, that of *Archimedes*,
diam. : circ. ::	113	: 355, ... *Metius*,
diam. : circ. ::	1	: 3·1416, that commonly used by artificers,
diam. : circ. ::	1	: 3·14159, that commonly used in works of science.
diam. : circ. ::	·318309	: 1 ; the same, in fact, as the last.

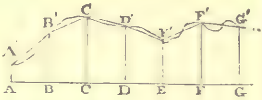
In some few cases, the approximation is carried to eight or ten decimal places: but the necessity for this is of comparatively rare occurrence. In *Hutton's Tables, p.* 360, the parts of the circle for degrees, minutes, and seconds, are put down to seven decimals.

2. The length of an arc of 1° is $\frac{6 \cdot 2831853 \ldots}{360}$, or ·0174533 ...

3. The factor $\frac{1}{4\pi} = \cdot 0795775 \ldots$ is often used in our practical calculations.

Other numbers relating to the circle and sphere are given on the last page of *Hutton's Tables, 7th Edition*, and constructions of a line approximately equal to the circumference have been given at *pp.* 400—1 of this volume.

14. *To find the area of an irregular figure by the method of equidistant ordinates.*

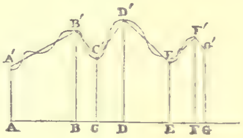


When a figure is bounded by an irregular outline A′ B′ C′ D′ E′ F′ G′ on one side, this method will enable us to obtain a tolerably close approximation to the area enclosed by it and a straight line AG. Set of AB = BC = CD = ... = FG, any number of equal parts. Denote each by h, and measure the several distances AA′ BB′, ... GG′ perpendicular to AG; and denote them severally by $a, b, c, \ldots\ldots g$. Then the area of the space enclosed by the straight lines A′A, AG, GG′ and the irregular boundary A′G′ will be expressed by

$$\left\{\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+d}{2}+\frac{d+e}{2}+\frac{e+f}{2}+\frac{f+g}{2}\right\}h,$$

$$\text{or, by } \left\{\frac{a+g}{2}+b+c+d+e+f\right\}h.$$

For these are the areas of A′ABB′, B′BCC′, ... ; and if the distance h be so taken as to render the collected excesses of one set of parts above the figure, *visibly* equal to the collected defects of the others from the several trapezoïds, the area will be found to a visible degree of accuracy.



When the figure is composed of boundaries along parts of which straight lines of considerable length can be drawn without varying much from the actual boundaries, a better method will be to take the extremities A′, B′, C′ .... G′ of these lines as the angles of the trapezoïds; to measure the lengths AB, BC, CD, ... FG of the intersections of the perpendiculars AA′, BB′, CC′, ... GG′ upon AG. Then the figure will be, calling these distances $h_1, h_2, \ldots\ldots h_n$,

$$\frac{a+b}{2}h_1+\frac{b+c}{2}h_2+\frac{c+d}{2}h_3+\ldots\ldots\frac{f+g}{2}h_n.$$

On these principles, the areas of fields bounded by irregular fences are estimated in land surveying: which will be illustrated under that head.

II. PROBLEMS ON PLANE SURFACES.

On the principles laid down in the preceding theorems the following series of problems admit of solution. The theorems for solution are referred to without specifying the rules in words, a step which is altogether unnecessary.

PROBLEM I. *To find the area of any parallelogram.*

Nos. 1, 2, 3, *page* .

1. The length of a parallelogram is 12·25 and its breadth is 8·5: what is its area? Ans. 12·25.8·5 = 104·125 = area.

2. The side of a square is 35 25 chs: required its area? Ans. 124ac 1r 1p.

3. Find the area of a rectangular board whose length is 12½ft, and whose breadth is 9in. Ans. 9⅜ ft.

4. Find the content of a piece of land in the form of a rhombus, its length being 6·2 chs, and breadth 5·45. Ans. 3ac 1r 20p.

5. Required the number of sq. yds in a rhomboid whose length is 37ft. and height 5ft 3in. Ans. $21\frac{7}{12}$ sq. yds.

6. The two diagonals of a parallelogram are 185·5 and 137·9, and they intersect under an angle of 42° 10′ 18″: what is the area? Ans 8587·55.

7. The side of a rhombus is $18\frac{3}{4}$, and one diagonal is $23\frac{1}{2}$: find the other diagonal and the area. Ans. diag. = 29·223278, area = 343·373.

8. If a, b, c be the distances of a tree from three of the angles of a square field, show that its area is expressed by

$$a^2 - 2ab\cos(45° + \phi) + b^2, \text{ where } \cos\phi = \frac{(a^2 - c^2)\cos 45° + b^2 \sec 45°}{2ab}.$$

9. The distances a, b, c, d of a point from the four angles of a rectangle are given to find its sides and area.

10. The stretching frame of a picture is 24in by 18; the frame of the picture (which is flat with a bevelled edge of $\frac{1}{2}$in wide, and inclined to the picture in an angle of 45°) extends over 2in of the canvas; and the visible area of the picture is equal to the visible surface of the frame: it is required to find the width of the frame, including the bevelled edge.

Problem II. *To find the area of a triangle.*

Rule I. No. 2, Cor. 2. page 471.

1. The base of a triangle is 625 and height 520 links: what is its area?
Ans. 625.260 = 162500lks = 1ac 2r 20p.

2. Find the area in yds of the triangle whose base is 40 and perpendicular 30ft. Ans. $66\frac{2}{3}$ sq. yds.

3. Required the number of yds in a triangle whose base and height are 49 and $25\frac{1}{4}$ft respectively. Ans. $68\frac{53}{72}$yds.

4. Required the area of a triangle whose base is 18ft 4in, and altitude 11ft 10in. Ans. 108ft. $5\frac{2}{3}$in.

Rule II. No. 2, Cor. 3, and No. 7, pages 472, 473.

1. The containing sides are 30 and 40, and included angle 28° 57′: what is the area?

By natural numbers.	By logarithms.
n sin 28° 57′ = ·4840462	t sin 28° 57′ = 9·6848868
$\frac{1}{2}$. 30.40 = 600	log 600 = 2·7781513
290·42772	log 290·4277 = 2·4630381

2. How many sq. yds are contained in the triangle, one of whose angles is 45°, and the containing sides 25 and $21\frac{3}{4}$ft respectively? Ans. 20·86947.

3. Given the base of a triangle equal to 476·25yds, and the angles at the base 27° 10′ 15″ and 35° 10′ 18″, to find the area and the three perpendiculars from the angles to the opposite sides.
Ans. area 33678, and perpendiculars 141·43, 217·47, 274·33.

4. The base of a triangle is 27·25 chains, the vertical angle is 57° 15′, and the difference of the angles at the base is 18° 18′. What is the area of the triangle? Ans. 328·972.

5. The difference of the segments into which the perpendicular divides the base is 10, the base itself is 50, and the vertical angle is 100°. What is the area of the triangle? Ans. 511·938.

Rule III. No. 6, page 473.

1. To find the area of a triangle whose sides are 20, 30, 40.

Here $2s = 20 + 30 + 40$, or $s = 45$, $s - a = 25$, $s - b = 15$, $s - c = 5$; and hence the area is $\sqrt{45.25.15.5} = 75\sqrt{15} = 290·4737$.

2. How many yards are there in a triangle whose sides are 30, 40, 50ft respectively? Ans. $66\frac{2}{3}$.

3. Find the area of a field whose sides are 2569, 4900, and 5025 links respectively. Ans 61ac 1r 39p.

4. The area of a triangle is 6, and two of its sides 3 and 5: find the third side and all the angles. Ans. 4 or $2\sqrt{13}$.

5. The sides of a triangle are in arithmetical progression, their sum is 27 linear feet, and the sum of their squares is 261 sq. ft: find the area in yds.

Ans. $\frac{3}{4}\sqrt{15}$.

6. The area of a triangle is 1000 and its sides in the ratio $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$: find the sides. Ans. 40·074, 50·093, 66·791.

7. Find the side of an equilateral triangle whose area is 100. Ans. 15·197.

8. The sides of a triangle are $\sqrt{3}, \sqrt{4}, \sqrt{5}$: what is the area? Ans. $\frac{1}{2}\sqrt{11}$.

9. A triangle has its sides each equal to $3\sqrt{3\sqrt{3}}$: what is its area? Ans. $20\frac{1}{4}$.

10. Find the area of the triangle whose sides are 4, $4+\sqrt{3}$ and $4-\sqrt{3}$.

Ans. $2\sqrt{3}$.

11. On the perpendicular of an equilateral triangle whose side is a, another equilateral triangle is described, and on the perpendicular of this another, and so on *ad inf.*: it is required to find the sum of the areas of all the triangles so described. Ans. $\frac{3}{4}a^2\sqrt{3}$.

12. The sides a, b, and the area Δ, are given to show that

$$\sin C = \frac{2\Delta}{ab},\ c=\sqrt{a^2+b^2\pm2\sqrt{a^2b^2-4\Delta^2}},\ \text{and}\ \sin B=\frac{2\Delta}{a\sqrt{a^2+b^2\pm2\sqrt{a^2b^2-4\Delta^2}}}$$

PROBLEM III. *To find the area of a trapezoïd.*

No. 3, *page* 472.

1. In a trapezoïd the parallel sides are 750 and 1225, and their distance 1540 links: what is the area? Ans. 15ac 0r 33p.

2. The greater and less ends of a plank are 15 and 11in, and its length 12ft 6in: what is its area? Ans. $13\frac{13}{24}$ft.

3. A quadrangular field ACDB has perpendiculars CP, DQ drawn to the side AB; and the following measures were taken from which to find its area: AP = 110, AQ = 745, AB = 1110, CP = 352, and DQ = 595 links respectively.

Ans. 4ac 1r 5 792p.

4. Find the area of the trapezoïd whose opposite parallel sides are 35 and 19; and the angles made by the oblique sides with the parallel sides are 42° 10′ 15″ and 73° 6′ 20″ respectively. Ans. 306·861.

5. From the triangle ABC, whose base AB is 40 and the perpendicular upon it is 60, to cut off a trapezoïd by a line parallel to AB which shall have the area 480. Ans. The breadth OP of the trapezoïd is $13\frac{3}{4}$ nearly.

6. What length must be cut from the broader end of a board which is 13 ft long, and whose ends are respectively 18in and 14in, to make 10ft square?

Ans. 7·1 nearly.

7. The breadth of a ditch at the top was 72ft, at the bottom $38\frac{2}{3}$, and the sloping side $26\frac{2}{3}$ and 20ft; and the top and bottom horizontal: find the area of the vertical section. Ans. $885\frac{1}{3}$ft.

8. The area of the section of the ditch being 154ft, and its depth $5\frac{1}{2}$ft; also the breadths at the top and bottom are as 9 to 5: what are those breadths?

Ans. 36 and 20ft.

Problem IV. *To find the area of any trapezium.*

Nos. 4, 5, *page* 472.

1. To find the area of a trapezium, one of whose diagonals is 42 and the perpendiculars upon it 16 and 18. Ans. $\frac{1}{2}(16+18)42=714$.

2. A diagonal is 65 ft, and the perpendiculars on it are 28 and $33\frac{1}{2}$ ft: how many yds does it contain? Ans. $222\frac{1}{12}$yds.

3. In a quadrangular field ABCD, owing to obstructions there could only be taken the following measures; BC = 265, AD = 220, AC = 378, AE = 100, CF = 70 yds, when DE and BF are perpendicular to AC: it is required to construct the figure and compute the area. Ans. 17ac 2r 21p.

4. The two diagonals of a trapezium are 31·2956 and 62·1598, and they intersect under an angle of 105° 18′ 25″: what is the area?

5. One angle of a trapezium is 107° 18′ 10″, and the diagonal divides it in the ratio of 7 to 5, whilst the sides containing it are 123·456 and 654·321, and the greater side makes the less angle with the diagonal, the diagonal itself being 1000: it is, from these data, required to find the areas of the two triangles into which the other diagonal divides the trapezium.

6. The four sides of a field taken in order are 25, 35, 31, and 19 poles, and the diagonals are equal: required the area of the field.

7. ABCD is a quadrangular field, whose sides taken in succession are AB = 15 ch 24 l, BC = 18 ch 86 l, CD = 9 ch 90 l, and DA = 11 ch 14 l; also the angles at A and C are 105° 28′ and 89° 54′: find its area. Ans. 17·5169ac.

8. The side AD of a quadrangular field ABCD was 311yds, and the angles were BAC = 44° 20′, CAD = 41° 19′, ADB = 24° 10′, and DBC = 37° 4′: required its area. Ans. 8·6531ac.

9. The sides of a quadrangular field taken in order are 26, 20, 16, and 10 poles, and the angle contained by the longest sides is 56°: what is its area?
Ans. 1ac 127·676p.

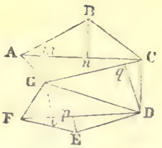
10. ABCD is a quadrilateral, whose angles B and C are right angles, as are likewise those formed by the diagonals BD, AC: the diagonals themselves are given, d and d_1, and it is required to find the area, the sides, and the remaining angles formed by the several lines of the figure.

11. The two diagonals AC, BD, of a trapezium bisect each other; one of the angles, B, is 30°, the side BC is 100, and twice the product of the diagonals is equal to the square of their difference: required the sides, angles, and area of the figure.

Problem V. *To find the area of any polygon.*

This will be effected if we divide it into triangles or quadrilaterals, or both, and take sufficient measures either of lines or lines and angles for the trigonometrical determination of all the parts requisite for the purpose. One or two examples of the manner of dividing the polygon will suffice to show the nature of the problem, whilst the actual computation is left for the student.

1. Suppose a regular pentagon whose side is 170 fathoms, to be fortified, and that the salient angle of the bastion is 71°, and its face 47 fathoms; required the flank and curtain, supposing the line of defence to be perpendicular to the flank: determine, also, how many acres would be contained within the boundaries of the fortification, supposing the work completed.
Ans. Flank = 25·65, curtain = 64 57, area = 26 051.



2. The polygon in the annexed figure had the following parts measured, from which to determine the area: viz. AC = 55, FD = 52, GC = 44, Gm = 13, Bn = 18, Go = 12, Ep = 8, and Dq = 23.

Ans. area = 1878·5.

3. The sides FE, AB, BC, were found to be 15·726, 25·182, and 23·629, and the angles were taken as follows: FED = 152° 10′, EFG = 65° 18′, FEG = 66° 28′, EGD = 31° 15′, CGD = 32° 18′, GCD = 58° 40′, AGC = 100° 5′, GAC = 41° 50′: it is required to find the area.

Problem VI.—*To find the area of a regular polygon.**

Nos. 8, 9, 10, *page* 474.

1. Required the area of a regular pentagon, each side being 25 ft.

Ans. 1075·298356.

2. Find the area of an equilateral triangle whose side is 20 ft.

Ans. 173·2058.

3. Required the area of a hexagon whose side is 20. Ans. 1039·23048.
4. Find the area of an octagon whose side is 20. Ans. 1931·37084.
5. Find the area of a decagon whose side is 20. Ans. 3077·68352

Prob. VII.—*To find the circumference and diameter of a circle from one another.*

No. 11, *page* 475.

The approximation 1 : 3·14159 is the number used for general scientific purposes; but for mere round numbers 7 : 22 or 113 : 335 are used as the ratio of the diameter to the circumference. In the same manner, circumference : diameter :: 1 : ·318309.

1. Find the circumference of a circle whose diameter is 20.

Ans. 7 : 22 :: 20 : 62$\frac{6}{7}$ as the roughest approximation.

2. If the circumference of the earth be 24877·4 miles, what is its diameter?

3·14159 : 1 :: 24877·4 : 7918·72
355 : 113 :: 24877·4 : 7918·72
1 : ·318309 :: 24877·4 : 7918·72

3. Required the circumference of a circle whose radius is 22$\frac{1}{8}$.

Ans. 139·01547.

---

* It is sometimes convenient to possess the values $n \cot \frac{\pi}{n}$ in a table; and likewise the value of the circumscribed radius. In the latter case the tabulated value is multiplied by the side of the polygon, and in the former by its square. The following is such a table to the duodecagon inclusive.

No. of Sides.	Names.	Areas, or Multipliers.	Radius of circum. circle.	No. of Sides.	Names.	Areas, or Multipliers.	Radius of circum. circle.
3	Trigon or triangle ...	0·4330127	0·5773503	8	Octagon	4·8284272	1·3065630
4	Tetragon or square .	1·0000000	0·7071068	9	Nonagon	6·1818242	1·4619022
5	Pentagon...	1·7204774	0·8506508	10	Decagon	7·6942088	1·6180340
6	Hexagon .... ,..	2·5980762	1·0000000	11	Undecagon	9·3656415	1·7747331
7	Heptagon	3·6339126	1·1523825	12	Duodecagon	11·1961524	1·9318517

To show its use we may take Ex. 1, in which we have $25^2 . 1·7204774 = 1075·29837$, the area.

4. The circumference of a circle is 64·4: what is the diameter?

Ans. 20·49916.

5. Two chords whose sum is 21 and difference 1, and the rectangle of the segments of either is 24, cut each other at right angles: it is required to find the circumference of the circle, and the distance of their point of intersection from the centre. Ans. circumf. $= 5\pi\sqrt{5}$, dist. $= \frac{1}{2}\sqrt{29}$.

6. Let C be the circumference of a circle, d its diameter, and c the chord of an arc, a: show that $c = \dfrac{16\,ad\,(C - a)}{5C^2 - 4a\,(C - a)}$ nearly.

7. If a be an arc of a circle to radius 1, $a^4 = 5.48^2 \cdot \dfrac{3 + \cos a - 4\cos\frac{1}{2}a}{237 - \cos a + 124\cos\frac{1}{2}a}$ nearly.

8. Show that $3a = \tan a + 2\sin a$, very nearly, when a is small.

PROBLEM VIII. *To find the length of an arc of the circle.*

Scholium, No. 13, page 477.

Multiply the number of degrees and parts expressing the arc by ·0174533, and by the radius of the circle; or, take out the numbers corresponding to the given number of degrees, minutes, and seconds successively from *Hutton's Tables*, p. 360; and then their sum, so multiplied, will be the length required.

1. Find the length of 30° in a circle whose radius is 9ft. Ans. 4·712388.
2. To diameter 10ft find the length of 12° 10′ 15″. Ans. 4·248422.
3. What portion of the arc of a circle is equal to the radius?

Ans. 57°·2957795. ....

PROBLEM IX. *To find the area of a circle.*

No. 12, and scholium, page 477.

Let r be the radius, d the diameter, c the circumference, and A the area: then is found from any one of the following equations.

$$A = \pi r^2 = \tfrac{1}{2}r.\tfrac{1}{2}c = \tfrac{1}{4}dc = d^2.\frac{\pi}{4} = \cdot0795775c^2.$$

1. Find the area of a circle whose circumference is 31·41593.

Ans. 78·5398.

2. Find the area of a circle whose diameter is 7. Ans. 38·484501.
3. How many square yards are there in a circle whose diameter is $3\frac{1}{2}$ft.

Ans. 1·069014.

4. Required the area of a circle whose circumference is 12ft.

Ans. 11·45916.

5. Find the area of a circle, the difference of whose diameter and circumference is 1056·64 ft. Ans. 191105·4ft.

6. If the centre of a circle whose diameter is 20 be in the circumference of another whose diameter is 40, what are the areas of the three included spaces?

Ans. 173·852, 140·308, and 1116·332.

7. What is the diameter of that circle which contains an acre? Ans. $78\frac{1}{2}$ yds.

8. If the area of a circle be 100, find the sides of the inscribed square, pentagon and hexagon.

PROBLEM X. *To find the area of an annulus.*

No. 12, Cor. 4, page 477.

1. The diameters of two concentric circles are 10 and 6: required the area of the annulus. Ans. 50·26552.

2. The bounding circles are 10 and 20 in diameter: what is the area of the annulus? Ans. 235·61947.

3. The circumference of a ring is 161in, and its width 1in: required its internal diameter and area. Ans. int. diam. $= 49·248$in, and area $= 157·8634$.

4. The radii of two concentric circles are in the ratio of 10 to 9, and the area of the ring is 375·562feet: find the diameters of the circles.

Ans. 50·1624, and 45·1462.

5. Let c be the outer circumference, and b the breadth of a ring: show that its area will be $(c - b\pi)\, b$.

PROBLEM XI. *To find the area of a sector of a circle.*

No. 12, *Cor.* 2, *page* 477.

(1) Multiply the radius by half the length of the arc.
(2) Take $\frac{1}{4}$th of the product of the arc and diameter.
(3) 360° : given arc :: πr^2 : area of sector.

1. Find the area of a sector of 18° to a diameter of 3ft. Ans. ·35343.

2. The radius is 10ft, and the arc 20ft: find the area and angular measure.

Ans. 100 and $114^\circ\, 35'\, 29''$.

3. A sector of $147^\circ\, 29'\, 18''$ has a radius of 25ft: what is its area?

Ans. 804·3986.

4. Find the area of a sector whose radius is 50 and arc $56^\circ\, 30'$.

Ans. 1232·6387.

5. The area of a sector is 100, and the length of its arc 20: what is the angle of the sector? Ans. $114^\circ\, 35'\frac{1}{2}$ nearly.

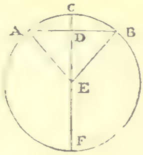
PROBLEM XII. *To find the area of a segment of a circle.*

No. 13, *page* 477.

(1) Find the difference or sum of the sector having the same arc and the triangle formed by the chord and the radii bounding the sector, according as the arc is less or greater than a semicircle.

(2) Segment $= \frac{1}{2}r^2\,\{a \mp \sin a\}$, which is only the same rule in another form.

1. The chord AB is 12, and the radius AE is 10: what is the area of the segment?



$$\sin AED, \text{ or } \sin AEC = \frac{AD}{AE} = ·6 = \sin 36^\circ\, 52'\, 11''·2.$$

Hence the arc ACB $= 73^\circ\, 44'\, 22''·4$; and we have

$360^\circ : 73^\circ\, 44'\, 22''·4 :: 10^2.\pi : 64·3504 =$ area of sector.

Also, for the triangle AEB, we have

$$ED = \sqrt{AE^2 - AD^2} = \sqrt{100 - 36} = 8,$$

and the area of the triangle is AD.DE $= 6.8 = 48$. Whence segment ACB $=$ sector AEB $-$ triangle AEB $= 16·3504$.

Or again, by the formula: we have as before arc ACB $= 73^\circ\, 44'\, 22''·4$; and by *Hutton's Tables, p.* 360,

arc 73° =	1·2740904	sin $73^\circ\, 44'$ =	·9599684
$44'$ =	127991	pp to $22''·4$ =	304
$22''$ =	1067		
$·4''$ =	3		·9599988
			1·2869965
	1·2869965		
			·3269977 for less segment.
			2·2469953 for greater segment.

Hence, less segment $= \frac{1}{2}r^2\,\{a - \sin a\} = 16·349885$.

and greater segment $= \frac{1}{2}r^2\,\{a + \sin a\} = 112·349765$.

2. The height CD is 18, and radius CE is 50: what is the area of the segment? Ans. 961·3532.

3. Required the area of each of the segments where the chord is 16, the diameter of the circle being 20. Ans. 44 728 or 269·432.

Scholium.

It occurs in many problems that we have to find the arc of the sector from knowing the area, together with the radius of the circle, or some other given line in the circle: or in other words, to solve the equation.

$$\theta \pm \sin\theta = \frac{b}{a}.$$

For this purpose no method seems so generally and easily applicable as the method of trial and error. One example is annexed: viz. to find θ from the equation

$$\theta - \sin\theta = 2·0943951.$$

Take as a conjectural approximation $\theta = 150°$: then

arc 150° =	2·6179939	arc 149° =	2·6005406
— sin 150° = —	·5000000	— sin 149° = —	·5150381
	2·1179939		2·0855025
too great by	·0235988	too small by	·0088926

Hence by *Trial and Error*, p. 202 of this work

$$c_1 = \frac{·0088926}{·0324914} = ·26° = 16' \text{ nearly.}$$

Taking next $\theta = 149° \, 16'$ we have a second correction.

arc 149° 16′ =	2 6051948	arc 149° 17′ =	2·6054857
— sin 149 16 = —	·5110431	— sin 149 17 = —	·5107930
	2·0941517		2·0946927
too small by	·0002434	too great by	·0002976

$$\text{Hence } c_2 = \frac{·0002434}{·0005410} = ·45' = 27'' \text{ nearly; and hence again}$$

$$\theta = 149° \, 16' \, 27'' \text{ nearly.}$$

PROBLEM XIII. *To measure any long and irregular figure.*

No. 14, *page* 478.

1. The breadths of an irregular figure at five equidistant places are 8·2, 7·4, ·2, 10·2, and 8·6, and the whole length is 39: what is the area? Ans. 343·2.

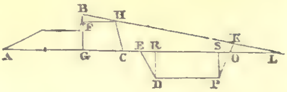
2. The length of the figure is 84, and the six equidistant ordinates are 17·4, 0 6, 14·2, 16 5, 20·1, and 24·4: what is the area? Ans. 1550·64.

3. The distances of seven points in the long side of a figure formed by straight lines from the first extremity of that side were 108, 104, 25, 10·6, 18·5, 56·2, and the ordinates of the angular points were 0, 12, 18, 25, 18, 5, 15, and 0: what was the area?

3. In the vertical section of a rampart AS is the horizontal base, and the horizontal distance, in feet, of the several angular points of the work reckoned on this line, together with the heights of those points above it, are ranged below, from which to find the area of the section and construct the figure: viz.—

Distance on AS, AB = 16, BD = 18, DH = 2, HK = 3, KL = 2, LP = 12, PS = 10.

Height above AS, BC = 12, DE = $12\frac{1}{2}$, HG = $13\frac{1}{2}$, KI = $13\frac{1}{2}$, LO = 18, PR = 16.



4. Let ABC be the profile, or perpendicular section of a breast-work, and EP that of the ditch. Now, suppose the area of the section ABC is 88 feet, the depth of the ditch RD 6 feet, ER = SO = 3 feet; what is the breadth of the ditch at top when the sections of the ditch and the breast-work are equal; that is, when the earth thrown out of the ditch is sufficient to make the breast-work?

5. And what must be the breadth of the ditch at top, the depth and width at bottom remaining the same, when the profile of the breast-work remains the same, and the earth, in consequence of removal, occupies $\frac{1}{12}$th more space than it did before it was taken out of the ditch?

MENSURATION OF SOLIDS.

The cube described upon the linear unit is always taken as the unit of volume. It will hence follow in precisely the same manner as for superficial measure, that the volume of a rectangular prism or cylinder is the product of the numerical measures of the base and altitude. For any two rectangular parallelopipedons are to one another in a ratio compounded of that of their bases and that of their altitudes; and the bases are compounded of the ratios of their sides containing one of the right-angles: hence they are compounded of the ratios of the three edges of the one to the three edges of the other, each to each respectively.

So far as the surfaces of solids *bounded by plane faces* are concerned, the surfaces are computed by the rules appropriate to them, as already explained in the mensuration of planes: but the surface of the sphere and its segments or zones will form a part of this section.

I. THE SURFACES AND VOLUMES OF FIGURES OF THREE DIMENSIONS.

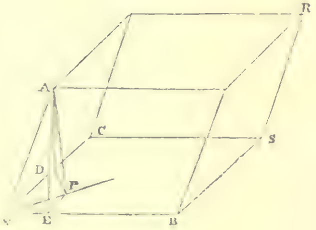
1. The *volume of a rectangular parallelopipedon* is expressed by the continued product of these edges which meet at one of the solid right-angles, as is evident from the foregoing introductory remarks.

2. The volume of *any parallelopipedon*, whose plane angles forming one of the solid angles are α, β, γ, and whose opposite edges are a, b, c, is expressed by

$$2abc\sqrt{\sin\sigma\,\sin(\sigma-\alpha)\,\sin(\sigma-\beta)\,\sin(\sigma-\gamma)}$$

where $2\sigma = \alpha + \beta + \gamma$.

Let Q be the solid angle, the three edges AQ, BQ, CQ, of which are denoted by a, b, c; and the three angles CQB, CQA, AQB are α, β, γ. From A draw the perpendicular AP to the plane CQB, and from P draw PD, PE perpendicular to QC, QB, and join AE, AD, and PQ. Then (*Geom. Planes, th.* 7), AEQ, ADQ, are right-angles, and AQP is the inclination of the line AQ to the plane CQB. Denote this by θ: then we have



$$QP = a\cos\theta,\ QD = a\cos\beta, \text{ and } QE = a\cos\gamma. \quad \text{Also,}$$

$$\cos PQC = \frac{QD}{QP} = \frac{\cos\beta}{\cos\theta}, \text{ and } \cos PQB = \frac{QE}{QP} = \frac{\cos\gamma}{\cos\theta}.$$

$$\text{But } \cos a = \cos(PQC + PQB) = \frac{\cos\beta\cos\gamma}{\cos^2\theta}, - \sqrt{\left(1 - \frac{\cos^2\beta}{\cos^2\theta}\right)\left(1 - \frac{\cos^2\gamma}{\cos^2\theta}\right)}$$

whence, transposing, squaring, and performing obvious reductions, it becomes at once

$$\sin\theta\sin a = \sqrt{1 - \cos^2 a - \cos^2\beta - \cos^2\gamma + 2\cos a\cos\beta\cos\gamma}.$$
$$= 2\sqrt{\sin\sigma\sin(\sigma - a)\sin(\sigma - \beta)\sin(\sigma - \gamma)}.$$

But the parallelopipedon QR is compounded of the base QS and altitude AP, and hence we have

$$\text{vol} = bc\sin a \,.\, a\sin\theta = abc\sin a\sin\theta.$$
$$= 2abc\sqrt{\sin\sigma\sin(\sigma - a)\sin(\sigma - \beta)\sin(\sigma - \gamma)}.$$

3. The *volume of any prism* is the product of one end into the distance of the two parallel ends. (*Geom. Pl. and Sol., th.* 30, *p.* 366).

4. The *volume of a cylinder* is expressed by the product of its base and altitude; and that of a *cone* by one-third of that product: as is clear from *Geom. of Pl. and Sol., theorems* 30, and *Cor.* 34.

5. The *curve surface of a right-cylinder* is the product of the perimeter of the base into the length of the axis, or by $2rh\pi$. For that surface is composed of an infinite number of infinitely narrow rectangles, all of the same length.

When the two ends are also required in the expression, the entire surface is expressed by $2r(r + h)\pi$. For $2r^2\pi$ = area of the base, and $2rh\pi$ = curve surface; hence the whole surface $2r^2\pi + 2rh\pi = 2r(r + h)\pi$.

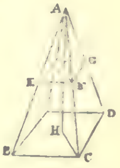
6. The *curve surface of a right-cone* is half the product of the perimeter of the base into one of the slant sides or edges. The reason is similar to the last.

Cor. The curve surface of a frustum of a cone whose bounding sections have a and b for radii, and their distance reckoned along the side of the cone is d, is $(a + b)d\pi$.

7. The *volumes of a cone and pyramid* are one-third of the volumes of the cylinder of a prism respectively of the same bases and altitudes.

8. The *volume of a truncated pyramid or cone* is expressed by $\frac{1}{3}h\{a^2 + ab + b^2\}$, where a and b are the areas of the two ends, and h is their distance.

For, let ABCD be the pyramid of which BCDEFG is the frustum, a^2 the area of BCD, b^2 the area of EFG, and h the height HI of the frustum. Denote AI by c: then by similar figures



$$c + h = \frac{ac}{b}, \text{ or } c = \frac{bh}{a-b} \text{ and } c + h = \frac{ah}{a-b}.$$

Again, vol frustum BDGE

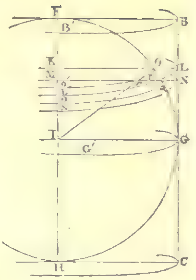
$$= \text{vol ABCD} - \text{vol AEFG} = \frac{1}{3}a^2(c + h) - \frac{1}{3}b^2 c$$

$$= \frac{1}{3}a^2\frac{ah}{a-b} - \frac{1}{3}b^2\frac{bh}{a-b} = \frac{1}{3}h\{a^2 + ab + b^2\}.$$

The same demonstration applies to the conic frustum.

Cor. If D, d, be the corresponding linear dimensions of the ends, m their appropriate multiplier, (so that mD^2, md^2, are their areas,) and δ^2 the difference of those areas: then the volume is expressed by $\frac{1}{3}mh\{3Dd + \delta^2\}$.

9. The *curve surface of a sphere* is equal to four times the area of one of its great circles.



Let the hemisphere be generated by the revolution of the quadrant FOG about one of its limiting radii FI, and let also the tangent GB at the other extremity G describe a cylinder of the same altitude. Cut both these by two planes, KLL', MNN', parallel to the common base IGG' of the cylinder and hemisphere; which will cut the sphere in two parallel circles KOO', MQQ', and the cylinder in two parallel circles KLL', MNN; and likewise any plane GIF through the axis FI will be cut by them in the lines KOL, MQN, and the planes of the ends of the cylinder in FB, IG. Draw in the plane GIF, the line OP perpendicular to MN, and join OI.

Now if the arc OQ be taken smaller and smaller continually, the angle formed by the *line* OQ with the radius IO may be made to differ from a right angle by an angle less than any assignable one, whilst it can never exceed a right angle. The right angle is, therefore, the ultimate one formed by the *arc* OP and the radius IO, since in that case the arc OQ and line OQ are coincident.

In this case, then, ROQ is a triangle right-angled at O, and OP perpendicular to its base: whence the triangle OPQ is similar to RPO, that is, to RMI and to OKI. Wherefore by the similar triangles OPQ, OKI, (since OP = LN, and IO = KL,)

$$\text{OQ} : \text{LN} :: \text{KL} : \text{KO} :: 2\pi.\text{KL} : 2\pi.\text{KO} :: \text{circ LL}' : \text{circ OO}'.^*$$
$$\text{or OQ.circ OO}' = \text{LN.circ LL}'.$$

But the zone LNN'L' of the cylinder is expressed by the circumference of the base multiplied by the altitude; and the zone OQQ'O' of the sphere is *ultimately* the surface of the frustum of a cone whose side is OQ, and whose two ends are OO' and QQ', and is expressed by $\frac{1}{2}$ OQ $\{$OO' + QQ'$\}$. Also ultimately OO' is equal to QQ', and hence the spherical zone is ultimately expressed by OQ.OO': whence also ultimately we have the spherical zone OQQ'O' equal to the cylindrical zone LNN'L'.

Again, since this is true wherever the point O is taken, it is true for every point in the quadrant FG; and hence all the elementary zones which compose the surface of the hemisphere are equal to all those which compose the surface of the cylinder, each to each. The entire surfaces of the hemisphere and cylinder, or any corresponding parts of them between parallel planes, must, therefore, be equal, each to each; and likewise, the entire spherical and the entire cylindrical surfaces of the same altitude, or any corresponding parts of them, will be equal, each to each.

Now S = cylindrical surface $= 2r \,.\, 2r\pi = 4r^2\pi =$ 4 area of the great circle GG'; which is the proposition enunciated.

Cor. 1. Whence $S = 2r \,.\, 2r\pi = d \,.\, d\pi =$ diam $\times$ circ.

Cor. 2. Any zone or segment of a sphere is expressed by the product of its altitude into the circumference of its great circle.

10. Putting V for the *volume of a sphere* and the remaining notation as before, we shall have

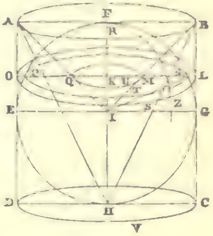
$$V = \pi d \,.\, \tfrac{1}{6} d^2 = \tfrac{1}{6}\pi d^3 = \tfrac{1}{6} dS = \tfrac{1}{6}\pi^{-2}(d\pi)^3.$$

For the sphere is two-thirds of its circumscribing cylinder (*Geom. of Pl. and Sol.*

* By these are meant the *entire* circumferences of the circles whose centres are K and M, though only parts of them are actually marked in the figure.

th. 37); that is, two-thirds of the base of the cylinder multiplied by its altitude; or again in symbols $V = \frac{2}{3} \cdot \frac{1}{4} \pi d^2 = \frac{1}{6} d^2 \cdot \pi d$, which is the first equation; and the others are simple transformations into factors expressive of the different way by which the volume may be found from the data.

11. Denote the height, KF, of a spherical segment by h, and the radius of its base by r_1; then the *volume of the segment is expressed* by $\frac{1}{6} \pi (3d - 2h)h^2$, or by $\frac{1}{6} \pi (3r_1^2 + h^2)h$; or again by $\frac{1}{3} \pi \{2r^3 \pm (2r^2 + r_1^2) \sqrt{r^2 - r_1^2}\}$ the upper or lower sign being used according as the segment is greater or less than a hemisphere.



For the cones generated by AIB, QIM, are similar; and hence,

$FI^3 : KI^3 :: (2FI)^3 : (2KI)^3 ::$ cone AIB : cone QIM,

or $d^3 : (d-2h)^3 :: \frac{1}{24} d^3\pi$: cone QIM $= \frac{1}{24} \pi (d-2h)^3$

Hence cone AIB — cone QIM $= \frac{1}{24} \pi \{d^3-(d-2h)^3\}$

$$= \frac{1}{12} \pi \{3d^2h - 6dh^2 + 4h^3\}$$

But the spherical segment is equal to the difference of the cone and cylinder (*Geom. Pl. and Sol., th.* 37); that is,

$$V = \tfrac{1}{4} \pi d^2h - \tfrac{1}{12} \pi \{3d^2h - 6dh^2 + 4h^3\}$$
$$= \tfrac{1}{6} \pi \{3d - 2h\} h^2 \ldots\ldots\ldots\ldots (1)$$

Again, $PK^2 = FK.KH$, or $r_1^2 = (d-h)h$; whence $d = \frac{r_1^2 + h^2}{h}$. Substitute this in (1), then

$$V = \tfrac{1}{6} \pi (3r_1^2 + h^2)h \ldots\ldots\ldots\ldots (2)$$

From $r_1^2 = (d - h)h$ we have $h = r \pm \sqrt{r^2 - r_1^2}$, which substituted in (1) or (2) gives

$$V = \tfrac{1}{3} \pi \{2r^3 \pm (2r^2 + r_1^2) \sqrt{r^2 - r_1^2}\} \ldots\ldots\ldots\ldots (3)$$

Cor. A *spherical zone* may be considered as the difference of two segments, the factors of $\frac{1}{3}\pi$ in which must be separately calculated, and their difference then multiplied by $\frac{1}{3}\pi$.

II. PROBLEMS ON SOLID FIGURES.

PROB. I. *To find the surface of a prism or cylinder.*

1. Find the surface of a cube, the side of which is 20ft. Ans. 2400ft.

2. Find the whole surface of a triangular prism whose length is 20ft, and each side of its ends 18in. Ans. 91·948ft.

3. Required the convex surface, and the whole surface, of a right cylinder, the length being 20ft, and diameter of the base 2ft.

Ans. 125·664ft, and 131·947ft.

4. A rectangular cistern whose length was 3ft 2in, breadth 2ft 8in, and depth 2ft 6in, was lined with lead at 3*d* per lb, and the thickness of the lead such as to weigh 7lb to the foot surface: what must be paid for it? Ans. 3*l* 5*s* $9\frac{3}{4}$*d*.

5. How many square feet of board are required to make a packing-case whose length is $3\frac{1}{2}$ft, breadth 2ft, and depth 20in? Ans. $32\frac{1}{3}$ft.

6. How many revolutions of a roller 40in in length and 30in in diameter, will be required in rolling a lawn 80ft in length and 50ft in breadth; the lawn being rolled first lengthways and then across?

PROB. II. *To find the surface of a right pyramid or right cone.*

Nos. 5, 6, *p.* 487.

1. What is the whole surface of a triangular pyramid, the slant edges being each 20ft, and each side of the base 3ft?

2. Required the convex surface of a cone, or circular pyramid, the slant height being 50ft, and the diameter of its base $8\frac{1}{2}$ft. Ans. 667·59.

3. Find the whole surface of the same cone.

4. Find the cost of lining a circular reservoir whose diameter at the top is 40yds, at the bottom $38\frac{2}{3}$yds, and whose side or slant depth is 11ft: the brick-work being executed at 3*s* 10*d* per square yard. Ans. 311*l* 18*s* 2*d*.

5. What quantity of canvas is required for a conical tent, eight feet high and thirteen feet wide at the bottom? Ans. $70\frac{1}{6}$ yds square, nearly.

PROBLEM III. *To find the surface of a frustum of a pyramid or cone.*

No. 6 *Cor. p.* 487.

1. How many square feet are in the surface of the frustum of a square pyramid, whose slant height is 10ft; also each side of the base or greater end being 3ft 4in, and each side of the less end 2ft 2in? Ans. 110ft.

2. Required the convex surface of the frustum of a cone, the slant height of the frustum being $12\frac{1}{2}$ft, and the circumferences of the two ends being as 5 to 7, and the area of the less one 2·86488ft. Ans. 90ft.

PROB. IV. *To find the volume of a right prism or cylinder.*

Nos. 3, 4, *p.* 487.

1. Find the solid content of a cube, whose side is 24 in. Ans. 13824.

2. How many cubic feet are in a block of marble, its length being 3ft 2in, breadth 2ft 8in, and thickness 2ft 6in? Ans. $21\frac{1}{9}$.

3. How many gallons of water will the cistern contain, whose dimensions are as 19, 16, 15, and the diagonal drawn through the cistern is 58·0345 in, when $277\frac{1}{4}$ cubic inches are contained in one gallon? Ans. 131·53.

4. Required the solidity of a triangular prism, whose length is 10ft, and the three sides of its triangular end or base are 3, 4, 5 feet. Ans. 60ft.

5. If the depth of an oblique parallelopiped be 8 ft, and the obtuse angle at the base be 138°, and the including sides be 10 and 15 ft: what is its volume in cubic yards? Ans. 25·7392.

6. If the mean velocity of water through a cylindrical pipe an inch and a half in diameter be 13in per second: what quantity would it supply in 12 hours?

PROBLEM V. *To find the volume of any pyramid or cone.*

No. 7, *page* 487.

1. Required the solidity of a square pyramid; each side of its base being 30, and its height 25. Ans. 7500.

2. Find the content of a triangular pyramid, whose height is 30, and each side of the base 3. Ans. 38·971143.

3. Find the content of a triangular pyramid, its height being 14ft 6in, and the three sides of its base 5, 6, 7ft. Ans. 71·0352.

4. What is the content of a pentagonal pyramid, its height being 12ft, and each side of its base 2ft? Ans. 27·5276.

5. What is the content of the hexagonal pyramid, whose height is 6·4ft, and each side of its base 6in? Ans. 1·38564ft.

6. Required the content of a cone, its height being $10\frac{1}{2}$ ft, and the circumference of its base 9ft. Ans. 22·56093.

7. The content of a cone is 67·7828, the radius of its base 1·43176. Find the slant height, and the length of its axis. Ans. 45·2086 and 45·3614.

PROBLEM VI. *To find the volume of the frustum of a cone or pyramid.**

No. 8, *page* 487.

1. To find the number of solid feet in a piece of timber, whose bases are squares, each side of the greater end being 15in, and each side of the less end 6in, and the length 24ft. Ans. $19\frac{1}{2}$.

2. Required the content of a pentagonal frustum, whose height is 5ft, each side of the base 18in; and each side of the top or less end 6 in.
Ans. 9·31925ft.

3. Find the content of a conic frustum, the altitude being 18, the greatest diameter 8, and the least diameter 4. Ans. 527·7888.

4. What is the solidity of the frustum of a cone, the altitude being 25, also the circumference at the greater end being 20, and at the less end 10?
Ans. 464·216.

5. If a cask, which is two equal conic frustums joined together at the bases, have its bung diameter 28in, the head diameter 20in, and length 40in; how many gallons of wine will it hold? Ans. 79·0613.

PROBLEM VII. *To find the surface of a sphere or spherical segment.*

No. 9, *page* 488.

$$Surface = cd = \pi d^2 = \frac{c^2}{\pi} = \cdot 318309c^2, \text{ and } Segment = ch.$$

1. Find the surface of a sphere whose diameter is 7. Ans. 153·93804.
2. Find it when the circumference is 22. Ans. 154·06156.
3. Required the surface of a globe of 24in diameter. Ans. 1809·557368.
4. Find the area of the surface of the globe, its diameter being taken $7957\frac{3}{4}$ miles and circumference 25000: also find it from each of these data taken separately. Ans. 198943750, 198943821, and 198943125, respectively.

---

* We may proceed rather differently when the ends are either circles or regular polygons. In this latter case, square one side of each polygon, and also multiply the one side by the other; add all these three products together; then multiply their sum by the tabular area proper to the polygon, and take one-third of the product from the mean area, to be multiplied by the length, to give the solid content. Also in the case of the frustum of a cone, the ends being circles, square the diameter or the circumference of each end, also multiply the same two dimensions together; then take the sum of the three products, and multiply it by the proper tabular number, viz. by ·7854 when the diameters are used, or by ·07958 in using the circumferences; then take one-third of the product to multiply by the length, for the content.

5. The axis of a sphere being 42in, what is the convex surface of a segment whose height is 9in? Ans. 1187·5248.

6. Required the convex surface of a zone whose breadth is 2ft cut from a sphere $12\frac{1}{2}$ft diameter? Ans. 78·539.

Problem VIII. *To find the volume of a sphere.*

No. 10, *page* 488.

$$Vol. = \frac{1}{6}d \times surface = \frac{1}{2}d^2c = \frac{1}{6}d^3 = \cdot 01688c^3.$$

1. Find the solid content of the globe of the earth, supposing its circumference to be 25000 miles. Ans. 263750000000 miles.

2. Supposing that a cubic inch of cast iron weighs ·269 of a lb avoird., what is the weight of an iron ball of 5·04 inches diameter?

3. The radii of a shell are R and r: show that its content is $\frac{4}{3}\pi\left\{R^3 - r^3\right\}$.

Problem IX. *To find the volume of a spherical segment or zone.*

No. 11, *page* 489.

$Vol. = (3d - 2h)h^2 . \frac{\pi}{6} = (3r_1^2 + h^2)h . \frac{\pi}{6}$, where r_1 is the radius of the section and h the height of the spherical segment.

1. To find the content of a spherical segment, of 2ft in height, cut from a sphere of 8ft diameter. Ans. 41·888ft.

2. What is the solidity of the segment of a sphere, its height being 9, and the diameter of its base 20? Ans. 1795·4244.

3. The radii of the faces of a zone are 12 and 9 respectively; and the thickness of the zone is 4 find the volume of the zone, and the radius of the sphere of which it forms a part.

LAND SURVEYING.

DESCRIPTION AND USE OF THE INSTRUMENTS.

1. *Of the chain.* Land is measured with a chain, called Gunter's Chain, from its inventor, the length of which is 4 poles, or 22 yards, or 66 feet. It consists of 100 equal links; and the length of each link is therefore $\frac{22}{100}$ of a yard, or $\frac{66}{100}$ of a foot, or as much as 7·92 inches*.

* Land is estimated in acres, roods, and perches. An acre is equal to 10 square chains, or as much as 10 chains in length and 1 chain in breadth. Or, in yards, it is $220 \times 22 = 4840$ square yards. Or, in poles, it is $40 \times 4 = 160$ square poles. Or, in links, it is $1000 \times 100 = 100000$ square links: these being all the same quantity.

Also an acre is divided into four parts called roods, and a rood into 40 parts called perches, which are square poles, or the square of a pole of $5\frac{1}{2}$ yards long, or the square of $\frac{1}{4}$ of a chain, or of 25 links, which is 625 square links. So that the divisions of land-measure will be thus:

625 square links = 1 pole or perch; 40 perches = 1 rood; 4 roods = 1 acre.

The lengths of lines measured with a chain, are best set down in links as integers, every chain in length being 100 links: and not in chains and decimals. Therefore, after the content is found, it will be in square links; then cut off five of the figures on the right-hand for decimals,

2. *Of the plane table.* This instrument consists of a plain rectangular board, of any convenient size: the centre of which, when used, is fixed by means of screws to a three-legged stand, having a ball and socket, or other joint, at the top, by means of which, when the legs are fixed on the ground, the table is inclined in any direction *.

---

and the rest will be acres. These decimals are then multiplied by 4 for roods, and the decimals of these again by 40 for perches.

Exam. Suppose the length of a rectangular piece of ground be 792 links, and its breadth 385; to find the area in acres, roods, and perches.

Here we have, $792.385 = 3{\cdot}0492 = 3$ac 0r $7{\cdot}872$p.

* To the table belong various parts, as follow.

1. A frame of wood, made to fit round its edges, and to be taken off, for the convenience of putting a sheet of paper on the table. One side of this frame is usually divided into equal parts, for drawing lines across the table, parallel or perpendicular to the sides; and the other side of the frame is divided into 360 degrees, to a centre in the middle of the table; by means of which the table may be used as a theodolite, &c.

1. A magnetic needle and compass, either screwed into the side of the table, or fixed beneath its centre, to point out its directions, and to be a check on the sights.

3. An index, which is a brass two-foot scale, with either a small telescope, or open sights set perpendicularly on the ends. These sights and one edge of the index are in the same plane, and that is called the fiducial edge of the index.

To use this instrument, take a sheet of paper which will cover it, and wet it to make it expand; then spread it flat on the table, pressing down the frame on the edges, to stretch it and keep it fixed there; and when the paper is become dry, it will, by contracting again, stretch itself smooth and flat from any cramps and unevenness. On this paper is to be drawn the plan or form of the ground measured.

Thus, begin at any proper part of the ground, and make a point on a convenient part of the paper or table, to represent that place on the ground; then fix in that point one leg of the compasses, or a fine steel pin, and apply to it the fiducial edge of the index, moving it round till through the sights you perceive some remarkable object, as the corner of a field, &c.; and from the station point draw a line with the point of the compasses along the fiducial edge of the index, which is called setting or taking the object: then set another object or corner, and draw its line; do the same by another; and so on, till as many objects are taken as may be thought fit. Then measure from the station towards as many of the objects as may be necessary, but not more, taking the requisite offsets to corners or crooks in the edges, laying the measures down on their respective lines on the table. Then at any convenient place measured to, fix the table in the same position, and set the objects which appear from that place; and so on, as before. Thus continue till the work is finished, measuring such lines only as are necessary, and determining as many as may be by intersecting lines of direction drawn from different stations.

Of shifting the paper on the plain table.

When one paper is full, and there is occasion for more, draw a line in any manner through the farthest point of the last station line, to which the work can be conveniently laid down; then take the sheet off the table, and fix another on, drawing a line over it, in a part most convenient for the rest of the work; then fold or cut the old sheet by the line drawn on it, applying the edge to the line on the new sheet, and as they lie in that position, continue the last station line on the new paper, placing on it the rest of the measure, beginning at where the old sheet left off. And so on from sheet to sheet.

When the work is done, and you would fasten all the sheets together into one piece, or rough plan, the aforesaid lines are to be accurately joined together, in the same manner as when the lines were transferred from the old sheets to the new ones. But it is to be noted, that if the said joining lines, on the old and new sheets, have not the same inclination to the side of the table, the needle will not point to the original degree when the table is rectified; and if the needle be required to respect still the same degree of the compass, the easiest way of drawing the line in the same position, is to draw them both parallel to the same sides of the table, by means of the equal divisions marked on the other two sides.

3. *Of the theodolite.* The theodolite is a brass circular ring, divided into 360 degrees, *etc.* and having an index with sights, or a telescope, placed on the centre, about which the index is moveable; also a compass fixed to the centre, to point out courses and check the sights; the whole being fixed by the centre on a stand of a convenient height for use *.

4. *Of the cross.* The cross consists of two pair of sights set at right angles to each other, on a staff having a sharp point at the bottom, to fix in the ground†.

PROBLEM I.

To measure a line or distance.

To measure a line on the ground with the chain: Having provided a chain, with ten small arrows or rods, to fix one into the ground, as a mark, at the end of every chain; two persons take hold of the chain, one at each end of it; and

---

* In using this instrument, an exact account, or field-book, of all measures and things necessary to be remarked in the plan, must be kept, from which to make out the plan on returning home from the ground.

Begin at such part of the ground, and measure in such directions as are judged most convenient: taking angles or directions to objects, and measuring such distances as appear necessary under the same restrictions as in the use of the plain table. And it is safest to fix the theodolite in the original position at every station, by means of fore and back objects, and the compass, exactly as in using the plain table; registering the number of degrees cut off by the index when directed to each object; and, at any station, placing the index at the same degree as when the direction towards that station was taken from the last preceding one, to fix the theodolite there in the original position.

The best method of laying down the aforesaid lines of direction, is to describe a pretty large circle; then quarter it, and lay on it the several numbers of degrees cut off by the index in each direction, and drawing lines from the centre to all these marked points in the circle. Then, by means of a parallel ruler, draw from station to station, lines parallel to the aforesaid lines drawn from the centre to the respective points in the circumference.

† The *cross* is very useful to measure small and crooked pieces of ground. The method is, to measure a base or chief line, usually in the longest direction of the piece, from corner to corner: and while measuring it, finding the places where perpendiculars would fall on this line, from the several corners and bends in the boundary of the piece, with the cross, by fixing it, by trials, on such parts of the line, as that through one pair of the sights both ends of the line may appear, and through the other pair the corresponding bends or corners; and then measuring the lengths of the said perpendiculars.

Remarks. Besides the fore-mentioned instruments, which are most commonly used, there are some others: as,

The *perambulator*, used for measuring roads, and other great distances, level ground, and by the sides of rivers. It has a wheel of $8\frac{1}{4}$ft, or half a pole, in circumference, by the turning of which the machine goes forward: and the distance measured is pointed out by an index, which is moved round by clock-work.

Levels, with telescopic or other sights, are used to find the level between place and place, or how much one place is higher or lower than another. And in measuring any sloping or oblique line, either ascending or descending, a small pocket level is useful for showing how many links for each chain are to be deducted, to reduce the line to the horizontal length.

An *offset staff* is a very useful instrument, for measuring the offsets and other short distances. It is 10 links in length, being divided and marked at each of the 10 links.

Ten *small arrows*, or rods of iron or wood, are used to mark the end of every chain length, in measuring lines. And sometimes pickets or staves with flags, are set up as marks or objects of direction.

Various *scales* are also used in protracting and measuring on the plan or paper; such as plane scales, line of chords, protractor, compasses, reducing scale, parallel and perpendicular rules, &c. Of plane scales, there should be several sizes, as a chain in 1 in, a chain in $\frac{3}{4}$ of an inch, a chain in $\frac{1}{2}$ an inch, &c. And of these, the best for use are those that are laid on the very edges of the ivory scale, to mark off distances without compasses.

all the 10 arrows are taken by one of them, who goes foremost, and is called the leader; the other being called the follower, for distinction's sake.

A picket, or station-staff, being set up in the direction of the line to be measured, if there do not appear some marks naturally in that direction, they measure straight towards it, the leader fixing down an arrow at the end of every chain, which the follower always takes up, as he comes at it, till all the ten arrows are used. They are then all returned to the leader, to use over again. And thus the arrows are changed from the one to the other at every 10 chains' length, till the whole line is finished; then the number of changes of the arrows shows the number of tens, to which the follower adds the arrows he holds in his hand, and the number of links of another chain over to the mark or end of the line. So, if there have been 3 changes of the arrows, and the follower hold 6 arrows, and the end of the line cut off 45 links more, the whole length of the line is set down in links thus, 3645.

When the ground is not level, but either ascending or descending; at every chain's length, lay the offset-staff, or link-staff, down in the slope of the chain, on which lay the small pocket level, to show how many links or parts the slope line is longer than the true level one; then draw the chain forward so many links or parts, which reduces the line to the horizontal direction.

PROBLEM II.

To take angles and bearings.

Let B and C be two objects, or two pickets set up perpendicular; and let it be required to take their bearings, or the angles formed between them at any station.

1. *With the plain table.*

The table being covered with a paper, and fixed on its stand; place it at the station A, and fix a fine pin, or a foot of the compasses, in a proper point of the paper, to represent the place A: close by the side of this pin lay the fiducial edge of the index, and turn it about, still touching the pin, till one object B can be seen through the sights: then by the fiducial edge of the index draw a line. In the same manner draw another line in the direction of the other object C; and it is done.

2. *With the theodolite.*

Direct the fixed sights along one of the lines, as AB, by turning the instrument about till the mark B is seen through these sights; and there screw the instrument fast. Then turn the moveable index round, till through its sights the other mark C is seen. Then the degrees cut by the index, on the graduated limb or ring of the instrument, show the quantity of the angle.

3. *With the magnetic needle and compass.*

Turn the instrument or compass so, that the north end of the needle point to the *fleur-de-lis*. Direct the sights to one mark as B, and note the degrees cut by the needle: direct the sights to the other mark C, and note again the degrees cut by the needle. Then, their sum or difference, as the case may be, will give the quantity of the angle BAC.

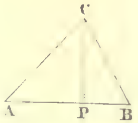
4. *By measurement with the chain.*

Measure one chain length, or any other length, along both directions, as to B and C. Then measure the distance BC, and it is done. This is easily trans-

ferred to paper, by making a triangle ABC with these three lengths, and then measuring the angle A.

PROBLEM III.

To survey a triangular field ABC.



1. *By the chain.* Having set up marks at the corners, which is to be done in all cases where there are not marks naturally; measure with the chain from A to P, where a perpendicular would fall from the angle C, and set up a mark at P, noting down the distance AP. Then complete the distance AB, by measuring from P to B. Having set down this measure, return to P, and measure the perpendicular PC. And thus, having the base and perpendicular, the area from them is easily found. Or, having the place P of the perpendicular, the triangle is easily constructed.

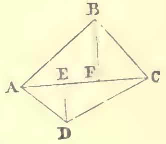
Or, when practicable, measure all the three sides with the chain, and note them down. From which the content is easily found, or the figure is constructed.

Ex. Suppose AP = 794, AB = 1321, and PC = 826, to find the area.

2. *By taking some of the angles.* Measure two sides AB, AC, and the angle A between them. Or measure one side AB, and the two adjacent angles A and B. From either of these ways the figure is easily planned; then by measuring the perpendicular CP on the plan, and multiplying it by half AB, the content is found.

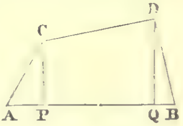
PROBLEM IV.

To measure a four-sided field.



1. *By the chain.* Measure along one of the diagonals, as AC; and either the two perpendiculars DE, BF, as in the last problem; or else the sides AB, BC, CD, DA. From either of these the figure may be planned and computed as before directed.

Ex. The following measures were taken, AE = 214, AF = 362, AC = 592, DE = 210, BF = 306.



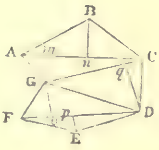
2. *Otherwise, by the chain.* Measure, on the longest side, the distances AP, AQ, AB; and the perpendiculars PC, QD. For example, AP = 110, AQ = 745, AB = 1110, and PC = 352, QD = 595.

3. *By taking some of the angles.* Measure the diagonal AC (see the last fig. but one), and the angles CAB, CAD, ACB, ACD. Or measure the four sides, and any one of the angles, as BAD.

Thus: AC=591, CAB=$37^\circ 20'$, CAD=$41^\circ 15'$, ACB=$72^\circ 25'$, ACD=$54^\circ 40'$.
Or thus: AB=486, BC=394, CD=410, DA=462, BAD=$78^\circ 35'$.

PROBLEM V.

To survey any field by the chain only.

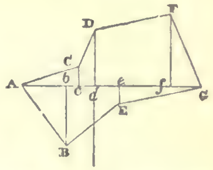


First method. Having set up marks at the corners, where necessary, of the proposed field ABCDEFG, walk over the ground, and consider how it can best be divided into triangles and trapeziums; and measure them separately, as in the last two problems. Thus, the following figure is divided into the two trapeziums

ABCG, GDEF, and the triangle GCD. Then, in the first trapezium, beginning at A, measure the diagonal AC, and the two perpendiculars Gm, Bn. Then the base GC, and the perpendicular Dq. Lastly, the diagonal DF, and the two perpendiculars pE, oG. All which measures write against the corresponding parts of a rough figure drawn to resemble the figure surveyed, or set them down in any other form you choose.

Thus: Am=135, An = 410, AC = 550; Cq = 152, CG = 440; Fo = 237, Fp = 288, FD = 520: mG = 130, nB = 180; qD = 230; oG = 120, pE = 80.

Or thus: Measure all the sides AB, BC, CD, DE, EF, FG, GA; and the diagonals AC, GD, GD, DF.



Second method. Many pieces of land may be very well surveyed, by measuring any base line, either within or without them, with the perpendiculars let fall on it from every corner. For they are by those means divided into several triangles and trapezoids, all whose parallel sides are perpendicular to the base line; and the sum of these triangles and trapeziums will be equal to the figure proposed if the base line fall within it; if not, the sum of the parts which are without, being taken from the sum of the whole which are both within and without, will leave the area of the figure proposed.

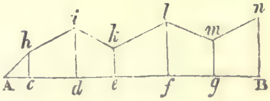
In pieces that are not very large, it will be sufficiently exact to find the points, in the base line, where the several perpendiculars will fall, by means of the *cross*, or even by judging by the eye only, and from thence measuring to the corners for the lengths of the perpendiculars. And it will be most convenient to draw the line so as that all the perpendiculars may fall within the figure.

Thus, in the annexed figure, beginning at A, and measuring along the line AG, the distances and perpendiculars on the right and left are as below.

Ab = 315, Ac = 440, Ad = 585, Ae = 610, Af = 990, AG = 1020, bB = 350, cC = 70, dD = 320, eE = 50, fF = 470, 0.

PROBLEM VI.

To measure the offsets.

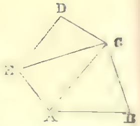


Let A*hiklmn* be a crooked hedge, a brook, or other irregular boundary. From A measure in a straight direction along the side of it to B. And in measuring along this line AB, observe when you are directly opposite any bends or corners of the boundary, as at c, d, e, ...; and from these measure the perpendicular offsets, *ch*, *di*, ..., with the offset-staff, if they are not very large, otherwise with the chain itself; and the work is done. The register, or field-book, may be as follows:—

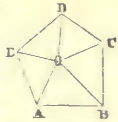
Offs. left.		Base line AB.	
	0	⊙	A
ch	62	45	A*c*
di	84	220	A*d*
ek	70	340	A*e*
fl	98	510	A*f*
gm	57	634	A*g*
B*n*	91	785	AB

PROBLEM VII.

To survey any field with the plain table.



1. *From one station.* Place the table at any angle, as C, from which all the other angles, or marks set up, can be seen; turn the table about till the needle point to the *fleur-de-lis;* and there screw it fast. Make a point for C on the paper on the table, and lay the edge of the index to C, turning it about C till through the sights you see the mark D; and by the edge of the index draw a dry or obscure line: then measure the distance CD, and lay that distance down on the line CD. Then turn the index about the point C, till the mark E be seen through the sights, by which draw a line, and measure the distance to E, laying it on the line from C to E. In like manner determine the positions of CA and CB, by turning the sights successively to A and B; and lay the lengths of those lines down. Then connect the points, by drawing the black lines CD, DE, EA, AB, BC, for the boundaries of the field.



2. *From a station within the field.* When all the other parts cannot be seen from one angle, choose some place O within, or even without, if more convenient, from which the other parts can be seen. Plant the table at O, then fix it with the needle north, and mark the point O on it. Apply the index successively to O, turning it round with the sights to each angle, A, B, C, D, E, drawing dry lines to them by the edge of the index; then measuring the distances OA, OB, &c. and laying them down on those lines. Lastly, draw the boundaries, AB, BC, CD, DE, EA.

3. *By going round the figure.* When the figure is a wood, or water, or when from some other obstruction you cannot measure lines across it; begin at any point A, and measure around it either within or without the figure, and draw the directions of all the sides thus: place the table at A; turn it with the needle to the north or *fleur-de-lis;* fix it, and mark the point A. Apply the index to A, turning it till you can see the point E, and there draw a line: then the point B, and there draw a line: then measure these lines, and lay them down from A to E and B. Next move the table to B, lay the index along the line AB, and turn the table about till you can see the mark A, and screw fast the table; in which position also the needle will again point to the *fleur-de-lis*, as it will do indeed at every station when the table is in the right position. Here turn the index about B till through the sights you see the mark C; there draw a line, measure BC, and lay the distance on that line after you have set down the table at C. Turn it then again into its proper position, and in like manner find the next line CD. And so on, quite around by E, to A again. Then the proof of the work will be the joining at A; for if the work be all right, the last direction EA on the ground, will pass exactly through the point A on the paper; and the measured distance will also reach exactly to A. If these do not coincide, or nearly so, some error has been committed, and the work must be examined over again.

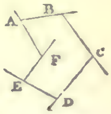
PROBLEM VIII.

To survey a field with the theodolite.

1. *From one point or station.* When all the angles can be seen from one point, as the angle C (*first fig. to last prob.*), place the instrument at C, and turn it about, till through the fixed sights you see the mark B, and there fix it. Then

urn the moveable index about till the mark A be seen through the sights, and ote the degrees cut on the instrument. Next turn the index successively to E nd D, noting the degrees cut off at each; which gives all the angles BCA, BCE, 3CD. Lastly, measure the lines CB, CA, CE, CD; and enter the measures in field-book, or rather, against the corresponding parts of a rough figure drawn y guess to resemble the field.

2. *From a point within or without.* Place the instrument at O (last fig.), and urn it about till the fixed sights point to any object, as A; and there screw it ast. Then turn the moveable index round till the sights point successively to he other points E, D, C, B, noting the degrees cut off at each of them; which ives all the angles round the point O. Lastly, measure the distances OA, OB,)C, OD, OE, noting them down as before, and the work is done.



3. *By going round the field.* By measuring round, either vithin or without the field, proceed thus. Having set up narks at B, C, near the corners as usual, plant the instrument t any point A, and turn it till the fixed index be in the direcion AB, and there screw it fast: then turn the moveable index o the direction AC; and the degrees cut off will be the angle A. Measure the ine AB, and plant the instrument at B, and there in the same manner observe he angle A. Then measure BC, and observe the angle C. Then measure the listance CD, and take the angle D. Then measure DE, and take the angle E. Then measure EF, and take the angle F. And lastly, measure the distance FA.

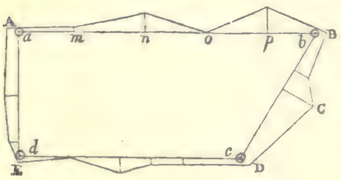
To prove the work; add all the inward angles, A, B, C, *etc.* together; for vhen the work is right, their sum will be equal to twice as many right angles as he figure has sides, wanting 4 right angles. But when there is an angle, as F, hat bends inwards, and you measure the external angle, which is less than two ight angles, subtract it from 4 right angles, or 360 degrees, to give the internal ngle greater than a semicircle or 180 degrees.

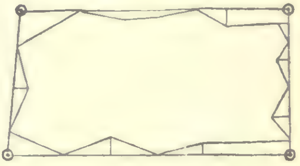
4. *Otherwise,* instead of observing the internal angles, we may take the exteral angles, formed without the figure by producing the sides farther out. And n this case, when the work is right, their sum altogether will be equal to 360 legrees. But when one of them, as F, runs inwards, subtract it from the sum of the rest, to leave 360 degrees.

PROBLEM IX.

To survey a field with crooked hedges, etc.

With any of the instruments, measure the lengths and positions of imaginary ines running as near the sides of the field as you can; and, in going along them, neasure the offsets in the manner before taught; then you will have the plan on he paper in using the plain table, drawing the crooked hedges through the ends of the offsets; but in surveying with the theodolite, or other instrument, set lown the measures properly in a field-book, or memorandum-book, and plan hem after returning from the field, by laying down all the lines and angles.





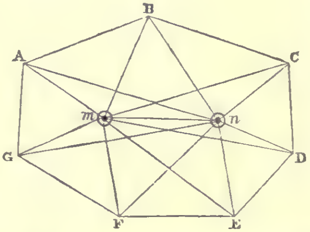
So, in surveying the piece ABCDE, set up marks, *a*, *b*, *c*, *d*, dividing it so as to have as few sides as may be. Then begin at any station, *a*, and measure the lines *ab*, *bc*, *cd*, *da*, taking their positions, or the angles, *a*, *b*, *c*, *d*; and, in going along the lines, measure all the offsets, as at *m*, *n*, *o*, *p*, &c. along every station-line.

And this is done either within the field, or without, as may be most convenient. When there are obstructions within, as wood, water, hills, or buildings, then measure without, as in the next following figure.

PROBLEM X.

To survey a field, or small estate, by two stations.

This is performed by choosing two stations from which all the marks and objects can be seen; then measuring the distance between the stations, and at each station taking the angles formed by every object from the station line or distance.



The two stations may be taken either within the bounds, or in one of the sides, or in the direction of two of the objects, or quite at a distance and without the bounds of the objects or part to be surveyed.

In this manner, not only grounds may be surveyed, without even entering them, but a map may be taken of the principal parts of a county, or the chief places of a town, or any part of a river or coast surveyed, or any other inaccessible objects; by taking two stations, on two towers, or two hills, or such-like.

PROBLEM XI.

To survey a large estate.

If the estate be very large, and contain a great number of fields, it cannot well be done by surveying all the fields singly, and then putting them together; nor can it be done by taking all the angles and boundaries that enclose it. For in these cases, any small errors will be so much increased, as to render it very much distorted. But proceed as below.

1. Walk over the estate two or three times, in order to get a perfect idea of it, or till you can keep the figure of it pretty well in mind. And to help your memory, draw an eye-draught of it on paper, at least of the principal parts of it, to guide you; setting the names within the fields in that draught.

2. Choose two or more eminent places in the estate, for stations, from which all the principal parts of it can be seen: selecting these stations as far distant from one another as convenient.

3. Take such angles, between the stations, as you think necessary, and measure the distances from station to station, always in a right line: these things must be done, till you get as many angles and lines as are sufficient for determining all the points of station. And in measuring any of these station-distances, mark accurately where these lines meet with any hedges, ditches, roads,

lanes, paths, rivulets, &c.; and where any remarkable object is placed, by measuring its distance from the station-line; and where a perpendicular from it cuts that line. And thus as you go along any main station-line, take offsets to the ends of all hedges, and to any pond, house, mill, bridge, &c. noting every thing down that is remarkable.

4. As to the inner parts of the estate, they must be determined, in like manner, by new station-lines; for, after the main stations are determined, and every thing adjoining to them, then the estate must be subdivided into two or three parts by new station-lines; taking inner stations at proper places, where you can have the best view. Measure these station-lines as you did the first, and all their intersections with hedges, and offsets to such objects as appear. Then proceed to survey the adjoining fields, by taking the angles that the sides make with the station-line, at the intersections, and measuring the distances to each corner, from the intersections. For the station-lines will be the bases to all the future operations; the situation of all parts being entirely dependent on them; and therefore they should be taken of as great length as possible; and it is best for them to run along some of the hedges or boundaries of one or more fields, or to pass through some of their angles. All things being determined for these stations, you must take more inner stations, and continue to divide and subdivide till at last you come to single fields: repeating the same work for the inner stations as for the outer ones, till all is done; and close the work as often as you can, and in as few lines as possible.

5. An estate may be so situated that the whole cannot be surveyed together; because one part of the estate cannot be seen from another. In this case, you may divide it into three or four parts, and survey the parts separately, as if they were lands belonging to different persons; and at last join them together.

6. As it is necessary to protract or lay down the work as you proceed in it, you must have a scale of a due length to do it by. To get such a scale, measure the whole length of the estate in chains; then consider how many inches long the map is to be; and from these will be known how many chains you must have in an inch; then make the scale accordingly, or choose one already made.

PROBLEM XII.

To survey a county, or large tract of land.

1. Choose two, three, or four eminent places, for stations; such as the tops of high hills or mountains, towers, or church steeples, which may be seen from one another; from which most of the towns and other places of note may also be seen; and so as to be as far distant from one another as possible. On these places raise beacons, or long poles, with flags of different colours flying at them, so as to be visible from all the other stations.

2. At all the places which you would set down in the map, plant long poles, with flags at them of several colours, to distinguish the places from one another; fixing them on the tops of church steeples, or the tops of houses; or in the centres of smaller towns and villages.

These marks then being set up at a convenient number of places, and such as may be seen from both stations; go to one of these stations, and, with an instrument to take angles, standing at that station, take all the angles between the other station and each of these marks. Then go to the other station, and take all the angles between the first station and each of the former marks, setting them down with the others, each against its fellow with the same colour. You

may, if convenient, also take the angles at some third station, which may serve to prove the work, if the three lines intersect in that point where any mark stands. The marks must stand till the observations are finished at both stations; and then they may be taken down, and set up at new places. The same operations must be performed at both stations, for these new places; and the like for others. The instrument for taking angles must be an exceeding good one, made on purpose with telescopic sights, and of a good length of radius.

3. And though it be not absolutely necessary to measure any distance, because, a stationary line being laid down from any scale, all the other lines will be proportional to it; yet it is better to measure some of the lines, to ascertain the distances of places in miles, and to know how many geometrical miles there are in any length; as also from thence to make a scale to measure any distance in miles. In measuring any distance, it will not be exact enough to go along the high roads; which, by reason of their turnings and windings, hardly ever lie in a right line between the stations; which must cause endless reductions, and require great trouble to make it a right line; for which reason it can never be exact. But a better way is to measure in a straight line with a chain, between station and station, over hills and dales, or level fields, and all obstacles. Only in case of water, woods, towns, rocks, banks, &c. where we cannot pass, such parts of the line must be measured by the methods of inaccessible distances; and besides, allowing for ascents and descents, when they are met with. A good compass, that shows the bearing of the two stations, will always direct us to go straight, when the two stations cannot be seen; and in the progress, if we can go straight, offsets may be taken to any remarkable places, likewise noting the intersection of the station-line with all roads, rivers, &c.

4. From all the stations, and in the whole progress, we must be very particular in observing sea-coasts, river-mouths, towns, castles, houses, churches, mills, trees, rocks, sands, roads, bridges, fords, ferries, woods, hills, mountains, rills, brooks, parks, beacons, sluices, floodgates, locks, *etc.*, and in general every thing that is remarkable.

5. After we have done with the first and main station-lines, which command the whole county; we must then take inner stations, at some places already determined; which will divide the whole into several partitions: and from these stations we must determine the places of as many of the remaining towns as we can. And if any remain in that part, we must take more stations, at some places already determined, from which we may determine the rest; and thus go through all the parts of the county, taking station after station, till we have determined the whole. And in general the station-distances must always pass through such remarkable points as have been determined before, by the former stations.

PROBLEM XIII.

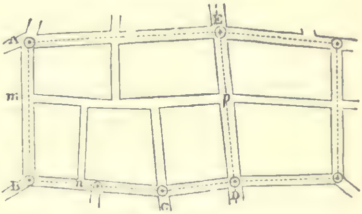
To survey a town or city.

This may be done with any of the instruments for taking angles, but best of all with the plain table, where every minute part is drawn while in sight. Instead of the common surveying, or Gunter's chain, it will be best, for this purpose, to have a chain 50 feet long, divided into 50 links of one foot each, and an offset-staff of 10 feet long.

Begin at the meeting of two or more of the principal streets, through which you can have the longest prospects, to get the longest station-lines: there having

fixed the instrument, draw lines of direction along those streets, using two men as marks, or poles set in wooden pedestals, or perhaps some remarkable places in the houses at the farther ends, as windows, doors, corners, &c. Measure these lines with the chain, taking offsets with the staff, at all corners of streets, bendings, or windings, and to all remarkable things, as churches, markets, halls, colleges, eminent houses, *etc.* Then remove the instrument to another station, along one of these lines; and there repeat the same process as before. And so on till the whole is finished.

Thus, fix the instrument at A, and draw lines in the direction of all the streets meeting there; then measure AB, noting the street on the left at *m*. At the second station, B, draw the directions of the streets meeting there; and measure from B to C, noting the places of the streets at *n* and *o* as you pass by them. At the third station, C, take the direction of all the streets meeting there, and measure CD. At D do the same, and measure DE, noting the place of the cross streets at *p*. And in the same manner go through all the principal streets. This done, proceed to the smaller and intermediate streets; and lastly to the lanes, alleys, courts, yards, and every part that it may be thought proper to represent in the plan.



PROBLEM XIV.

To lay down the plan of any survey.

If the survey was taken with the plain table, we have a rough plan of it already on the paper which covered the table. But if the survey was with any other instrument, a plan of it is to be drawn from the measures that were taken in the survey; and first of all a rough plan on paper.

To do this, you must have a set of proper instruments, for laying down both lines and angles, as scales of various sizes; scales of chords, protractors, perpendicular and parallel rulers, etc. Diagonal scales are best for the lines, because they extend to three figures, or chains, and links, which are 100 parts of chains. But in using the diagonal scale, a pair of compasses must be employed, to take off the lengths of the principal lines very accurately. But a scale with a thin edge divided is much readier for laying down the perpendicular offsets to crooked hedges, and for marking the places of those offsets on the station-line; which is done at only one application of the edge of the scale to that line, and then pricking off all at once the distances along it. Angles are to be laid down, either with a good scale of chords, which is perhaps the most accurate way, or with a large protractor, which is much readier when many angles are to be laid down at one point, as they are pricked off all at once round the edge of the protractor.

In general, all lines and angles must be laid down on the plan in the same order in which they were measured in the field, and in which they are written in the field-book; laying down first the angles for the position of lines, next the lengths of the lines, with the places of the offsets, and then the lengths of the offsets themselves, all with dry or obscure lines; then a black line drawn through the extremities of all the offsets will be the edge or bounding line of the field. After the principal bounds and lines are laid down, and made to fit or close

properly, proceed next to the smaller objects, till you have entered every thing that ought to appear in the plan, as, for instance, houses, brooks, trees, hills, gates, stiles, roads, lanes, mills, bridges, and woodlands.

The north side of a map or plan is commonly placed uppermost, and a meridian is somewhere drawn, with the compass or *fleur-de-lis* pointing north. Also, in a vacant part, a scale of equal parts or chains is drawn, with the title of the map in conspicuous characters, and embellished with a compartment. Hills are shadowed, to distinguish them in the map. Colour the hedges with different colours; represent hilly grounds by broken hills and valleys; draw single dotted lines for foot-paths, and double ones for horse or carriage-roads. Write the name of each field and remarkable place within it, and, if you choose, its contents in acres, roods, and perches.

In a very large estate, or a county, draw vertical and horizontal lines through the map, denoting the spaces between them by letters placed at the top, and bottom, and sides, for readily finding any field or other object mentioned in a table.

In mapping counties, and estates that have uneven grounds of hills and valleys, reduce all oblique lines, measured up-hill and down-hill to horizontal straight lines, if that was not done during the survey, before they were entered in the field-book, by making a proper allowance to shorten them; for which purpose there is commonly a small table engraven on some of the instruments for surveying.

PROBLEM XV.

To survey and plan by the new method.

In the former method of measuring a large estate, the accuracy depends both on the correctness of the instruments, and on the care in taking the angles. To avoid the errors incident to such a multitude of angles, other methods have of late years been used by a few skilful surveyors. The most practical, expeditious, and correct, seems to be the following, which is performed, without taking angles, by measuring with the chain only.

Choose two or more eminences as grand stations, and measure a principal base-line from one station to another; noting every hedge, brook, or other remarkable object, as you pass by it; measuring also such short perpendicular lines to the bends of hedges as may be near at hand. From the extremities of this base-line, or from any convenient parts of the same, go off with other lines to some remarkable object situated towards the sides of the estate, without regarding the angles they make with the base line or with one another; still remembering to note every hedge, brook, or other object that you pass by. These lines, when laid down by intersections, will, with the base-line, form a grand triangle on the estate; several of which, if need be, being thus measured and laid down, you may proceed to form other smaller triangles and trapezoïds on the sides of the former; and so on till you finish with the inclosures individually. By which means a kind of skeleton of the estate may first be obtained, and the chief lines serve as the bases of such triangles and trapezoïds as are necessary to fill up all the interior parts.

The field-book is ruled into three columns, as usual. In the middle one are set down the distances on the chain line, at which any mark, offset, or other observation, is made: and in the right and left-hand columns are entered the offsets and observations made on the right and left-hand respectively of the

chain-line; sketching on the sides the shape or resemblance of the fences or boundaries.

It is of great advantage, both for brevity and perspicuity, to begin at the bottom of the leaf, and write upwards; denoting the crossing of fences by lines drawn across the middle column, or only a part of such a line on the right and left opposite the figures, to avoid confusion; and the corners of fields, and other remarkable turns in the fences where offsets are taken to, by lines joining in the manner the fences do; as will be best seen by comparing the book with the plan annexed to the field-book, in four engraved pages, following p. 506.

The letter in the left-hand corner at the beginning of every line is the mark or place measured *from;* and that at the right-hand corner at the end is the mark measured *to.* But when it is not convenient to go exactly from a mark, the place measured from is described *such a distance* from *one mark* towards *another;* and where a former mark is not measured to, the exact place is ascertained by saying, turn to the right or left-hand, *such a distance to such a mark,* it being always understood that those distances are taken in the chain-line.

The characters used are, ⌈ for *turn to the right-hand;* ⌉ for *turn to the left-hand;* and ⌒ placed over an offset, to show that it is not taken at right angles with the chain-line, but in the direction of some straight fence; being chiefly used when crossing their directions: which is a better way of obtaining their true places than by offsets at right angles.

When a line is measured whose position is determined, either by former work (as in the case of producing a given line, or measuring from one known place or mark to another) or by itself (as in the third side of the triangle), it is called a *fast line,* and a double line across the book is drawn at the conclusion of it; but if its position is not determined (as in the second side of the triangle), it is called a *loose line,* and a single line is drawn across the book. When a line becomes determined in position, and is afterwards continued farther, a double line half through the book is drawn.

When a loose line is measured, it becomes absolutely necessary to measure some other line that will determine its position. Thus, the first line *ah* or *bh,* being the base of a triangle, is always determined; but the position of the second side *hj* does not become determined till the third side *jb* is measured; then the position of both is determined, and the triangle may be constructed.

At the beginning of a line, to fix a loose line to the mark or place measured from, the sign of turning to the right or left-hand must be added, as at *h* in the second, and *j* in the third line; otherwise a stranger, when laying down the work, may as easily construct the triangle *hjb* on the wrong side of the line *ah* as on the right one; but this error cannot be fallen into, if the sign above named be carefully observed.

In choosing a line to fix a loose one, care must be taken that it does not make a very acute or obtuse angle; as in the triangle *p*B*r*, by the angle at B being very obtuse, a small deviation from truth, even the breadth of a point at *p* or *r*, would make the error at B, when constructed, very considerable; but by constructing the triangle *p*B*q*, such a deviation is of no consequence.

Where the words *leave off* are written in the field-book, it signifies that the taking of offsets is from thence discontinued; and of course something is wanting between that and the next offset, to be afterwards determined by measuring some other line.

The field-book for this method, and the plan drawn from it, are contained in the four following pages, engraven on copper-plates; answerable to which the pupil is to draw a plan from the measures in the field-book, of a larger size, viz.

to a scale of a double size will be convenient, such a scale being also found on most instruments. In doing this, begin at the commencement of the field-book, or bottom of the first page, and draw the first line *ah* in any direction at pleasure, and then the next two sides of the first triangle *bhj* by sweeping intersected arcs; and so all the triangles in the same manner, after each other in their order; and afterwards setting the perpendicular and other offsets at their proper places, and through the ends of them drawing the bounding fences.

Note. That the field-book begins at the bottom of the first page, and reads up to the top; hence it goes to the bottom of the next page, and to the top; and thence it passes from the bottom of the third page to the top, which is the end of the field-book. The several marks measured to or from, are here denoted by the letters of the alphabet, first the small ones, *a*, *b*, *c*, *d*, . . . and after them the capitals, *A*, *B*, *C*, *D*, . . . But, instead of these letters, some surveyors use the numbers in order, 1, 2, 3, 4, &c.*

* In surveying with the plain table, a field-book is not used, as every thing is drawn on the table immediately when it is measured. But in surveying with the theodolite, or any other instrument, some kind of a field-book must be used, to write down in it a register or account of all that is done and occurs relative to the survey in hand.

This book every one contrives and rules as he thinks fittest for himself. The following is a specimen of a form which has been much used by country surveyors. It is ruled in three columns, as below.

Here ⊙ 1 is the first station, where the angle or bearing is 105° 25′. On the left, at 73 links in the distance or principal line, is an offset of 92; and at 610 an offset of 24 to a cross hedge. On the right, at 0, or the beginning, an offset 25 to the corner of the field; at 248 Brown's boundary hedge commences; at 610 an offset 35; and at 954, the end of the first line, the 0 denotes its terminating in the hedge. And so on for the other stations.

A line is drawn under the work, at the end of every station-line, to prevent confusion.

Form of this field-book.

Offsets and Remarks on the left.	Stations, Bearings and Distances.	Offsets and Remarks on the right.
	⊙ 1	
	105° 25′	
80	00	25 corner
92	73	
	248	Brown's hedge
a cross hedge 24	610	35
	954	00
	⊙ 2	
	53° 10′	
house corner 51	25	21
	120	29 a tree
34	734	40 a stile
	⊙ 3	
	67° 20′	
	61	35
a brook 30	248	
	639	16 a spring
footpath 16	810	
cross hedge 18	973	20 a pond

Then

PROBLEM XVI.

To compute the contents of fields.

1. Compute the contents of the figures as divided into triangles, or trapeziums, by the proper rules for these figures laid down in measuring, multiplying the perpendiculars by the diagonals or bases, both in links, and divide by 2; the quotient is acres, after having cut off five figures on the right for decimals. Then bring these decimals to roods and perches, by multiplying first by 4, and then by 40. An example of which is given in the description of the chain, p. 462.

2. In small and separate pieces, it is usual to compute their contents from the measures of the lines taken in surveying them, without making a correct plan of them.

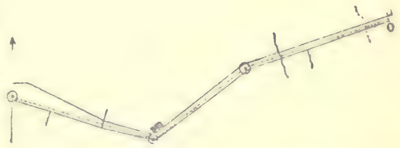
3. In pieces bounded by very crooked and winding hedges, measured by offsets, all the parts between the offsets are most accurately measured separately as small trapezoïds.

4. Sometimes such pieces as that last mentioned are computed by finding a mean breadth, by adding all the offsets together, and dividing the sum by the number of them, accounting that for one of them where the boundary meets the station-line (which increases the number of them by 1, for the divisor, though it does not increase the sum or quantity to be divided); then multiply the length by that mean breadth.

5. But in larger pieces and whole estates, consisting of many fields, it is the common practice to make a rough plan of the whole, and from it compute the contents, quite independent of the measures of the lines and angles that were taken in surveying. For then new lines are drawn in the fields on the plans, so as to divide them into trapeziums and triangles, the bases and perpendiculars of which are measured on the plan by means of the scale from which it was drawn, and so multiplied together for the contents. In this way the work is very expeditiously done, and sufficiently correct; for such dimensions are taken as afford the most easy method of calculation: and among a number of parts, thus taken and applied to a scale, though it be likely that some of the parts will be taken a small matter too little, and others too great, yet they will, on the whole, in all probability, very nearly balance one another, and give a sufficiently accurate

---

Then the plan, on a small scale, drawn from the above field-book, will be as in the following figure. But the pupil may draw a plan of 3 or 4 times the size on his paper book. The dotted lines denote the three measured lines, and the black lines the boundaries on the right and left.



But some skilful surveyors now make use of a different method for the field-book, namely, beginning at the bottom of the page and writing upwards; sketching also a neat boundary on either hand, resembling the parts near the measured lines as they pass along; an example of which was given in the new method of surveying, in the preceding pages.

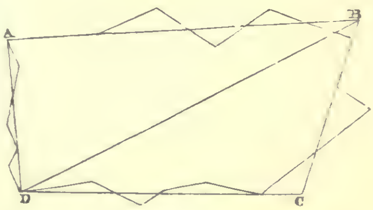
In smaller surveys and measurements, a good way of setting down the work is to draw by the eye, on a piece of paper, a figure resembling that which is to be measured: and so writing the dimensions, as they are found, against the corresponding parts of the figure. This method may, also, be practised to a considerable extent, even in the larger surveys.

result. After all the fields and particular parts are thus computed separately, and added all together into one sum; calculate the whole estate independent of the fields, by dividing it into large and arbitrary triangles and trapeziums, and add these also together. Then if this sum be equal to the former, or nearly so, the work is right; but if the sums have any considerable difference, it is wrong, and they must be examined, and recomputed, till they nearly agree.

6. But the chief art in computing consists in finding the contents of pieces bounded by curved or very irregular lines, or in reducing such crooked sides of fields or boundaries to straight lines, that shall enclose the same or equal area with those crooked sides, and so obtain the area of the curved figure by means of the right-lined one, which will commonly be a trapezium. Now this reducing the crooked sides to straight ones is very easy, and accurately performed in this manner :—apply the straight edge of a thin, clear piece of lantern-horn to the crooked line, which is to be reduced, in such a manner, that the small parts cut off from the crooked figure by it, may be equal to those which are taken in; which equality of the parts included and excluded you will presently be able to judge of very nicely by a little practice: then with a pencil, or point of a tracer, draw a line by the straight edge of the horn. Do the same by the other sides of the field or figure. So shall you have a straight-sided figure equal to the curved one; the content of which, being computed as before directed, will be the content of the crooked figure proposed.

Or, instead of the straight edge of the horn, a horse-hair, or fine thread, may be applied across the crooked sides in the same manner; and the easiest way of using the thread is to string a small slender bow with it, either of wire, or cane, or whalebone, or such like slender elastic matter; for the bow keeping it always stretched, it can be easily and neatly applied with one hand, while the other is at liberty to make two marks by the side of it, to draw the straight line by.

Example. Thus, let it be required to find the contents of the same figure as in prob. 9. p. 499, to a scale of 4 chains to an inch.



Draw the 4 dotted straight lines AB, BC, CD, DA, cutting off equal quantities on both sides of them, which they do as near as the eye can judge; so is the crooked figure reduced to an equivalent right-lined one of 4 sides, ABCD. Then draw the diagonal BD, which, by applying a proper scale to it, measures suppose 1256. Also the perpendicular, or nearest distance from A to this diagonal, measures 456; and the distance of C from it is 428.

Then, half the sum of 456 and 428, multiplied by the diagonal 1256, gives 555152 square links, or 5 acres, 2 roods, 8 perches, the content of the trapezium, or of the irregular crooked piece.

As a general example of this practice, let the contents be computed of all the fields separately in the foregoing plan facing page 506, and, by adding the contents altogether, the whole sum or content of the estate will be found nearly equal to $103\frac{1}{2}$ acres. Then, to prove the work, divide the whole plan into two parts, by a pencil-line drawn across it any way near the middle, as from the corner *l* on the right, to the corner near *s* on the left; then, by computing these two large parts separately, their sum must be nearly equal to the former sum, when the work is all right.

The content of irregular fields, farms, &c. when planned, may be readily and

correctly found by the process of *weighing* *. If the plan be not upon paper, or fine drawing pasteboard of uniform texture, let it be transferred upon such. Then cut the figure separately close upon its boundaries, and cut out from the same paper or pasteboard *a square* of known dimensions, according to the scale employed in drawing the plan. Weigh the two separately in an accurate balance, and the ratio of the weight will be the same as that of the superficial contents.

If great accuracy be required, cut the plan into four portions, called 1, 2, 3, 4. First, weigh 1 and 2 together, 3 and 4 together, and take their sum. Then weigh 1 and 3 together, 2 and 4 together, and take their sum. Lastly, weigh 1 and 4 together, 2 and 3 together, and take their sum. The mean of the three aggregate weights thus obtained, compared with the weight of the standard square, will give the ratio of their surfaces very nearly.

PROBLEM XVII.

To transfer a plan to another paper.

After the rough plan is completed, and a fair one is wanted; this may be done by any of the following methods.

First method. Lay the rough plan on the clean paper, keeping them always pressed flat and close together, by weights laid on them. Then, with the point of a fine pin or pricker, prick through all the corners of the plan to be copied. Take them asunder, and connect the pricked points on the clean paper, with lines, and it is done. This method is only to be practised in plans of such figures as are small and tolerably regular, or bounded by right lines.

Second method. Rub the back of the rough plan over with black-lead powder, and lay this blacked part on the clean paper on which the plan is to be copied, and in the proper position. Then, with the blunt point of some hard substance, as brass, or such-like, trace over the lines of the whole plan, pressing the tracer so much, as that the black-lead under the lines may be transferred to the clean paper; after which, take off the rough plan, and trace over the leaden marks with common ink, or with Indian ink. Or, instead of blacking the rough plan, we may keep constantly a blacked paper to lay between the plans.

Third method. This is by means of squares. This is performed by dividing both ends and sides of the plan which is to be copied into any convenient number of equal parts, and connecting the corresponding points of division with lines; which will divide the plan into a number of small squares. Then divide the paper on which the plan is to be copied into the same number of squares, each equal to the former when the plan is to be copied of the same size, but greater or less than the others, in the proportion in which the plan is to be increased or diminished, when of a different size. Lastly, copy into the clean squares the parts contained in the corresponding squares of the old plan; and you will have the copy, either of the same size, or greater or less in any proportion. See p. 399.

Fourth method. By the instrument called a pentagraph, which also copies the plan in any size required; for this purpose, also, Professor Wallace's eidograph may be advantageously employed.

Fifth method. A very neat process, at least for copying from a fair plan, is this: procure a copying frame of glass, made in this manner; namely, a large

---

* By a method like this, Dr. Long found the quantities of land and water on our globe to be very nearly as 2 to 5. He cut up the gores of a globe for the purpose.

square of the best plate-glass, set in a broad frame of wood, which can be raised up to any angle, when the lower side of it rests on a table. Set this frame up to any angle before you, facing a strong light; fix the old plan and clean paper together, with several pins quite around, to keep them together, the clean paper being laid uppermost, and over the face of the plan to be copied. Lay them, with the back of the old plan, on the glass; namely, that part which you intend to begin at to copy first; and by means of the light shining through the papers, you will very distinctly perceive every line of the plan through the clean paper. In this state, then, trace all the lines on the paper with a pencil. Having drawn that part which covers the glass, slide another part over the glass, and copy it in the same manner. Then another part; and so on, till the whole is copied. Then take them asunder, and trace all the pencil lines over with a fine pen and Indian ink, or with common ink. You may thus copy the finest plan, without injuring it.

ARTIFICERS' WORK AND TIMBER MEASURE.

Artificers compute their work in different ways: the chief distinctions of which are the following:—

1. *Glazing* and *masonry* by the foot square *.

2. *Painting*, *plaistering*, *paving*, and *paperhanging* by the yard square.

3. *Flooring*, *partitioning*, *roofing*, and *tiling* by the square of 100 feet, or a square whose side is 10 feet.

4. The *removal of earth*, as in forming roads and railways, the *purchase of stone*, and other works on which volume is concerned, the measures are either the cubic foot or the cubic yard.

All works, whether of superficial or solid measure, are computed by the rules proper to the figure of the magnitude concerned, and therefore come under one or other of the methods already explained for the mensuration of surfaces and solids. The only peculiarity of the operations as distinct from those already laid down, is the computation of the value of the work done or the materials supplied. The particular customary allowances to be made are detailed in the notes to the several kinds of work in which they occur.

I. BRICKLAYERS' WORK.

Brickwork is estimated at the rate of a brick and a half thick: but if a wall be more or less than this standard thickness, it must be reduced to it, as follows:—

Multiply the superficial content of the wall by the number of half-bricks in the thickness, and divide the product by 3 †.

---

* This is only the common mode of expressing the magnitude which has been previously designated as the "square foot."

† The dimensions of a building may be taken by measuring half round on the outside and half round on the inside: the sum of these two gives the compass of the wall, which, multiplied by the height, gives the content of the materials.

Chimneys are commonly measured as if they were solid, on account of the trouble, deducting

EXAMPLES.

1. How many rods of standard brickwork are contained in a wall whose length or compass is 57 ft 3 in, and height 24 ft 6 in; the wall being $2\frac{1}{2}$ bricks thick? Ans. 8 rods, $17\frac{3}{4}$ yards.

2. Required the content of a wall 62 ft 6 in long, 14 ft 8 in high, and $2\frac{1}{2}$ bricks thick? Ans. 169·753 yards.

3. A triangular gable is raised $17\frac{1}{2}$ ft high, on an end-wall whose length is 24 ft 9 in, the thickness being 2 bricks: required the content.
Ans. 32·08$\frac{1}{4}$ yards.

4. The end-wall of a house is 28 ft 10 in long, and 55 ft 8 in high, to the eaves; 20 ft high is $2\frac{1}{2}$ bricks thick, other 20 ft high is 2 bricks thick, and the remaining 15 ft 8 in, is $1\frac{1}{2}$ brick thick; above which is a triangular gable, of 1 brick thick, which rises 42 courses of bricks: what is the content in standard measure? Ans. 253·626 yards.

5. Required the number of bricks necessary to build a wall of $2\frac{1}{2}$ bricks thick, the superficial area being 2346 feet.

---

only the vacuity from the earth to the mantle. All windows, doors, etc. are to be deducted from the contents of the walls in which they are placed.

The dimensions of a common bare brick are, $8\frac{1}{2}$ inches long, 4 broad, and $2\frac{1}{2}$ thick; but, on account of the half-inch joint of mortar, when laid in brickwork, every dimension is to be counted half an inch more; thus making its length 9, its breadth $4\frac{1}{2}$, and thickness 3 inches. Hence, every 4 courses of brickwork measure 1ft in height.

450 stock bricks weigh about a ton, and 2 hods of mortar make nearly a bushel.

The standard rod requires 4500 bricks of the usual size, including waste.

1 rod of brickwork requires 27 bushels of chalk lime, and 3 loads of road drift or sand.

Taking 4500 for the bricks employed, including waste, in a standard rod of 272 feet face, and $1\frac{1}{2}$ brick thick; the following table will serve to determine the number of bricks required in any proposed case.

Area of the face of the wall, in feet.	Number of bricks required for 1, 2, 3, 4, ... feet at the respective thicknesses.				
	1 brick.	$1\frac{1}{2}$ brick.	2 bricks.	$2\frac{1}{2}$ bricks.	3 bricks.
1	11·02947	16·54412	22·05883	27·57353	33·08824
2	22·05883	33·08824	44·11765	55·14707	66·17648
3	33·08824	49·63236	66·17648	82·72060	99·26472
4	44.11765	66·17648	88·23531	110·29413	132·35296
5	55·14706	82·72060	110·29414	137·86766	165·44120
6	66·17648	99·26472	132·35296	165·44121	198·52944
7	77·20589	115·80884	154·41180	193·01473	231·61768
8	88·23531	132·35296	176·47062	220·58827	264·70592
9	99·26472	148·89708	198·52945	248·16180	297·79416

The left-hand column exhibiting the area of the face of a wall in feet, the numbers of bricks required for 1 brick thick, $1\frac{1}{2}$ brick thick, etc. are shown in the corresponding horizontal column under the appropriate heading. For greater numbers, being 10 times, 100 times, 1000 times, etc. the number of square feet specified in any part of the left-hand column, take 10 times, 100 times, 1000 times, etc. the number given under the proper head.

Thus, 5 sq. ft of 2 bricks thick will require 110·294 bricks; 50, 1102·940; 500, 11029·400; and so on.

For much valuable and really practical information, on artificer's work, the reader may refer to *Maynard's edition of Hutton's Mensuration.*

For 2000 take	1000 times the number		for 2	55147·07
300 -	100 times	- -	for 3	8272·06
40 -	10 times	- -	for 4	1102·94
6 -	- -	- -	for 6	165·44
	Total number of bricks required		-	64687·51.

II. MASONS' WORK.

To masonry belong all sorts of stone-work; and the measure made use of is a foot, either superficial or solid *.

EXAMPLES.

1. Required the content of a wall, 53 ft 6 in long, 12 ft 3 in high, and 2 ft thick. Ans. $1310\frac{3}{4}$ ft.

2. What is the content of a wall, the length being 24 ft 3 in, height 10 ft 9 in, and 2 ft thick? Ans. 521·375 ft.

3. Required the value of a marble slab, at 8*s* per ft; the length being 5 ft 7 in, and breadth 1 ft 10 in. Ans. 4*l* 1*s* $10\frac{1}{2}$*d*.

4. In a chimney-piece, suppose the length of the mantle and slab, each 4 ft 6 in, the breadth of both together 3 ft 2 in, the length of each jamb 4 ft 4 in, and the breadth of both together 1 ft 9 in. Required the superficial content? Ans. 21 ft 10 in.

III. CARPENTERS' AND JOINERS' WORK.

To this branch belongs all the wood-work of a house, such as flooring, partitioning, roofing, etc. †.

---

* Walls, columns, blocks of stone or marble, etc. are measured by the cubic foot; and pavements, slabs, chimney-pieces, etc. by the superficial or square foot.

Cubic or solid measure is used for the materials, and square measure for the workmanship.

In the solid measure, the true length, breadth, and thickness are measured, and multiplied together. In the superficial, the length and breadth are taken of every part of the projection which is seen without the general upright face of the building.

A ton of Portland stone is about 16 cubic feet; of Bath stone, 17; of granite, $13\frac{1}{2}$; of marble, at a medium, 13 cubic feet.

† Large and plain articles are usually measured by the square foot or yard, etc.; but enriched mouldings, and some other articles, are often estimated by running or lineal measure; and some things are rated by the piece.

In measuring of joists, take the dimensions of one joist, considering that each end is let into the wall about $\frac{2}{3}$ of the thickness, and multiply its content by the number of them.

Partitions are measured from wall to wall for one dimension, and from floor to floor, as far as they extend, for the other.

The measure of centering for cellars is found by making a string pass over the surface of the arch for the breadth, and taking the length of the cellar for the length: but in groin centering, it is usual to allow double measure, on account of their greater trouble.

In

EXAMPLES.

1. Required the content of a floor, 48ft 6in long, and 24ft 3in broad?
Ans. 11 sq. $76\frac{1}{8}$ ft.

2. A floor being 36ft 3in long, and 16ft 6in broad, how many squares are in it? Ans. 5sq $98\frac{1}{8}$ft.

3. How many squares of partitioning are there in 173ft 10in in length, and 10ft 7in in height? Ans. 18·3973 sq.

4. What was the cost of roofing a house at 10*s* 6*d* a square; the length within the walls being 52ft 8in, and the breadth 30ft 6in; reckoning the roof $\frac{3}{2}$ of the flat? Ans. 12*l* 12*s* $11\frac{3}{4}$*d*.

5. Required the cost, at 6*s* per square yard, of the wainscoting of a room; the height, including the cornice and mouldings, being 12ft 6in, and the whole compass 83ft 8in; also the three window-shutters being each 7ft 8in by 3ft 6in, and the door 7ft by 3ft 6in, which being worked on both sides must be reckoned work and half work. Ans. 36*l* 12*s* $2\frac{1}{2}$*d*.

IV. SLATERS' AND TILERS' WORK.

In this work, the content of a roof is found by multiplying the length of the ridge by the girt from eaves to eaves; making allowance in this girt for the double row of slates at the bottom, or for how much one row of slates or tiles is laid over another*.

In roofing, the dimensions, as to length, breadth, and depth, are taken as in flooring joists, and the contents computed the same way.

In floor-boarding, multiply the length by the breadth of the room.

For stair-cases, take the breadth of all the steps, by making a line ply close over them, from the top to the bottom; and multiply the length of this line by the length of a step, for the whole area. By the length of a step is meant the length of the front and the returns at the two ends; and by the breadth is to be understood the girts of its two outer surfaces, or the tread and riser.

For the balustrade, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel-post, for the one dimension; and twice the length of the baluster on the landing, with the girt of the hand-rail, for the other dimension.

For wainscoting, take the compass of the room for the one dimension; and the height from the floor to the ceiling; making the string ply close into all the mouldings, for the other.

For doors, multiply the height into the breadth, for the area. If the door be panneled on both sides, take double its measure for the workmanship; but if one side only be panneled, take the area and its half for the workmanship. *For the surrounding architrave*, girt it about the uppermost part for its length; and measure over it, as far as it can be seen when the door is open, for the breadth. *Window-shutters*, *bases*, *etc.* are measured in like manner.

In measuring of joiners' work, the string is made to ply close into all mouldings, and to every part of the work over which it passes.

Note. 64 cubic feet of fir, 60 of elm, 45 of ash, 39 of oak, make each a ton, at a medium.

Battens are 7 inches, deals 9, and planks 11 inches wide.

* When the roof is of a true pitch, that is, forming a right angle at the top; then the breadth of the building, with its half added, is the girt over both sides nearly.

In angles formed in a roof, running from the ridge to the eaves, when the angle bends inwards, it is called a valley; but when outwards, it is called a hip.

Deductions are made for chimney-shafts or window-holes.

6 inch gage, 1 square requires 760 plain tiles
7 .... 1 .... 660 ....
8 .... 1 .... 576 ....

Five

EXAMPLES.

1. Required the content of a slated roof, the length being 45ft 9in, and the whole girt 34ft 3in. Ans. $174\frac{5}{48}$yds.

2. To how much amounts the tiling of a house, at 25*s* 6*d* per square; the length being 43ft 10in, and the breadth on the flat 27ft 5in; also the eaves projecting 16in on each side, and the roof of true pitch? Ans. 24*l* 9*s* $8\frac{1}{4}$*d*.

V. PLASTERERS' WORK.

Plasterers' work is of two kinds, which are measured separately; namely, ceiling, which is plastering on laths; and rendering, which is plastering on walls *.

EXAMPLES.

1. Find the content of a ceiling which is 43ft 3in long, and 25ft 6in broad. Ans. $122\frac{1}{2}$ yds.

2. Required the cost of the ceiling of a room at 10*d* per yd; the length being 21ft 8in, and the breadth 14ft 10in. Ans. 1*l* 9*s* $8\frac{3}{4}$*d*.

3. The length of a room is 18ft 6in, the breadth 12ft 3in, and height 10ft 6in; what is the amount of ceiling and rendering, the former at 8*d* and the latter at 3*d* per yd: allowing for the door of 7ft by 3ft 8in, and a fire-place of 5ft square? Ans. 1*l* 13*s* $3\frac{1}{2}$*d*.

4. Required the quantity of plastering in a room, the length being 14ft 5in, breadth 13ft 2in, and height 9ft 3in to the under side of the cornice, which girts $8\frac{1}{2}$in, and projects 5in from the wall on the upper part next the ceiling; deducting only for a door 7ft by 4.
Ans. 53yds 5ft $3\frac{1}{2}$in of rendering, 18yds 5ft 6in of ceiling, and 39ft $0\frac{1}{12}$in of cornice.

VI. PAINTERS' WORK.

Painters' work is computed in square yards. Every part is measured where the colour lies; and the measuring line is forced into all the mouldings and corners.

Windows are painted at so much a piece: and it is usual to allow double measure for carved mouldings and other ornamental works.

Five hundred feet in length of laths make a bundle; and is the quantity usually allowed to a square of tiling.

A square of Westmoreland slates will weigh half a ton; of Welsh rag from $\frac{3}{4}$ of a ton to a ton: and a square of pantiling weighs about $7\frac{1}{2}$ cwt.

* The contents are estimated either by the foot or the yard, or the square of 100 feet. Enriched mouldings, etc. are rated by running or lineal measure.

Deductions are made for chimneys, doors, windows, and other apertures.

3 cwts. of lime, 4 loads of sand, and 10 bushels of hair, are allowed to 200 yards of rendering.

1 bundle of laths, and 500 of nails, are allowed to cover $4\frac{1}{2}$ square yards.

1 barrel of cement is 5 bushels, and weighs 3 cwt. 1 rod of brickwork in cement requires 36 bushels of cement and 36 bushels of sand.

EXAMPLES.

1. How many yards of painting are there in a room which is 65ft 6in in compass, and 12ft 4in high? Ans. $89\frac{41}{54}$yds.

2. The length of a room being 20ft, its breadth 14ft 6in, and height 10ft 4in: how many yards of painting are there, deducting a fire-place of 4ft by 4ft 4in, and two windows, each 6ft by 3ft 2in? Ans. $73\frac{2}{27}$yds.

3. Required the cost of painting a room of the following dimensions at 6*d* a yd: viz. the length 24ft 6in, the breadth 16ft 3in, and the height 12ft 9in; the door 7ft by 3ft 6in, and the fire-place 5ft by 5ft 6in; also the shutters to the two windows each 7ft 9in by 3ft 6in, the breaks of the windows 8ft 6in high by 1ft 3in deep, and the window-cills and soffits determinable from the dimensions already given. Ans. 3*l* 3*s* $10\frac{3}{4}$*d*.

VII. GLAZIERS' WORK.

Glaziers take their dimensions, either in feet, inches, and parts, or feet, tenths, and hundredths; and they compute their work in square feet *.

EXAMPLES.

1. How many square feet are there in the window which is 4·25ft long, and 2·75ft broad? Ans. $11\frac{2}{3}$ft.

2. What will the glazing a triangular sky-light cost at 10*d* per foot; the base being 12ft 6in, and the height 6ft 9in? Ans. 1*l* 15*s* $1\frac{3}{4}$*d*.

3. There is a house with three tiers of windows, three windows in each tier, their common breadth 3ft 11in: and their height are 7ft 10in, 6ft 8in, and 5ft 4in respectively. Required the expense of glazing at 14*d* per foot. Ans. 13*l* 11*s* $10\frac{1}{2}$*d*.

4. Required the expense of glazing the windows of a house at 13*d* a foot; there being three stories, and three windows in each story; the heights of which are respectively 7ft 9in, 6ft 6in, and 5ft $3\frac{1}{4}$in, and of an oval window over the door 1ft $10\frac{1}{2}$in: also the common breadth of all the windows 3ft 9in. Ans. 12*l* 5*s* 6*d*.

VIII. PAVERS' WORK.

Pavers' work is done by the square yard: and the content is found by multiplying the length by the breadth.

EXAMPLES.

1. What cost the paving a foot-path, at 3*s* 4*d* a yard; the length being 35ft 4in, and breadth 8ft 3in? Ans. 5*l* 7*s* $11\frac{1}{2}$*d*.

---

* In taking the length and breadth of a window, the cross bars between the squares are included. Windows also of round or oval forms are measured as square, measuring them to their greatest length and breadth, on account of the waste in cutting the glass.

2. What was the expense of paving a court, at 3*s* 2*d* per yd; the length being 27ft 10in, and the breadth 14ft 9in? Ans. 7*l* 4*s* $5\frac{1}{4}$*d*.

3. What will be the expense of paving a rectangular court-yard, whose length is 63ft, and breadth 45ft; in which there is laid a foot-path of 5ft 3in broad, running the whole length, with broad stones, at 3*s* a yd; the rest being paved with pebbles at 2*s* 6*d* a yd? Ans. 40*l* 5*s* $10\frac{1}{2}$*d*.

IX. PLUMBERS' WORK.

PLUMBERS' work is rated at so much a pound; or else by the hundred weight of 112 pounds*.

EXAMPLES.

1. Required the weight of the lead which is 39ft 6in long, and 3ft 3in broad, at $8\frac{1}{2}$lbs to the square foot. Ans. $1091\frac{3}{16}$lbs.

2. Find the cost of covering and guttering a roof with lead, at 18*s* per cwt; the length of the roof being 43ft, and breadth, or girt over it, 32ft; the guttering 57ft long, and 2ft wide; the former 9·831lb, and the latter 7·373lb to the square foot. Ans. 115*l* 9*s* $1\frac{1}{2}$*d*.

X. TIMBER MEASURING.

PROBLEM I.

To find the area, or superficial content, of a board or plank.

MULTIPLY the length by the mean breadth, when the breadths of each end are equal; but when the board is tapering, add the breadths at the two ends together, and take half the sum for the mean breadth; or, if convenient, take the breadth in the middle.

By the sliding rule†.

Set 12 on B to the breadth in inches on A; then against the length in feet on B, is the content on A, in feet and fractional parts.

---

* Sheet lead, used in roofing, guttering, &c. weighs from 6lb. to 10lb. to the square foot; and pipe of an inch bore is commonly 13 or 14lb. to the yard in length.

A square foot an eighth of an inch thick, weighs 7·38 or $7\frac{3}{8}$lb. nearly; a quarter of an inch thick $14\frac{3}{4}$lb., and so on.

† The Carpenter's or Sliding Rule is an instrument much used in measuring of timber and artificers' works, both for taking the dimensions, and computing the contents.

The instrument consists of two equal pieces, each a foot in length, which are connected together by a folding joint.

One side or face of the rule is divided into inches, and eighths, or half-quarters. On the same face also are several plane scales divided into twelfth parts by diagonal lines; which are used in planning dimensions that are taken in feet and inches. The edge of the rule is commonly divided decimally, or into tenths; namely, each foot into ten equal parts again; so that by means of this last scale, dimensions are taken in feet, tenths, and hundredths, and multiplied as common decimal numbers, which is the best way. On

EXAMPLES.

1. What is the value of a plank, at $1\frac{1}{2}d$ per foot, whose length is 12ft 6in, and mean breadth 11in? Ans. 1*s* 5*d*.

2. Required the content of a board, whose length is 11ft 2in, and breadth 1ft 10in. Ans. 20ft $5\frac{3}{4}$in.

3. What is the value of a plank, which is 12ft 9in long, and 1ft 3in broad, at $2\frac{1}{2}d$ a ft? Ans. 3*s* $3\frac{3}{4}d$.

4. Required the value of 5 oaken planks, at 3*d* per ft, each of them being $17\frac{1}{2}$ft long; and their several breadths as follows, namely, two of $13\frac{1}{2}$in in the middle, one of $14\frac{1}{2}$in in the middle, and the two remaining ones, each 18in at the broader end, and $11\frac{1}{4}$in at the narrower? Ans. 1*l* 5*s* $9\frac{1}{2}d$.

PROBLEM II.

To find the solid content of squared or four-sided timber.

Multiply the mean breadth by the mean thickness, and the product again by the length, for the content nearly.

By the sliding rule.

C D D C

As length : 12 or 10 : : quarter girt : solidity.

That is, as the length in feet on C, is to 12 on D, when the quarter girt is in inches, or to 10 on D, when it is in tenths of feet; so is the quarter girt on D, to the content on C.

If the tree taper regularly from the one end to the other; either take the mean breadth and thickness in the middle, or take the dimensions at the two ends, and half their sum will be the mean dimensions: which multiplied as above, will give the content nearly.

If the piece do not taper regularly, but be unequally thick in some parts and small in others; take several different dimensions, add them all together, and divide their sum by the number of them, for the mean dimensions.

EXAMPLES.

1. The length of a piece of timber is 18ft 6in, the breadths at the greater and less end 1ft 6in and 1ft 3in, and the thickness at the greater and less end 1ft 3in and 1ft; required the content. Ans. 28ft 7in.

On the one part of the other face are four lines, marked A, B, C, D; the two middle ones B and C being on a slider, which runs in a groove made in the stock. The same numbers serve for both these two middle lines, the one being above the numbers, and the other below.

These four lines are logarithmic ones, and the three A, B, C, which are all equal to one another, are double lines, as they proceed twice over from 1 to 10. The other or lowest line, D, is a single one, proceeding from 4 to 40. It is also called the girt-line, from its use in computing the contents of trees and timber; and on it are marked WG at 17·15, and AG at 18·95, the wine and ale gage points, to make this instrument serve the purpose of a gaging rule.

On the other part of this face there is a table of the value of a load, or 50 cubic feet of timber, at all prices, from 6 pence to 2 shillings a foot.

When 1 at the beginning of any line is accounted 1, then the 1 in the middle will be 10, and the 10 at the end 100; but when 1 at the beginning is counted 10, then the 1 in the middle is 100, and the 10 at the end 1000; and so on. And all the smaller divisions are altered proportionally.

2. What is the content of the piece of timber, whose length is $24\frac{1}{2}$ft, and the mean breadth and thickness each 1·04ft? Ans. $26\frac{1}{2}$ft.

3. Required the content of a piece of timber, whose length is 20·38ft, and its ends unequal squares, the side of the greater being $19\frac{1}{8}$in, and the side of the less $9\frac{7}{8}$in. Ans. 29·7562ft.

4. Required the content of the piece of timber, whose length is 27·36ft; at the greater end the breadth is 1·78ft and thickness 1·23ft; and at the less end the breadth is 1·04ft, and thickness 0·91ft. Ans. 41·278ft.

PROBLEM III.

To find the solidity of round or unsquared timber.

Multiply the square of the quarter girt, or of $\frac{1}{4}$ of the mean circumference by the length, for the content.

By the sliding rule.

As the length upon C : 12 or 10 upon D :: quarter girt in 12ths or 10ths, on D : content on C.

1. When the tree is tapering, take the mean dimensions as in the former problems, either by girting it in the middle, for the mean girt, or at the two ends, and taking half the sum of the two; or by girting it in several places, then adding all the girts together, and dividing the sum by the number of them, for the mean girt: but when the tree is very irregular, divide it into several lengths, and find the content of each part separately.

2. This rule, which is commonly used, gives the answer about $\frac{1}{4}$ less than the true quantity in the tree, or nearly what the quantity would be, after the tree is hewed square in the usual way; so that it seems intended to make an allowance for the squaring of the tree.

On this subject, however, *Hutton's Mensuration, pt. v. sect.* 4, may be advantageously consulted.

EXAMPLES.

1. A piece of round timber being 9ft 6in long, and its mean quarter girt 42in; what is the content? Ans. $116\frac{3}{8}$ft.

2. The length of a tree is 24ft, its girt at the thicker end 14ft, and at the smaller end 2ft; required the content. Ans. 96ft.

3. What is the content of a tree whose mean girt is 3·15ft, and length 14ft 6in? Ans. 8·9922ft.

4. Required the content of a tree, whose length is $17\frac{1}{4}$ft, and which girts in five different places as follows, namely, in the first place 9·43ft, in the second 7·92ft, in the third 6·15ft, in the fourth 4·74ft, and in the fifth 3·16ft.

Ans. 42·519525.

PRACTICAL EXERCISES IN MENSURATION.

1. What difference is there between a floor 28ft long by 20 broad, and two others, each of half the dimensions: and what do all three come to at 45*s* per square of 100ft? Ans. dif. 280ft; amount 18 guineas.

2. An elm plank is 14ft 3in long, and I would have just a square yard slit off it; at what distance from the edge must the line be struck? Ans. $7\frac{11}{19}$in.

3. A ceiling contains 114yds 6ft of plastering, and the room is 28ft broad; what is the length of it? Ans. $36\frac{6}{7}$ft.

4. A common joist is 7in deep, and $2\frac{1}{2}$in thick; but I want a scantling just as big again, that shall be 3in thick; what will the other dimension be?
Ans. $11\frac{2}{3}$in.

5. A wooden trough, length 102in, and depth 21in, cost me 3*s* 2*d* painting, within, at 6*d* per yd: what was the width? Ans. $27\frac{1}{4}$in.

6. If my court-yard be 47ft 9in square, and I have laid a foot-path with Purbeck-stone, of 4ft wide, along one side of it; what will paving the rest with flints come to at 6*d* per square yd? Ans. 5*l* 16*s* $0\frac{1}{2}$*d*.

7. A ladder, 36ft long, may be so placed, that it shall reach a window 30·7ft from the ground on one side of the street; and, by only turning it over, without moving the foot out of its place, it will do the same by a window 18·9ft high on the other side; what is the breadth of the street, and the angle of elevation of the second window from the first?

Ans. the street is 49·4414ft wide; and the elevation is 13° 25′ 24″.

8. The paving of a triangular court, at 18*d* per ft, came to 100*l*; the longest of the three sides was 88ft; required the sum of the other two equal sides?
Ans. 106·85ft.

9. The perambulator, or surveying-wheel, is so contrived, as to turn twice in the length of a pole, or $16\frac{1}{2}$ft; required the diameter. Ans. 2·626ft.

10. In turning a one-horse chaise within a ring of a certain diameter, it was observed, that the outer wheel made two turns, while the inner made but one: the wheels were both 4ft high; and, supposing them fixed at the statutable distance of 5ft asunder on the axle-tree, what was the circumference of the track described by the outer wheel? Ans. 62·832ft.

11. What is the side of that equilateral triangle, whose area cost as much paving at 8*d* a ft, as the palisading the three sides did at a guinea a yd?
Ans. 72·746ft.

12. A roof, which is 24ft 8in by 14ft 6in, is to be covered with lead at 8lb per square ft; find the price at 18*s* per cwt. Ans. 22*l* 19*s* $10\frac{1}{2}$*d*.

13. Having a rectangular marble slab, 58in by 27, I would have a square foot cut off parallel to the shorter edge; I would then have the like quantity divided from the remainder parallel to the longer side; and this alternately repeated, till there shall not be the quantity of a foot left; what will be the dimensions of the remaining piece *? Ans. 20·7in by 6·086.

14. Given two sides of an obtuse-angled triangle, which are 20 and 40 poles; required the third side, that the triangle may contain just an acre of land?
Ans. 58·876 or 23·099.

15. How many bricks will it take to build a wall, 10ft high, and 500ft long, of a brick and half thick, reckoning the brick 10 inches long, and four courses to the foot in height? Ans. 72000.

16. How many bricks will build a square pyramid of 100ft on each side at the base, and also 100ft perpendicular height, the dimensions of a brick being supposed 10in long, 5in broad, and 3in thick? Ans. 3840000.

17. If, from a right-angled triangle, whose base is 12, and perpendicular 16ft,

---

* This question may be solved neatly by an algebraical process, as may be seen in the Ladies' Diary for 1823. In applying the formulæ there found, the term to stop at is that whose *ordinal number* is the number of *entire feet* in the slab: which in the present case is 10, since 58.27 = $10\frac{7}{8}$ft.

a line be drawn parallel to the perpendicular, cutting off a triangle whose area is 24ft; required the sides of this triangle. Ans. 6, 8, and 10.

18. If a round pillar, 7in across, have 4ft of stone in it; of what diameter is the column, of equal length, that contains 10 times as much? Ans. 22·136in.

19. A circular fish-pond is to be made in a garden, that shall take up half an acre; required the length of the cord that strikes the circle? Ans. $27\frac{3}{4}$yds.

20. When a roof is of a true pitch, the rafters are $\frac{3}{4}$ of the breadth of the building. Now supposing the eaves-boards to project 10in on a side, what will the new ripping a house cost, that measures 32ft 9in long, by 22ft 9in broad on the flat, at 15*s* per square? Ans. 8*l* 15*s* $9\frac{1}{2}$*d*.

21. A cable, which is 3ft long and 9in in compass, weighs 22lb; required the weight of a fathom of a cable which measures a foot round? Ans. $78\frac{2}{9}$lb.

22. A plumber has put 28lb per square foot into a cistern, 74in and twice the thickness of the lead long, 26in broad, and 40 deep; he has also put three stays across it within, 16in deep, of the same strength, and reckons 22*s* per cwt for work and materials: a mason has in return paved him a workshop, 22ft 10in broad, with Purbeck-stone, at 7*d* per ft; and upon the balance finds there is 3*s* 6*d* due to the plumber: what was the length of the workshop, supposing sheet lead $\frac{1}{10}$ of an inch thick to weigh 5·899lb per ft? Ans. 32·2825ft.

23. The girt or outside circumference of a vessel is 44in, the hoop is 1in thick, and the height of the vessel is 24in; required its content in imperial gallons. Ans. 9·7892 gallons.

24. If 20ft of iron railing weigh half a ton, when the bars are an inch and quarter square; what will 50ft come to at $3\frac{1}{2}$*d* per lb, the bars being but $\frac{7}{8}$ of an inch square? Ans. 20*l* 0*s* 2*d*.

25. It is required to find the thickness of the lead in a pipe, of an inch and quarter bore, which weighs 14lb per yd in length; the cubic foot of lead weighing 11325 ounces? Ans. ·20737in.

26. Supposing the expense of paving a semicircular plot, at 2*s* 4*d* per ft, come to 10*l*; what is its diameter? Ans. 14·7737ft.

27. What is the length of a chord which cuts off one-third of the area from a circle whose diameter is 289? Ans. 278·6716.

28. My plumber has set me up a cistern, and, his shopbook being burnt, he has no means of bringing in the charge, and I do not wish to take it down to have it weighed; but by measure he finds it contains $64\frac{3}{10}$ square feet, and that it is precisely $\frac{1}{8}$ of an inch in thickness. If lead was then wrought at 2*l* per fother of $19\frac{1}{2}$cwt., can we from these items make out the bill, allowing $6\frac{2}{9}$ oz for the weight of a cubic inch of lead? Ans. 4*l* 11*s* 2*d*.

29. What will the diameter of a globe be, when the solidity and superficial content are expressed by the same number? Ans. 6.

30. A sack, that would hold 3 bushels of corn, is $22\frac{1}{2}$in broad when empty; what will another sack contain, which, being of the same length, has twice its breadth? Ans. 12 bushels.

31. A carpenter is to put an oaken curb to a round well, at 8*d* per foot square; the breadth of the curb is to be 8in, and the diameter within $3\frac{1}{2}$ft: what will be the expense? Ans. 5*s* $9\frac{3}{4}$*d*.

32. A gentleman has a garden 100ft long, and 80ft broad; and a gravel walk is to be made of an equal width half round it: determine both by *construction* and *calculation* the breadth of the walk, to take up just half the ground.

Ans. 25·968ft.

33. The top of a may-pole, being broken off by a blast of wind, struck the ground at 15ft from the foot of the pole; what was the height of the whole may-pole, supposing the length of the broken piece to be 39ft? Ans. 75ft.

34. Seven men bought a grinding-stone of 60in diameter, each paying $\frac{1}{7}$ part of the expense; what part of the diameter must each grind down for his share? Also, exhibit the solution by a *geometrical construction*.

Ans. the 1st 4·4508, 2d 4·8400, 3d 5·3535, 4th 6·0765, 5th 7·2079, 6th 9·3935, 7th 22·6778in.

35. A maltster has a kiln, that is 16ft 6in square; but he wants to pull it down, and build a new one, that may dry three times as much as the old one; what must be the length of its side? Ans. 28ft 7in.

36. How many 3in cubes may be cut out of a 12in cube? Ans. 64.

37. How long must the tether of a horse be, that will allow him to graze an acre of ground? Ans. 39¼yds.

38. What will the cost of painting a conical spire come to at 8*d* per yd; the height being 118ft, and the circuit of the base 64ft? Ans. 14*l* 0*s* 8¾*d*.

39. The diameter of an old standard corn bushel is 18½in, and its depth 8in; what must be the diameter of that bushel whose depth is 7½in? Ans. 19·1067in.

40. The ball on the top of St. Paul's church is 6ft diameter; what did gilding it cost at 3½*d* per square inch? Ans. 237*l* 10*s* 1*d*.

41. What will a frustum of a marble cone come to at 12*s* per ft; the diameter of the greater end being 4ft, that of the less end 1½ft, and the length of the slant side 8ft? Ans. 30*l* 1*s* 10¼*d*.

42. Divide a cone into three equal parts by sections parallel to the base, and find the heights of the three parts, that of the whole cone being 20in.

Ans. the upper 13·867, the middle 3·605, the lower 2·528.

43. A gentleman has a bowling-green, 300ft long, and 200ft broad, which he wishes to raise 1ft higher, by means of the earth to be dug out of a ditch surrounding it: to what depth must the ditch be dug, supposing its breadth to be every where 8ft? Ans. $7\frac{23}{86}$ft.

44. How high above the earth must a person be raised, that he may see $\frac{1}{3}$ of its surface: and under what angle will the earth then appear?

Ans. to the height of the earth's diameter; angle 38° 56′ 32″.

45. A cubic foot of brass is to be drawn into wire of $\frac{1}{40}$inch in diameter; what will the length of the wire be, allowing no loss in the metal?

Ans. 97784·797yds, or 55mls 984·797yds.

46. Of what diameter must the bore of a cannon be, which is cast for a ball of 24lb weight, so that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball? Ans. 5·647in.

47. Supposing the diameter of an iron 9lb ball to be 4in, it is required to find the diameter of the several balls weighing 1, 2, 3, 4, 6, 12, 18, 24, 32, 36, and 42lb, and the calibre of their guns, allowing $\frac{1}{50}$ of the calibre, or $\frac{1}{49}$ of the ball's diameter, for windage.

Answer.

Wt. ball.	Diameter ball.	Calibre gun.	Wt. ball.	Diameter ball.	Calibre gun.
1	1·9230	1·9622	12	4·4026	4·4924
2	2·4228	2·4723	18	5·0397	5·1425
3	2·7734	2·8301	24	5·5469	5·6601
4	3·0526	3·1149	32	6·1051	6·2297
6	3·4943	3·5656	36	6·3496	6·4792
9	4·0000	4·0816	42	6·6844	6·8208

48. Supposing the windage of all mortars to be $\frac{1}{80}$ of the calibre, and the diameter of the hollow part of the shell to be $\frac{7}{10}$ of the calibre of the mortar: it

is required to determine the diameter and weight of the shell, and the quantity or weight of powder requisite to fill it, for each of the several sorts of mortars, namely, the 13, 10, 8, 5·8, and 4·6in mortar.

Answer.

Calibre mort.	Diameter ball.	Wt. shell empty.	Wt. of powder.	Wt. shell filled.
4·6	4·523	8·320	0·583	8·903
5·8	5·703	16·677	1·168	17·845
8	7·867	43·764	3·065	46·829
10	9·833	85·476	5·986	91·462
13	12 783	187·791	13·151	200·942

49. If a heavy sphere, whose diameter is 4in, be let fall into a conical glass, full of water, whose diameter is 5, and altitude 6in; it is required to determine how much water wili be displaced. Ans. 26·272 cubic in, or nearly $\frac{35}{47}$pint.

50. The dimensions of a sphere and cone being the same as in the last question, and the cone only $\frac{1}{3}$ full of water; what part of the axis of the sphere is immersed in the water? Ans. ·546 parts of an inch.

51. (1) If R and r be the radii of two spheres inscribed in a cone, so that the greater may touch the less, and that planes are drawn to touch the spheres at their intersections with the axis of the cone: it is required to prove that the volumes of the three cones thus cut off by the planes, and estimated from the vertex, are respectively expressed by

$$\frac{2\pi}{3}\cdot\frac{r^5}{R(R-r)},\ \frac{2\pi}{3}\cdot\frac{R^2r^2}{R-r},\ \text{and}\ \frac{2\pi}{3}\cdot\frac{R^5}{r(R-r)}.$$

(2) It is likewise required to prove that if any number of spheres are inscribed in a cone to touch each other in succession, that they will be in geometrical progression.

52. If a person, in an air balloon, ascend vertically from London, to such height that he can just see Oxford in the horizon; required his height above the earth, supposing its circumference to be 25000 miles, and the distance between London and Oxford 49·5933 miles? Ans. nearly $\frac{311}{1000}$m, or 547yds 1ft.

53. In a garrison there are three remarkable objects, A, B, C, the distances of which from one to another are known to be, AB = 213, AC = 424, and BC = 262 yds. I am desirous of knowing my position and distance at a place or station S, from whence I observed the angle ASB = 13° 30′, and the angle CSB = 29° 50′, both by geometry and trigonometry, the point S being on the same side of AC with B. Ans. AS = 605·7122, BS = 429·6814, CS = 524·2365.

54. Required the same as in the last question, when the point B is on the other side of AC, supposing AB = 9, AC = 12, and BC = 6 furlongs; also the angle ASB = 33° 45′, and the angle BSC = 22° 30′.

Ans. AS = 10·65, BS = 15·64, CS = 14·01.

55. It is required to determine the magnitude of a cube of standard gold, which shall be equal to £960000000; supposing a guinea to weigh 5dwts 9½grs.

Ans. 23·549ft.

56. The ditch of a fortification is 1000ft long, 9ft deep, 20ft broad at the bottom, and 22 at the top; how much water will fill the ditch?

Ans. 1177867gall nearly.

57. If the diameter of the earth be 7930 miles, and that of the moon 2160

iles; required the ratio of their surfaces, and also of their volume, supposing tem both to be spherical.

Ans. the surfaces are as $13\frac{1}{2}$ to 1 nearly; and the volumes as $49\frac{1}{2}$ to 1 nearly.

58. A rectangular cistern whose length, breadth, and depth, internally, were ft 10in, 2ft 1in, and 2ft 9in respectively, was rested on props at the corners, f 4in high; but by accident one of the props was knocked out of its place: ow much less water would the cistern hold when it was brought with that orner to rest upon the ground; and how much less, still, when two adjacent rops were removed, either those under the side or under the end?

59. Let the section of the breast-work be as in *Ex.* 4, *p.* 486, and EO the readth of the ditch at top be 20ft; the slopes of the ditch unequal so that ER RD : : 2 : 3 and SO : SP : : 2 : 4; what must be the depth of the ditch, so hat the earth thrown out shall form a glacis whose height is 3ft and base OL is 4ft? Ans. 6·8ft.

60. If the area of the profile ABHC be 100ft; and BF = 1, FH = 6, BG = 0, GR = 13, RD = 6, and ER = 3ft: what must be the breadth of the ditch o that its section EDPS shall be equal to the profile ABHC and OKL (the ection of the glacis) together, when the slopes BH, KL are in the same plane, nd the slopes ED, OP, are equal? Ans. 25·778 ft.

61. (1) The four sides of a trapezium are $6\frac{1}{2}$, $15\frac{3}{4}$, 12, and 9 respectively, the first wo of these sides make a right angle: required the area of the quadrilateral.

(2) When the same four sides form a quadrilateral inscriptible in a circle, ind its area, angles, and diagonals.

62. Find the ratio of the surfaces of the torrid zone, the two temperate, and he two frigid zones, of the earth; supposing the two tropics to be 23° 28′ from he equator, and the two polar circles to be 23° 28′ from their respective poles.

63. A cone, whose altitude is 63, and diameter of the base 32, is to be cut, y sections parallel to the base, into four portions of equal curved surface: equired the respective distances from the vertex, measured on the slant side, at hich the sections are to be made.

64. The solid content of a spherical shell, is equal to that of a conic frustrum, he areas of whose two ends are respectively equal to the exterior and interior urve surfaces of the shell, and whose height is equal to the shell's thickness.

65. A sphere is to any circumscribing polyhedron, as the surface of the sphere o the surface of the polyhedron.

66. The surface of a sphere is double the curve surface of an inscribed cy-inder whose height and diameter of the base are equal. Also, the surface of a phere is to the curve surface of an equilateral inscribed cone, as 8 to 3.

67. If a cone be cut by a vertical section, the segment of the base cut off by hat section, is to the corresponding segment of the same surface, as the radius of he base to the slant side of the cone.

68. The volume of a regular octahedron inscribed in a sphere is to the cube f the radius as 4 to 3.

69. If α, β, γ, be the angles under which any three diameters of a sphere o radius a intersect, and $\sigma = \frac{1}{2}(\alpha + \beta + \gamma)$: show that the volume of the arallelopiped which is formed by tangent planes at the extremities of their liameter, is expressed by $4a^3\left\{\sin\sigma\sin(\sigma-\alpha)\sin(\sigma-\beta)\sin(\sigma-\gamma)\right\}^{-\frac{1}{2}}$.

70. Find the volumes of the pyramids which envelope a triangular and a quare pile of balls, respectively, the side of the lower course in each being n.

71. A cannon-ball whose radius is r, may be touched by four, or by eight, or y twenty shells of equal radii, R, and each of which touches three of the remain-

ing ones; or it may be touched by six equal shells of radii R_1, each of which touches four of the remaining ones; or, again, it may be touched by twelve equal shells of radii R_2. each of which touches five of the others: it is required to prove these contacts, and assign the several external radii R, R_1, R_2 of these shells in terms of r.

72. If a rectangular pile of six inch balls of 150 in length and 40 in breadth be roofed over, the roof being in close contact with the balls: how much empty space would be enclosed?

NOTES.

NOTE I. *Synthetic Division*, p. 129.

A different and much more simple investigation of this process has occurred to me since the sheet on this subject was printed off. It is as follows:—

To divide $Ax^n + Bx^{n-1} + Cx^{n-2} + \dots$ by $x^m + a_1x^{m-1} + a_2x^{m-2} + \dots$

Assume the quotient to be $Ax^{n-m} + A_1x^{n-m-1} + A_2x^{n-m-2} + \dots$; in which $A_1, A_2 \dots$ are unknown: then, since quotient $\times$ divisor $=$ dividend, let this multiplication be made.

$$
\begin{array}{lllllll}
\text{quot}^t = & A & + A_1 & + A_2 & + A_3 & + \dots & \\
\text{div}^r = & 1 & + a_1 & + a_2 & + a_3 & + \dots & \\
\hline
& A & + A_1 & + A_2 & + A_3 & + \dots & \\
& & a_1A & + a_1A_1 & + a_1A_2 & + a_1A_3 & + \dots \\
& & & a_2A & + a_2A_1 & + a_2A_2 & + a_2A_3 + \dots \\
& & & & a_3A & + a_3A_1 & + a_3A_2 + a_3A_3 \dots \\
& & & & & \dots\dots & \dots\dots
\end{array}
$$

$$
\begin{array}{l|lllllll}
\text{div}^d = & A & + B & + C & + D & + \dots & & \\
-a_1 & & -a_1A & -a_1A_1 & -a_1A_2 & -a_1A_3 & - \dots & \\
-a_2 & & & -a_2A & -a_2A_1 & -a_2A_2 & -a_2A_3 & - \dots \\
-a_3 & & & & -a_3A & -a_3A_1 & -a_3A_2 & -a_3A_3 - \dots \\
\vdots & & & & & \dots\dots & \dots\dots & \dots\dots \\
\hline
\vdots & A & + A_1 & + A_2 & + A_3 & + \dots & &
\end{array}
$$

In this we have worked by detached coefficients as usual. The first part of the operation shows the manner in which the coefficients of the divisor, which are known, are combined with those of the quotient, which are unknown, in forming those of the dividend; and, conversely, the second operation shows the formation of the addends to coefficients of the dividend to form those of the quotient, to be precisely the same as before, except that *all the signs are changed.* Moreover, when A (which is the same in the dividend and quotient) is known, we can form the *diagonal column* composed of $-a_1A$, $-a_2A$, $-a_3A$, $-a_4A, \dots$; and thence we obtain A_1, and, consequently, the next diagonal column $-a_1A_1$, $-a_2A_1$, $-a_3A_1$, $-a_4A_1, \dots$; then, similarly for A_2 and the next diagonal column, as far as it is necessary to carry the process.

The change of sign of the coefficients of the divisor, it is obvious, is a consequence of the converse nature of the operation of finding the coefficients of dividend from those of the quotient and divisor, to that of finding the quotient from the divisor and dividend. The entire result is so strictly in accordance with the prescribed rule (p. 128) that any further detail in addition to what has been given would be superfluous.

NOTE II. *Small arcs, p.* 436.

THE tabular sine and tangent of a very small angle, or *vice versâ*, which cannot be obtained from the tables on account of the rapid variations which those functions undergo at that stage, is often required to be found with great accuracy. They are, however, found in the following manner, without much labour.

1. *To find* tab sin x *when* x *is very small.*

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots$$

Now, as the length of an arc of 1°, or of $\frac{\pi}{180}$, is ·01745329 .... the third term of the series for this arc has no effective figure within the first ten decimal places; and hence, *à fortiori*, the series for a smaller arc can have no effective figure within the first seven decimal places. The series will be, then, *effectively* reduced in this case to its first two terms, and we shall have

$$\sin x = x \left\{1 - \frac{x^2}{1.2.3}\right\} = x \left\{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4}\right\}^{\frac{1}{3}} = x \sqrt[3]{\cos x}.$$

But when it is very small, $\cos x$ varies very slowly, and may be taken, for this purpose, with sufficient accuracy from the tables: whence the value of $\sin x$ can also be computed from

$$\text{tab} \sin x = \log x + 10 + \tfrac{1}{3}(\text{tab} \cos x - 10).$$

Let the arc x contain p seconds, or $x = \frac{\pi}{180.60.60} \cdot p''$: then

$$\log x = \log p + \log \pi - \log (180.60^2) = \log p - 5{\cdot}3144251 \ldots$$

Substituting this in the preceding general formula, we have

$$\text{tab} \sin x = \log p + 10 - 5{\cdot}3144251 + \tfrac{1}{3}(\text{tab} \cos x - 10)$$
$$= \log p + 4{\cdot}6855749 - \tfrac{1}{3} \text{ ac tab} \cos x.$$

2. *To find* tab tan x *when* x *is very small.*

By the preceding we have $\sin x = x \sqrt[3]{\cos x}$, and hence $\tan x = \frac{\sin x}{\cos x} = \frac{x}{\sqrt[3]{\cos^2 x}}$: wherefore $\text{tab} \tan x = \log x + \frac{2}{3}(10 - \text{tab} \cos x)$; and, proceeding as before, $\text{tab} \tan x = \log p + 4{\cdot}6855749 + \frac{2}{3} \text{ ac tab} \cos x$.

3. *Given* tab sin x *or* tab tan x *to find* x *itself.*

Here, by merely reversing the former processes, we have

$$\log p = \text{tab} \sin x + 5{\cdot}3144251 + \tfrac{1}{3} \text{ ac tab} \cos x - 10,$$
$$\log p = \text{tab} \tan x + 5{\cdot}3144251 - \tfrac{2}{3} \text{ ac tab} \cos x - 10:$$

from either of which, according to the data, the value of x is found *.

* The rules indicated by these formulæ were first given, in a verbal form, by Dr. Maskelyne, in the *Introduction to Taylor's Tables*. Many investigations of them have since been given; but the above, whilst they are the most generally adopted ones, are amongst the most simple. This method of investigation itself was originally given in *Woodhouse's Trigonometry*.

Note III. *p.* 423.

The notation for powers of trigonometrical functions.

In comparing the notation for *powers* with that for *inverse functions,* it cannot have escaped the student's attention that the same notation is used in two different senses; and a degree of confusion might arise from it, if he were not apprised of the reason, and of the relations of the two things signified by it.

The *strictly correct* notation for powers is $(\cos x)^n$, $(\tan \theta)^m$, *etc.:* but the increased space required for writing it in this way, as well as the additional trouble, has caused it to be written $\cos^n x$, $\tan^m\theta$, *etc.*, by some authors; and $\cos x^n$, $\tan \theta^m$, *etc.*, by others. Now, so long as no idea of the *successive trigonometrical functions*—such as the cosine of a cosine, or the trigonometrical functions of any *power of an arc*—was entertained, mathematicians very naturally abridged the notation as far as possible, so as not to create doubt in the mind as to the signification of the expression. The introduction of the notation for inverse functions has, however, interfered with the former of these notations; whilst the latter has never been extensively used: and, in strict accuracy, we should be compelled to adopt $(\cos x)^n$, $(\tan \theta)^m$, *etc.*, as our standard notation; and especially, should the inquiries of mathematicians ever lead to results involving the successive trigonometrical functions, direct and inverse, of any expressions for the arc, in the same manner that they have already led to the consideration of successive logarithmic functions (*p.* 262). Instances of this, however, are so rare, that it would be difficult to quote one in any subject of even a moderately elementary character. We have hence, for the present, adhered to the old notation: though it was necessary to point out the circumstance for the satisfaction of the inquiring student.

It may be added, too, that $\sin^1 x$ does really, in reference to the thing signified, bear the same meaning, though founded on a different view of the subject, as $\sin x$; and hence, as a fundamental principle, there is no real difference in this stage (the utmost extent to which the notations under the two aspects coalesce) of the inquiry, between the quantities signified.

Note IV. *The angular unit taken as the arc equal to radius, p.* 422.

The angular unit employed by the Greeks and by all the moderns, till the period of the French Revolution, was the ninetieth part of the quadrant: but by the French, the quadrant was divided centesimally, the hundredth part being the unit; with, however, a general impression that these were more advantageously considered as minor divisions of the quadrant, taken as the standard-unit. Amongst many distinguished men of science, however, the *radius* of the circle by which the angle was estimated, has been considered the most advantageous standard-unit: and in some important applications of trigonometry it is undoubtedly the case, though in reference to the ordinary ones for which tables have already been computed, it would be altogether useless till very extensive tables adapted to this division shall have been published. The main applications, indeed, of this unit, are to purposes for which, from particular circumstances, the existing tables are inadequate.

Denote by a and α the *length* and the *number of degrees* of the arc subtending the angle A ; and let the radius r contain ρ degrees of that circle. Then

$$\rho^\circ : 180^\circ :: r : r\pi, \text{ and } A^\circ : 180^\circ :: a : r\pi.$$

$$\text{Whence } \rho^\circ = \frac{180^\circ}{\pi} = 57^\circ{\cdot}2957795 \ldots . = 206264''{\cdot}8 \text{ nearly;}$$

$$\text{and } \frac{a}{r} = \frac{A^\circ}{\rho^\circ} = \frac{A^\circ}{57^\circ{\cdot}2957795\ldots} = {\cdot}01745329\ldots A^\circ.$$

In reference to this mode of estimating angles, $\rho = 57{\cdot}^\circ 2957795 \ldots$ is to be considered the unit: and the angle is then said to be *estimated in circular measure.*

Note V, pp. 451—6.

Certain conditions amongst the data of plane triangles.

In some of the examples given for solution, the student will have discovered a certain degree of uncertainty in the results which he obtained. This arises from the great relative variations of the trigonometrical functions compared with those of the angles themselves, in certain parts of the tables. In actual trigonometrical observations, the classes of conditions which require these computations to be employed, will, where practicable, be avoided: but numerous instances arise in practice where they cannot be avoided, and hence to remove the resulting uncertainty, other methods of solution have been devised, one or two of which are given in this note, with one or two other particulars relating to the subject.

1. *There are given a, b, C, where b is very small in comparison with a, to find the remaining parts of the triangle.*

$$\text{Putting for } \cos C \text{ its value } \frac{1}{2}\left\{e^{C\sqrt{-1}} + e^{-C\sqrt{-1}}\right\}$$

$$c^2 = a^2 - 2ab\cos C + b^2 = a^2\left\{1 + \frac{b}{a}\,e^{C\sqrt{-1}}\right\}\left\{1 - \frac{b}{a}\,e^{-C\sqrt{-1}}\right\}$$

and taking $\log_e$ of both sides, and expanding, we have

$$2\log_e c = 2\log_e a - \frac{b}{a}\left\{e^{C\sqrt{-1}} + e^{-C\sqrt{-1}}\right\} - \frac{b^2}{2a^2}\left\{e^{2C\sqrt{-1}} + e^{2C\sqrt{-1}}\right\}\ldots$$

$$\text{or } \log_e c = \log_e a - \frac{b}{a}\cos C - \frac{b^2}{2a^2}\cos 2C - \frac{b^3}{3a^2}\cos 3C - \ldots\ldots$$

whence c can be found.

Again, $\tan B = \dfrac{b \sin C}{a - b\cos C}$; or expressed in exponentials

$$\frac{e^{2B\sqrt{-1}} - 1}{e^{2B\sqrt{-1}} + 1} = \frac{b\left\{e^{C\sqrt{-1}} + e^{-C\sqrt{-1}}\right\}}{2a - b\left\{e^{C\sqrt{-1}} + e^{-C\sqrt{-1}}\right\}}; \text{ and hence also}$$

$$e^{2B\sqrt{-1}} = \frac{a - be^{-C\sqrt{-1}}}{a + be^{C\sqrt{-1}}} = \frac{1 - \frac{b}{a}\,e^{-C\sqrt{-1}}}{1 + \frac{b}{a}\,e^{C\sqrt{-1}}}.$$

Take $\log_e$ of both sides as before, and reduce: then

$$B = \frac{b}{a}\sin C + \frac{b^2}{2a^2}\sin 2C + \frac{b^3}{3a^3}\sin 3C + \ldots.$$

where B is estimated in circular measure. Or again, since $\sin 1'' = 1''$ very nearly, the value of B, in *seconds*, is

$$B = \frac{b\sin C}{a\sin 1''} + \frac{b^2\sin 2C}{2a^2\sin 1''} + \frac{b^3\sin 3C}{3a^3\sin 1''} + \ldots.$$

2. *Given the very acute angles A, B, and the side c to find the remaining parts.*

Since A and B are very small, we have very nearly

$$\sin A = A - \frac{A^3}{1.2.3};\ \cos A = 1 - \frac{3A^2}{1.2.3};\ \sin B = B - \frac{B^3}{1.2.3};\ \cos B = 1 - \frac{3B^2}{1.2.3}.$$

Hence, $\sin C = \sin(A+B) = \sin A \cos B + \cos A \sin B = (A+B) - \frac{(A+B)^3}{1.2.3}$.

Wherefore, $a = \frac{c \sin A}{\sin(A+B)} = \frac{cA}{A+B}\left\{1 + \frac{B(2A+B)}{1.2.3}\right\}$, and

$$b = \frac{c \sin B}{\sin(A+B)} = \frac{cB}{A+B}\left\{1 + \frac{A(2B+A)}{1.2.3}\right\}.$$

3. *Given the two sides a, b, and the included angle C, which is very obtuse, to find the other parts of the triangle.*

Put $C = 180° - \alpha$: then, since C is very obtuse, α is very small, and

$$\cos \alpha = 1 - \frac{\alpha^2}{1.2} \text{ nearly: whence}$$

$$c^2 = a^2 - 2ab \cos C + b^2 = a^2 + b^2 + 2ab\left\{1 - \frac{\alpha^2}{1.2}\right\} = (a+b)^2 - ab\,\alpha^2.$$

Whence $c = \sqrt{(a+b)^2 - ab\,\alpha^2} = a + b - \frac{ab}{a+b}\,\frac{\alpha^2}{2}$ nearly.

Again, $\sin A = \frac{a}{c}\sin C = \frac{a}{c}\sin\alpha = \frac{a}{c}\left\{\alpha - \frac{\alpha^3}{1.2.3}\right\}$ nearly; and

$$\frac{a}{c} = \frac{a}{a + b - \frac{ab}{a+b}\cdot\frac{\alpha^2}{2}} = \frac{a}{a+b}\left\{1 - \frac{ab}{(a+b)^2}\cdot\frac{\alpha^2}{2}\right\}^{-1}$$

$$= \frac{a}{a+b}\left\{1 + \frac{ab}{(a+b)^2}\cdot\frac{\alpha^2}{2}\right\} \text{ very nearly.}$$

Inserting in $\sin A = \frac{a}{c}\sin\alpha$, these values, we have

$$\text{Sin } A = \frac{a}{a+b}\left\{1 + \frac{ab}{(a+b)^2}\cdot\frac{\alpha^2}{2}\right\}\left\{\alpha - \frac{\alpha^3}{1.2.3}\right\} = \frac{a\alpha}{a+b}\left\{1 + \frac{ab\,\alpha^2}{2(a+b)^2} - \frac{\alpha^2}{1.2.3}\right\}$$

$$= \frac{a\,\alpha}{a+b}\left\{1 - \frac{a^2 - ab + b^2}{(a+b)^2}\cdot\frac{\alpha^2}{1.2.3}\right\}.$$

Also $\sin A = A - \frac{A^3}{1.2.3} = A - \frac{\sin^3 A}{1.2.3}$ nearly; or $A = \sin A + \frac{\sin^2 A}{1.2.3}$,

$$\text{or } A = \frac{a\,\alpha}{a+b}\left\{1 - \frac{a^2-ab+b^2}{2(a+b)^2}\cdot\frac{\alpha^2}{1.2.3}\right\} + \left\{\frac{a}{a+b}\right\}^3\cdot\frac{\alpha^3}{1.2.3}$$

$$= \frac{a\,\alpha}{a+b}\left\{1 + \frac{(a-b)\,b}{(a+b)^2}\cdot\frac{\alpha^2}{1.2.3}\right\} \text{ nearly.}$$

In the same manner we have

$$B = \frac{b\alpha}{a+b}\left\{1 - \frac{(a-b)\,a}{(a+b)^2}\cdot\frac{\alpha^2}{1.2.3}\right\} \text{ nearly.}$$

These angles being understood to be in *circular measure*, as explained in the preceding note.

For further information on subjects of this nature, the reader is referred to *Bonnycastle's Trigonometry*, *Cagnoli Trigonométrie*, several of the authors on Geodesy, especially *Puissant*, and to the second volume of this work.

No.	Square.	Cube.	Sq. Root.	Cube Root	No.	Square.	Cube.	Sq. Root.	Cube Root
1	1	1	1·0000000	1·000000	73	5329	389017	8·5440037	4·179339
2	4	8	1·4142136	1·259921	74	5476	405224	8·6023253	4·198336
3	9	27	1·7320508	1·442250	75	5625	421875	8·6602540	4·217163
4	16	64	2·0000000	1·587401	76	5776	438976	8·7177979	4·235824
5	25	125	2·2360680	1·709976	77	5929	456533	8·7749644	4·254321
6	36	216	2·4494897	1·817121	78	6084	474552	8·8317609	4·272659
7	49	343	2·6457513	1·912931	79	6241	493039	8·8881944	4·290840
8	64	512	2·8284271	2·000000	80	6400	512000	8·9442719	4·308869
9	81	729	3·0000000	2·080084	81	6561	531441	9·0000000	4·326749
10	100	1000	3·1622777	2·154435	82	6724	551368	9·0553851	4·344481
11	121	1331	3·3166248	2·223980	83	6889	571787	9·1104336	4·362071
12	144	1728	3·4641016	2·289429	84	7056	592704	9·1651514	4·379519
13	169	2197	3·6055513	2·351335	85	7225	614125	9·2195445	4·396830
14	196	2744	3·7416574	2·410142	86	7396	636056	9·2736185	4·414005
15	225	3375	3·8729833	2·466212	87	7569	658503	9·3273791	4·431048
16	256	4096	4·0000000	2·519842	88	7744	681472	9·3808315	4·447960
17	289	4913	4·1231056	2·571282	89	7921	704969	9·4339811	4·464745
18	324	5832	4·2426407	2·620741	90	8100	729000	9·4868330	4·481405
19	361	6859	4·3588989	2·668402	91	8281	753571	9·5393920	4·497941
20	400	8000	4·4721360	2·714418	92	8464	778688	9·5916630	4·514357
21	441	9261	4·5825757	2·758924	93	8649	804357	9·6436508	4·530655
22	484	10648	4·6904158	2·802039	94	8836	830584	9·6953597	4·546836
23	529	12167	4·7958315	2·843867	95	9025	857375	9·7467943	4·562903
24	576	13824	4·8989795	2·884499	96	9216	884736	9·7979590	4·578857
25	625	15625	5·0000000	2·924018	97	9409	912673	9·8488578	4·594701
26	676	17576	5·0990195	2·962496	98	9604	941192	9·8994949	4·610436
27	729	19683	5·1961524	3·000000	99	9801	970299	9·9498744	4·626065
28	784	21952	5·2915026	3·036589	100	10000	1000000	10·0000000	4·641589
29	841	24389	5·3851648	3·072317	101	10201	1030301	10·0498756	4·657009
30	900	27000	5·4772256	3·107232	102	10404	1061208	10·0995049	4·672329
31	961	29791	5·5677644	3·141381	103	10609	1092727	10·1488916	4·687548
32	1024	32768	5·6568542	3·174802	104	10816	1124864	10·1980390	4·702669
33	1089	35937	5·7445626	3·207534	105	11025	1157625	10·2469508	4·717694
34	1156	39304	5·8309519	3·239612	106	11236	1191016	10·2956301	4·732623
35	1225	42875	5·9160798	3·271066	107	11449	1225043	10·3440804	4·747459
36	1296	46656	6·0000000	3·301927	108	11664	1259712	10·3923048	4·762203
37	1369	50653	6·0827625	3·332222	109	11881	1295029	10·4403065	4·776856
38	1444	54872	6·1644140	3·361975	110	12100	1331000	10·4880885	4·791420
39	1521	59319	6·2449980	3·391211	111	12321	1367631	10·5356538	4·805895
40	1600	64000	6·3245553	3·419952	112	12544	1404928	10·5830052	4·820284
41	1681	68921	6·4031242	3·448217	113	12769	1442897	10·6301458	4·834588
42	1764	74088	6·4807407	3·476027	114	12996	1481544	10·6770783	4·848808
43	1849	79507	6·5574385	3·503398	115	13225	1520875	10·7238053	4·862944
44	1936	85184	6·6332496	3·530348	116	13456	1560896	10·7703296	4·876999
45	2025	91125	6·7082039	3·556893	117	13689	1601613	10·8166538	4·890973
46	2116	97336	6·7823300	3·583048	118	13924	1643032	10·8627805	4·904868
47	2209	103823	6·8556546	3·608826	119	14161	1685159	10·9087121	4·918685
48	2304	110592	6·9282032	3·634241	120	14400	1728000	10·9544512	4·932424
49	2401	117649	7·0000000	3·659306	121	14641	1771561	11·0000000	4·946087
50	2500	125000	7·0710678	3·684031	122	14884	1815848	11·0453610	4·959676
51	2601	132651	7·1414284	3·708430	123	15129	1860867	11·0905365	4·973190
52	2704	140608	7·2111026	3·732511	124	15376	1906624	11·1355287	4·986631
53	2809	148877	7·2801099	3·756286	125	15625	1953125	11·1803399	5·000000
54	2916	157464	7·3484692	3·779763	126	15876	2000376	11·2249722	5·013298
55	3025	166375	7·4161985	3·802952	127	16129	2048383	11·2694277	5·026526
56	3136	175616	7·4833148	3·825862	128	16384	2097152	11·3137085	5·039684
57	3249	185193	7·5498344	3·848501	129	16641	2146689	11·3578167	5·052774
58	3364	195112	7·6157731	3·870877	130	16900	2197000	11·4017543	5·065797
59	3481	205379	7·6811457	3·892996	131	17161	2248091	11·4455231	5·078753
60	3600	216000	7·7459667	3·914868	132	17424	2299968	11·4891253	5·091643
61	3721	226981	7·8102497	3·936497	133	17689	2352637	11·5325626	5·104469
62	3844	238328	7·8740079	3·957891	134	17956	2406104	11·5758369	5·117230
63	3969	250047	7·9372539	3·979057	135	18225	2460375	11·6189500	5·129928
64	4096	262144	8·0000000	4·000000	136	18496	2515456	11·6619038	5·142563
65	4225	274625	8·0622577	4·020726	137	18769	2571353	11·7046999	5·155137
66	4356	287496	8·1240384	4·041240	138	19044	2628072	11.7473401	5·167649
67	4489	300763	8·1853528	4·061548	139	19321	2685619	11·7898261	5·180101
68	4624	314432	8·2462113	4·081655	140	19600	2744000	11·8321596	5·192494
69	4761	328509	8·3066239	4·101566	141	19881	2803221	11·8743422	5·204828
70	4900	343000	8·3666003	4·121285	142	20164	2863288	11·9163753	5·217103
71	5041	357911	8·4261498	4·140818	143	20449	2924207	11·9582607	5·229321
72	5184	373248	8·4852814	4·160168	144	20736	2985984	12·0000000	5·241483

No.	Square.	Cube.	Sq. Root.	Cube Root	No.	Square.	Cube.	Sq. Root.	Cube Root
145	21025	3048625	12·0415946	5·253588	217	47089	10218313	14·7309199	6·009245
146	21316	3112136	12·0830460	5·265637	218	47524	10360232	14·7648231	6·018462
147	21609	3176523	12·1243557	5·277632	219	47961	10503459	14·7986486	6·027650
148	21904	3241792	12·1655251	5·289572	220	48400	10648000	14·8323970	6·036811
149	22201	3307949	12·2065556	5·301459	221	48841	10793861	14·8660687	6·045943
150	22500	3375000	12·2474487	5·313293	222	49284	10941048	14·8996644	6·055049
151	22801	3442951	12·2882057	5·325074	223	49729	11089567	14·9331845	6·064127
152	23104	3511808	12·3288280	5·336803	224	50176	11239424	14·9666295	6·073178
153	23409	3581577	12·3693169	5·348481	225	50625	11390625	15·0000000	6·082202
154	23716	3652264	12·4096736	5·360108	226	51076	11543176	15·0332964	6·091199
155	24025	3723875	12·4498996	5·371685	227	51529	11697083	15·0665192	6·100170
156	24336	3796416	12·4899960	5·383213	228	51984	11852352	15·0996689	6·109115
157	24649	3869893	12·5299641	5·394691	229	52441	12008989	15·1327460	6·118033
158	24964	3944312	12·5698051	5·406120	230	52900	12167000	15·1657509	6·126926
159	25281	4019679	12·6095202	5·417501	231	53361	12326391	15·1986842	6·135792
160	25600	4096000	12·6491106	5·428835	232	53824	12487168	15·2315462	6·144634
161	25921	4173281	12·6885775	5·440122	233	54289	12649337	15·2643375	6·153449
162	26244	4251528	12·7279221	5·451362	234	54756	12812904	15·2970585	6·162240
163	26569	4330747	12·7671453	5·462556	235	55225	12977875	15·3297097	6·171006
164	26896	4410944	12·8062485	5 473704	236	55696	13144256	15·3622915	6·179747
165	27225	4492125	12·8452326	5·484807	237	56169	13312053	15·3948043	6·188463
166	27556	4574296	12·8840987	5·495865	238	56644	13481272	15·4272486	6·197154
167	27889	4657463	12·9228480	5·506878	239	57121	13651919	15·4596248	6·205822
168	28224	4741632	12·9614814	5·517848	240	57600	13824000	15·4919334	6·214465
169	28561	4826809	13·0000000	5·528775	241	58081	13997521	15·5241747	6·223084
170	28900	4913000	13·0384048	5·539658	242	58564	14172488	15·5563492	6·231680
171	29241	5000211	13·0766968	5·550499	243	59049	14348907	15·5884573	6·240251
172	29584	5088448	13·1148770	5·561298	244	59536	14526784	15·6204994	6·248800
173	29929	5177717	13·1529464	5·572055	245	60025	14706125	15·6524758	6·257325
174	30276	5268024	13·1909060	5·582770	246	60516	14886936	15·6843871	6·265827
175	30625	5359375	13·2287566	5·593445	247	61009	15069223	15·7162336	6·274305
176	30976	5451776	13·2664992	5·604079	248	61504	15252992	15·7480157	6·282761
177	31329	5545233	13·3041347	5·614672	249	62001	15438249	15·7797338	6·291195
178	31684	5639752	13·3416641	5·625226	250	62500	15625000	15·8113883	6·299605
179	32041	5735339	13·3790882	5·635741	251	63001	15813251	15·8429795	6·307994
180	32400	5832000	13·4164079	5·646216	252	63504	16003008	15·8745079	6·316360
181	32761	5929741	13·4536240	5·656653	253	64009	16194277	15·9059737	6·324704
182	33124	6028568	13·4907376	5·667051	254	64516	16387064	15·9373775	6·333026
183	33489	6128487	13·5277493	5·677411	255	65025	16581375	15·9687194	6·341326
184	33856	6229504	13·5646600	5·687734	256	65536	16777216	16·0000000	6·349604
185	34225	6331625	13·6014705	5·698019	257	66049	16974593	16·0312195	6·357861
186	34596	6434856	13·6381817	5·708267	258	66564	17173512	16·0623784	6·366097
187	34969	6539203	13·6747943	5·718479	259	67081	17373979	16·0934769	6·374311
188	35344	6644672	13·7113092	5·728654	260	67600	17576000	16·1245155	6·382504
189	35721	6751269	13·7477271	5·738794	261	68121	17779581	16·1554944	6·390676
190	36100	6859000	13·7840488	5·748897	262	68644	17984728	16·1864141	6·398828
191	36481	6967871	13·8202750	5·758965	263	69169	18191447	16·2172747	6·406958
192	36864	7077888	13·8564065	5·768998	264	69696	18399744	16·2480768	6·415069
193	37249	7189057	13·8924440	5·778996	265	70225	18609625	16·2788206	6·423158
194	37636	7301384	13·9283883	5·788960	266	70756	18821096	16·3095064	6·431228
195	38025	7414875	13·9642400	5·798890	267	71289	19034163	16·3401346	6·439277
196	38416	7529536	14·0000000	5·808786	268	71824	19248832	16·3707055	6·447306
197	38809	7645373	14·0356688	5·818648	269	72361	19465109	16·4012195	6·455315
198	39204	7762392	14·0712473	5·828477	270	72900	19683000	16·4316767	6·463304
199	39601	7880599	14·1067360	5·838272	271	73441	19902511	16·4620776	6·471274
200	40000	8000000	14·1421356	5·848035	272	73984	20123648	16·4924225	6·479224
201	40401	8120601	14·1774469	5·857766	273	74529	20346417	16·5227116	6·487154
202	40804	8242408	14·2126704	5·867464	274	75076	20570824	16·5529454	6·495065
203	41209	8365427	14·2478068	5·877131	275	75625	20796875	16·5831240	6·502957
204	41616	8489664	14·2828569	5·886765	276	76176	21024576	16·6132477	6·510830
205	42025	8615125	14·3178211	5·896368	277	76729	21253933	16·6433170	6·518684
206	42436	8741816	14·3527001	5·905941	278	77284	21484952	16·6733320	6·526519
207	42849	8869743	14·3874946	5·915482	279	77841	21717639	16·7032931	6·534335
208	43264	8998912	14·4222051	5·924992	280	78400	21952000	16·7332005	6·542133
209	43681	9129329	14·4568323	5·934473	281	78961	22188041	16·7630546	6·549912
210	44100	9261000	14·4913767	5·943922	282	79524	22425768	16·7928556	6·557672
211	44521	9393931	14·5258390	5·953342	283	80089	22665187	16·8226038	6·565414
212	44944	9528128	14·5602198	5·962732	284	80656	22906304	16·8522995	6·573139
213	45369	9663597	14·5945195	5·972093	285	81225	23149125	16·8819430	6·580844
214	45796	9800344	14·6287388	5·981424	286	81796	23393656	16·9115345	6·588532
215	46225	9938375	14·6628783	5·990726	287	82369	23639903	16·9410743	6·596202
216	46656	10077696	14·6969385	6·000000	288	82944	23887872	16·9705627	6·603854

No.	Square.	Cube.	Sq. Root.	Cube Root	No.	Square.	Cube.	Sq. Root.	Cube Root
289	83521	24137569	17·0000000	6·611489	361	130321	47045881	19·0000000	7·120367
290	84100	24389000	17·0293864	6·619106	362	131044	47437928	19·0262976	7·126936
291	84681	24642171	17·0587221	6·626705	363	131769	47832147	19·0525589	7·133492
292	85264	24897088	17·0880075	6·634287	364	132496	48228544	19·0787840	7·140037
293	85849	25153757	17·1172428	6·641852	365	133225	48627125	19·1049732	7·146569
294	86436	25412184	17·1464282	6·649400	366	133956	49027896	19·1311265	7·153090
295	87025	25672375	17·1755640	6·656930	367	134689	49430863	19·1572441	7·159599
296	87616	25934336	17·2046505	6·664444	368	135424	49836032	19·1833261	7·166096
297	88209	26198073	17·2336879	6·671940	369	136161	50243409	19·2093727	7·172581
298	88804	26463592	17·2626765	6·679420	370	136900	50653000	19·2353841	7 179054
299	89401	26730899	17·2916165	6·686883	371	137641	51064811	19·2613603	7·185516
300	90000	27000000	17·3205081	6·694329	372	138384	51478848	19·2873015	7·191966
301	90601	27270901	17·3493516	6·701759	373	139129	51895117	19·3132079	7·198405
302	91204	27543608	17·3781472	6·709173	374	139876	52313624	19·3390796	7·204832
303	91809	27818127	17·4068952	6·716570	375	140625	52734375	19·3649167	7·211248
304	92416	28094464	17·4355958	6·723951	376	141376	53157376	19·3907194	7·217652
305	93025	28372625	17·4642492	6·731316	377	142129	53582633	19·4164878	7·224045
306	93636	28652616	17·4928557	6·738664	378	142884	54010152	19·4422221	7·230427
307	94249	28934443	17·5214155	6·745997	379	143641	54439939	19·4679223	7·236797
308	94864	29218112	17·5499288	6·753313	380	144400	54872000	19·4935887	7·243156
309	95481	29503629	17·5783958	6·760614	381	145161	55306341	19·5192213	7·249504
310	96100	29791000	17·6068169	6·767899	382	145924	55742968	19·5448203	7·255841
311	96721	30080231	17·6351921	6·775169	383	146689	56181887	19·5703858	7·262167
312	97344	30371328	17·6635217	6·782423	384	147456	56623104	19·5959179	7·268482
313	97969	30664297	17·6918060	6·789661	385	148225	57066625	19·6214169	7·274786
314	98596	30959144	17·7200451	6·796884	386	148996	57512456	19·6468827	7·281079
315	99225	31255875	17·7482393	6·804092	387	149769	57960603	19·6723156	7·287362
316	99856	31554496	17·7763888	6·811285	388	150544	58411072	19·6977156	7·293633
317	100489	31855013	17·8044938	6·818462	389	151321	58863869	19·7230829	7·299894
318	101124	32157432	17·8325545	6·825624	390	152100	59319000	19·7484177	7·306144
319	101761	32461759	17·8605711	6·832771	391	152881	59776471	19·7737199	7·312383
320	102400	32768000	17·8885438	6·839904	392	153664	60236288	19·7989899	7·318611
321	103041	33076161	17·9164729	6·847021	393	154449	60698457	19·8242276	7·324829
322	103684	33386248	17·9443584	6·854124	394	155236	61162984	19·8494332	7·331037
323	104329	33698267	17·9722008	6·861212	395	156025	61629875	19·8746069	7·337234
324	104976	34012224	18·0000000	6·868285	396	156816	62099136	19·8997487	7·343420
325	105625	34328125	18·0277564	6·875344	397	157609	62570773	19·9248588	7·349597
326	106276	34645976	18·0554701	6·882389	398	158404	63044792	19·9499373	7·355762
327	106929	34965783	18·0831413	6·889419	399	159201	63521199	19·9749844	7·361918
328	107584	35287552	18·1107703	6·896435	400	160000	64000000	20 0000000	7·368063
329	108241	35611289	18·1383571	6·903436	401	160801	64481201	20·0249844	7·374198
330	108900	35937000	18·1659021	6·910423	402	161604	64964808	20·0499377	7·380323
331	109561	36264691	18·1934054	6·917396	403	162409	65450827	20·0748599	7·386437
332	110224	36594368	18·2208672	6·924356	404	163216	65939264	20·0997512	7·392542
333	110889	36926037	18·2482876	6·931301	405	164025	66430125	20·1246118	7·398636
334	111556	37259704	18·2756669	6·938232	406	164836	66923416	20·1494417	7·404721
335	112225	37595375	18·3030052	6·945150	407	165649	67419143	20·1742410	7·410795
336	112896	37933056	18·3303028	6·952053	408	166464	67917312	20·1990099	7·416859
337	113569	38272753	18·3575598	6·958943	409	167281	68417929	20·2237484	7·422914
338	114244	38614472	18·3847763	6·965820	410	168100	68921000	20·2484567	7·428959
339	114921	38958219	18·4119526	6·972683	411	168921	69426531	20·2731349	7·434994
340	115600	39304000	18·4390889	6·979532	412	169744	69934528	20·2977831	7·441019
341	116281	39651821	18·4661853	6·986368	413	170569	70444997	20·3224014	7·447034
342	116964	40001688	18·4932420	6·993191	414	171396	70957944	20·3469899	7·453040
343	117649	40353607	18·5202592	7·000000	415	172225	71473375	20·3715488	7·459036
344	118336	40707584	18·5472370	7·006796	416	173056	71991296	20·3960781	7·465022
345	119025	41063625	18·5741756	7·013579	417	173889	72511713	20·4205779	7·470999
346	119716	41421736	18·6010752	7·020349	418	174724	73034632	20·4450483	7·476966
347	120409	41781923	18·6279360	7·027106	419	175561	73560059	20·4694895	7·482924
348	121104	42144192	18·6547581	7·033850	420	176400	74088000	20·4939015	7·488872
349	121801	42508549	18·6815417	7·040581	421	177241	74618461	20·5182845	7·494811
350	122500	42875000	18·7082869	7·047299	422	178084	75151448	20·5426386	7·500741
351	123201	43243551	18·7349940	7·054004	423	178929	75686967	20·5669638	7·506661
352	123904	43614208	18·7616630	7·060697	424	179776	76225024	20·5912603	7·512571
353	124609	43986977	18·7882942	7·067377	425	180625	76765625	20·6155281	7·518473
354	125316	44361864	18·8148877	7·074044	426	181476	77308776	20·6397674	7·524365
355	126025	44738875	18·8414437	7·080699	427	182329	77854483	20·6639783	7·530248
356	126736	45118016	18·8679623	7·087341	428	183184	78402752	20·6881609	7·536122
357	127449	45499293	18·8944436	7·093971	429	184041	78953589	20·7123152	7·541987
358	128164	45882712	18·9208879	7·100588	430	184900	79507000	20·7364414	7·547842
359	128881	46268279	18·9472953	7·107194	431	185761	80062991	20·7605395	7·553689
360	129600	46656000	18·9736660	7·113787	432	186624	80621568	20·7846097	7·559526

No.	Square.	Cube.	Sq. Root.	Cube Root	No.	Square.	Cube.	Sq. Root.	Cube Root
433	187489	81182737	20·8086520	7·565355	505	255025	128787625	22·4722051	7·963374
434	188356	81746504	20·8326667	7·571174	506	256036	129554216	22·4944438	7·968627
435	189225	82312875	20·8566536	7·576985	507	257049	130323843	22·5166605	7·973873
436	190096	82881856	20·8806130	7·582786	508	258064	131096512	22·5388553	7·979112
437	190969	83453453	20·9045450	7·588579	509	259081	131872229	22·5610283	7·984344
438	191844	84027672	20·9284495	7·594363	510	260100	132651000	22·5831796	7·989570
439	192721	84604519	20·9523268	7·600138	511	261121	133432831	22·6053091	7·994788
440	193600	85184000	20·9761770	7·605905	512	262144	134217728	22·6274170	8·000000
441	194481	85766121	21·0000000	7·611663	513	263169	135005697	22·6495033	8·005205
442	195364	86350888	21·0237960	7·617412	514	264196	135796744	22·6715681	8·010403
443	196249	86938307	21·0475652	7·623152	515	265225	136590875	22·6936114	8·015595
444	197136	87528384	21·0713075	7·628884	516	266256	137388096	22·7156334	8·020779
445	198025	88121125	21·0950231	7·634607	517	267289	138188413	22·7376340	8·025957
446	198916	88716536	21·1187121	7·640321	518	268324	138991832	22·7596134	8·031129
447	199809	89314623	21·1423745	7·646027	519	269361	139798359	22·7815715	8·036293
448	200704	89915392	21·1660105	7·651725	520	270400	140608000	22·8035085	8·041451
449	201601	90518849	21·1896201	7·657414	521	271441	141420761	22·8254244	8·046603
450	202500	91125000	21·2132034	7·663094	522	272484	142236648	22·8473193	8·051748
451	203401	91733851	21·2367606	7·668766	523	273529	143055667	22·8691933	8·056886
452	204304	92345408	21·2602916	7·674430	524	274576	143877824	22·8910463	8·062018
453	205209	92959677	21·2837967	7·680086	525	275625	144703125	22·9128785	8·067143
454	206116	93576664	21·3072758	7·685733	526	276676	145531576	22·9346899	8·072262
455	207025	94196375	21·3307290	7·691372	527	277729	146363183	22·9564806	8·077374
456	207936	94818816	21·3541565	7·697002	528	278784	147197952	22·9782506	8·082480
457	208849	95443993	21·3775583	7·702625	529	279841	148035889	23·0000000	8·087579
458	209764	96071912	21·4009346	7·708239	530	280900	148877000	23·0217289	8·092672
459	210681	96702579	21·4242853	7·713845	531	281961	149721291	23·0434372	8·097759
460	211600	97336000	21·4476106	7·719443	532	283024	150568768	23·0651252	8·102839
461	212521	97972181	21·4709106	7·725032	533	284089	151419437	23·0867928	8·107913
462	213444	98611128	21·4941853	7·730614	534	285156	152273304	23·1084400	8·112980
463	214369	99252847	21·5174348	7·736188	535	286225	153130375	23·1300670	8·118041
464	215296	99897344	21·5406592	7·741753	536	287296	153990656	23·1516738	8·123096
465	216225	100544625	21·5638587	7·747311	537	288369	154854153	23·1732605	8·128145
466	217156	101194696	21·5870331	7·752861	538	289444	155720872	23·1948270	8·133187
467	218089	101847563	21·6101828	7·758402	539	290521	156590819	23·2163735	8·138223
468	219024	102503232	21·6333077	7·763936	540	291600	157464000	23·2379001	8·143253
469	219961	103161709	21·6564078	7·769462	541	292681	158340421	23·2594067	8·148276
470	220900	103823000	21·6794834	7·774980	542	293764	159220088	23·2808935	8·153294
471	221841	104487111	21·7025344	7·780490	543	294849	160103007	23·3023604	8·158305
472	222784	105154048	21·7255610	7·785993	544	295936	160989184	23·3238076	8·163310
473	223729	105823817	21·7485632	7·791487	545	297025	161878625	23·3452351	8·168309
474	224676	106496424	21·7715411	7·796974	546	298116	162771336	23·3666429	8·173302
475	225625	107171875	21·7944947	7·802454	547	299209	163667323	23·3880311	8·178289
476	226576	107850176	21·8174242	7·807925	548	300304	164566592	23·4093998	8·183269
477	227529	108531333	21·8403297	7·813389	549	301401	165469149	23·4307490	8·188244
478	228484	109215352	21·8632111	7·818846	550	302500	166375000	23·4520788	8·193213
479	229441	109902239	21·8860686	7·824294	551	303601	167284151	23·4733892	8·198175
480	230400	110592000	21·9089023	7·829735	552	304704	168196608	23·4946802	8·203132
481	231361	111284641	21·9317122	7·835169	553	305809	169112377	23·5159520	8·208082
482	232324	111980168	21·9544984	7·840595	554	306916	170031464	23·5372046	8·213027
483	233289	112678587	21·9772610	7·846013	555	308025	170953875	23·5584380	8·217966
484	234256	113379904	22·0000000	7·851424	556	309136	171879616	23·5796522	8·222898
485	235225	114084125	22·0227155	7·856828	557	310249	172808693	23·6008474	8·227825
486	236196	114791256	22·0454077	7·862224	558	311364	173741112	23·6220236	8·232746
487	237169	115501303	22·0680765	7·867613	559	312481	174676879	23·6431808	8·237661
488	238144	116214272	22·0907220	7·872994	560	313600	175616000	23·6643191	8·242571
489	239121	116930169	22·1133444	7·878368	561	314721	176558481	23·6854386	8·247474
490	240100	117649000	22·1359436	7·883735	562	315844	177504328	23·7065392	8·252371
491	241081	118370771	22·1585198	7·889095	563	316969	178453547	23·7276210	8·257263
492	242064	119095488	22·1810730	7·894447	564	318096	179406144	23·7486842	8·262149
493	243049	119823157	22·2036033	7·899792	565	319225	180362125	23·7697286	8·267029
494	244036	120553784	22·2261108	7·905129	566	320356	181321496	23·7907545	8·271904
495	245025	121287375	22·2485955	7·910460	567	321489	182284263	23·8117618	8·276773
496	246016	122023936	22·2710575	7·915783	568	322624	183250432	23·8327506	8·281635
497	247009	122763473	22·2934968	7·921099	569	323761	184220009	23·8537209	8·286493
498	248004	123505992	22·3159136	7·926408	570	324900	185193000	23·8746728	8·291344
499	249001	124251499	22·3383079	7·931710	571	326041	186169411	23·8956063	8·296190
500	250000	125000000	22·3606798	7·937005	572	327184	187149248	23·9165215	8·301030
501	251001	125751501	22·3830293	7·942293	573	328329	188132517	23·9374184	8·305865
502	252004	126506008	22·4053565	7·947574	574	329476	189119224	23·9582971	8·310694
503	253009	127263527	22·4276615	7·952848	575	330625	190109375	23·9791576	8·315517
504	254016	128024064	22·4499443	7·958114	576	331776	191102976	24·0000000	8·320335

No.	Square.	Cube.	Sq. Root.	Cube Root	No.	Square.	Cube.	Sq. Root.	Cube Root
577	332929	192100033	24·0208243	8·325147	649	421201	273359449	25·4754784	8·657946
578	334084	193100552	24·0416306	8·329954	650	422500	274625000	25·4950976	8·662391
579	335241	194104539	24·0624188	8·334755	651	423801	275894451	25·5147016	8·666831
580	336400	195112000	24·0831891	8·339551	652	425104	277167808	25·5342907	8·671266
581	337561	196122941	24·1039416	8·344341	653	426409	278445077	25·5538647	8·675697
582	338724	197137368	24·1246762	8·349126	654	427716	279726264	25·5734237	8·680124
583	339889	198155287	24·1453929	8·353905	655	429025	281011375	25·5929678	8·684546
584	341056	199176704	24·1660919	8·358678	656	430336	282300416	25·6124969	8·688963
585	342225	200201625	24·1867732	8·363447	657	431649	283593393	25·6320112	8·693376
586	343396	201230056	24·2074369	8·368209	658	432964	284890312	25·6515107	8·697784
587	344569	202262003	24·2280829	8·372967	659	434281	286191179	25·6709953	8·702188
588	345744	203297472	24·2487113	8·377719	660	435600	287496000	25·6904652	8·706588
589	346921	204336469	24·2693222	8·382465	661	436921	288804781	25·7099203	8·710983
590	348100	205379000	24·2899156	8·387206	662	438244	290117528	25·7293607	8·715373
591	349281	206425071	24·3104916	8·391942	663	439569	291434247	25·7487864	8·719760
592	350464	207474688	24·3310501	8·396673	664	440896	292754944	25·7681975	8·724141
593	351649	208527857	24·3515913	8·401398	665	442225	294079625	25·7875939	8·728518
594	352836	209584584	24·3721152	8·406118	666	443556	295408296	25·8069758	8·732892
595	354025	210644875	24·3926218	8·410833	667	444889	296740963	25·8263431	8·737260
596	355216	211708736	24·4131112	8·415542	668	446224	298077632	25·8456960	8·741625
597	356409	212776173	24·4335834	8·420246	669	447561	299418309	25·8650343	8·745985
598	357604	213847192	24·4540385	8·424945	670	448900	300763000	25·8843582	8·750340
599	358801	214921799	24·4744765	8·429638	671	450241	302111711	25·9036677	8·754691
600	360000	216000000	24·4948974	8·434327	672	451584	303464448	25·9229628	8·759038
601	361201	217081801	24·5153013	8·439010	673	452929	304821217	25·9422435	8·763381
602	362404	218167208	24·5356883	8·443688	674	454276	306182024	25·9615100	8·767719
603	363609	219256227	24·5560583	8·448360	675	455625	307546875	25·9807621	8·772053
604	364816	220348864	24·5764115	8·453028	676	456976	308915776	26·0000000	8·776383
605	366025	221445125	24·5967478	8·457691	677	458329	310288733	26·0192237	8·780708
606	367236	222545016	24·6170673	8·462348	678	459684	311665752	26·0384331	8·785030
607	368449	223648543	24·6373700	8·467000	679	461041	313046839	26·0576284	8·789347
608	369664	224755712	24·6576560	8·471647	680	462400	314432000	26·0768096	8·793659
609	370881	225866529	24·6779254	8·476289	681	463761	315821241	26·0959767	8·797968
610	372100	226981000	24·6981781	8·480926	682	465124	317214568	26·1151297	8·802272
611	373321	228099131	24·7184142	8·485558	683	466489	318611987	26·1342687	8·806572
612	374544	229220928	24·7386338	8·490185	684	467856	320013504	26·1533937	8·810868
613	375769	230346397	24·7588368	8·494806	685	469225	321419125	26·1725047	8·815160
614	376996	231475544	24·7790234	8·499423	686	470596	322828856	26·1916017	8·819447
615	378225	232608375	24·7991935	8·504035	687	471969	324242703	26·2106848	8·823731
616	379456	233744896	24·8193473	8·508642	688	473344	325660672	26·2297541	8·828010
617	380689	234885113	24·8394847	8·513243	689	474721	327082769	26·2488095	8·832285
618	381924	236029032	24·8596058	8·517840	690	476100	328509000	26·2678511	8·836556
619	383161	237176659	24·8797106	8·522432	691	477481	329939371	26·2868789	8·840823
620	384400	238328000	24·8997992	8·527019	692	478864	331373888	26·3058929	8·845085
621	385641	239483061	24·9198716	8·531601	693	480249	332812557	26·3248932	8·849344
622	386884	240641848	24·9399278	8·536178	694	481636	334255384	26·3438797	8·853598
623	388129	241804367	24·9599679	8·540750	695	483025	335702375	26·3628527	8·857849
624	389376	242970624	24·9799920	8·545317	696	484416	337153536	26·3818119	8·862095
625	390625	244140625	25·0000000	8·549880	697	485809	338608873	26·4007576	8·866337
626	391876	245314376	25·0199920	8·554437	698	487204	340068392	26·4196896	8·870576
627	393129	246491883	25·0399681	8·558990	699	488601	341532099	26·4386081	8·874810
628	394384	247673152	25·0599282	8·563538	700	490000	343000000	26·4575131	8·879040
629	395641	248858189	25·0798724	8·568081	701	491401	344472101	26·4764046	8·883266
630	396900	250047000	25·0998008	8·572619	702	492804	345948408	26·4952826	8·887488
631	398161	251239591	25·1197134	8·577152	703	494209	347428927	26·5141472	8·891706
632	399424	252435968	25·1396102	8·581681	704	495616	348913664	26·5329983	8·895920
633	400689	253636137	25·1594913	8·586205	705	497025	350402625	26·5518361	8·900130
634	401956	254840104	25·1793566	8·590724	706	498436	351895816	26·5706605	8·904337
635	403225	256047875	25·1992063	8·595238	707	499849	353393243	26·5894716	8·908539
636	404496	257259456	25·2190404	8·599748	708	501264	354894912	26·6082694	8·912737
637	405769	258474853	25·2388589	8·604252	709	502681	356400829	26·6270539	8·916931
638	407044	259694072	25·2586619	8·608753	710	504100	357911000	26·6458252	8·921121
639	408321	260917119	25·2784493	8·613248	711	505521	359425431	26·6645833	8·925308
640	409600	262144000	25·2982213	8·617739	712	506944	360944128	26·6833281	8·929490
641	410881	263374721	25·3179778	8·622225	713	508369	362467097	26·7020598	8·933669
642	412164	264609288	25·3377189	8·626706	714	509796	363994344	26·7207784	8·937843
643	413449	265847707	25·3574447	8·631183	715	511225	365525875	26·7394839	8·942014
644	414736	267089984	25·3771551	8·635655	716	512656	367061696	26·7581763	8·946181
645	416025	268336125	25·3968502	8·640123	717	514089	368601813	26·7768557	8·950344
646	417316	269586136	25·4165301	8·644585	718	515524	370146232	26·7955220	8·954503
647	418609	270840023	25·4361947	8·649044	719	516961	371694959	26·8141754	8·958658
648	419904	272097792	25·4558441	8·653497	720	518400	373248000	26·8328157	8·962809

No.	Square.	Cube.	Sq. Root.	Cube Root	No.	Square.	Cube.	Sq. Root.	Cube Root
721	519841	374805361	26·8514432	8·966957	793	628849	498677257	28·1602557	9·256022
722	521284	376367048	26·8700577	8·971101	794	630436	500566184	28·1780056	9·259911
723	522729	377933067	26·8886593	8·975241	795	632025	502459875	28·1957444	9·263797
724	524176	379503424	26·9072481	8·979377	796	633616	504358336	28·2134720	9·267680
725	525625	381078125	26·9258240	8·983509	797	635209	506261573	28·2311884	9·271559
726	527076	382657176	26·9443872	8·987637	798	636804	508169592	28·2488938	9·275435
727	528529	384240583	26·9629375	8·991762	799	638401	510082399	28·2665881	9·279308
728	529984	385828352	26·9814751	8·995883	800	640000	512000000	28·2842712	9·283178
729	531441	387420489	27·0000000	9·000000	801	641601	513922401	28·3019434	9·287044
730	532900	389017000	27·0185122	9·004113	802	643204	515849608	28·3196045	9·290907
731	534361	390617891	27·0370117	9·008223	803	644809	517781627	28·3372546	9·294767
732	535824	392223168	27·0554985	9·012329	804	646416	519718464	28·3548938	9·298624
733	537289	393832837	27·0739727	9·016431	805	648025	521660125	28·3725219	9·302477
734	538756	395446904	27·0924344	9·020529	806	649636	523606616	28·3901391	9·306328
735	540225	397065375	27·1108834	9·024624	807	651249	525557943	28·4077454	9·310175
736	541696	398688256	27·1293199	9·028715	808	652864	527514112	28·4253408	9·314019
737	543169	400315553	27·1477439	9·032802	809	654481	529475129	28·4429253	9·317860
738	544644	401947272	27·1661554	9·036886	810	656100	531441000	28·4604989	9·321697
739	546121	403583419	27·1845544	9·040965	811	657721	533411731	28·4780617	9·325532
740	547600	405224000	27·2029410	9·045042	812	659344	535387328	28·4956137	9·329363
741	549081	406869021	27·2213152	9·049114	813	660969	537367797	28·5131549	9·333192
742	550564	408518488	27·2396769	9·053183	814	662596	539353144	28·5306852	9·337017
743	552049	410172407	27·2580263	9·057248	815	664225	541343375	28·5482048	9·340839
744	553536	411830784	27·2763634	9·061310	816	665856	543338496	28·5657137	9·344657
745	555025	413493625	27·2946881	9·065368	817	667489	545338513	28·5832119	9·348473
746	556516	415160936	27·3130006	9·069422	818	669124	547343432	28·6006993	9·352286
747	558009	416832723	27·3313007	9·073473	819	670761	549353259	28·6181760	9·356095
748	559504	418508992	27·3495887	9·077520	820	672400	551368000	28·6356421	9·359902
749	561001	420189749	27·3678644	9·081563	821	674041	553387661	28·6530976	9·363705
750	562500	421875000	27·3861279	9·085603	822	675684	555412248	28·6705424	9·367505
751	564001	423564751	27·4043792	9·089639	823	677329	557441767	28·6879766	9·371302
752	565504	425259008	27·4226184	9·093672	824	678976	559476224	28·7054002	9·375096
753	567009	426957777	27·4408455	9·097701	825	680625	561515625	28·7228132	9·378887
754	568516	428661064	27·4590604	9·101726	826	682276	563559976	28·7402157	9·382675
755	570025	430368875	27·4772633	9·105748	827	683929	565609283	28·7576077	9·386460
756	571536	432081216	27·4954542	9·109767	828	685584	567663552	28·7749891	9·390242
757	573049	433798093	27·5136330	9·113782	829	687241	569722789	28·7923601	9·394021
758	574564	435519512	27·5317998	9·117793	830	688900	571787000	28·8097206	9·397796
759	576081	437245479	27·5499546	9·121801	831	690561	573856191	28·8270706	9·401569
760	577600	438976000	27·5680975	9·125805	832	692224	575930368	28·8444102	9·405339
761	579121	440711081	27·5862284	9·129806	833	693889	578009537	28·8617394	9·409105
762	580644	442450728	27·6043475	9·133803	834	695556	580093704	28·8790582	9·412869
763	582169	444194947	27·6224546	9·137797	835	697225	582182875	28·8963666	9·416630
764	583696	445943744	27·6405499	9·141787	836	698896	584277056	28·9136646	9·420387
765	585225	447697125	27·6586334	9·145774	837	700569	586376253	28·9309523	9·424142
766	586756	449455096	27·6767050	9·149758	838	702244	588480472	28·9482297	9·427894
767	588289	451217663	27·6947648	9·153737	839	703921	590589719	28·9654967	9·431642
768	589824	452984832	27·7128129	9·157714	840	705600	592704000	28·9827535	9·435388
769	591361	454756609	27·7308492	9·161687	841	707281	594823321	29·0000000	9·439131
770	592900	456533000	27·7488739	9·165656	842	708964	596947688	29·0172363	9·442870
771	594441	458314011	27·7668868	9·169622	843	710649	599077107	29·0344623	9·446607
772	595984	460099648	27·7848880	9·173585	844	712336	601211584	29·0516781	9·450341
773	597529	461889917	27·8028775	9·177544	845	714025	603351125	29·0688837	9·454072
774	599076	463684824	27·8208555	9·181500	846	715716	605495736	29·0860791	9·457800
775	600625	465484375	27·8388218	9·185453	847	717409	607645423	29·1032644	9·461525
776	602176	467288576	27·8567766	9·189402	848	719104	609800192	29·1204396	9·465247
777	603729	469097433	27·8747197	9·193347	849	720801	611960049	29·1376046	9·468966
778	605284	470910952	27·8926514	9·197290	850	722500	614125000	29·1547595	9·472682
779	606841	472729139	27·9105715	9·201229	851	724201	616295051	29·1719043	9·476396
780	608400	474552000	27·9284801	9·205164	852	725904	618470208	29·1890390	9·480106
781	609961	476379541	27·9463772	9·209096	853	727609	620650477	29·2061637	9·483814
782	611524	478211768	27·9642629	9·213025	854	729316	622835864	29·2232784	9·487518
783	613089	480048687	27·9821372	9·216950	855	731025	625026375	29·2403830	9·491220
784	614656	481890304	28·0000000	9·220873	856	732736	627222016	29·2574777	9·494919
785	616225	483736625	28·0178515	9·224791	857	734449	629422793	29·2745623	9·498615
786	617796	485587656	28·0356915	9·228707	858	736164	631628712	29·2916370	9·502308
787	619369	487443403	28·0535203	9·232619	859	737881	633839779	29·3087018	9·505998
788	620944	489303872	28·0713377	9·236528	860	739600	636056000	29·3257566	9·509685
789	622521	491169069	28·0891438	9·240433	861	741321	638277381	29·3428015	9·513370
790	624100	493039000	28·1069386	9·244335	862	743044	640503928	29·3598365	9·517051
791	625681	494913671	28·1247222	9·248234	863	744769	642735647	29·3768616	9·520730
792	627264	496793088	28·1424946	9·252130	864	746496	644972544	29·3938769	9·524406

No.	Square.	Cube.	Sq. Root.	Cube Root	No.	Square.	Cube.	Sq. Root.	Cube Root
865	748225	647214625	29·4108823	9·528079	933	870489	812166237	30·5450487	9·771484
866	749956	649461896	29·4278779	9·531750	934	872356	814780504	30·5614136	9·774974
867	751689	651714363	29·4448637	9·535417	935	874225	817400375	30·5777697	9·778462
868	753424	653972032	29·4618397	9·539082	936	876096	820025856	30·5941171	9·781947
869	755161	656234909	29·4788059	9·542744	937	877969	822656953	30·6104557	9·785429
870	756900	658503000	29·4957624	9·546403	938	879844	825293672	30 6267857	9·788909
871	758641	660776311	29·5127091	9·550059	939	881721	827936019	30·6431069	9·792386
872	760384	663054848	29·5296461	9·553712	940	883600	830584000	30·6594194	9·795861
873	762129	665338617	29·5465734	9·557363	941	885481	833237621	30·6757233	9·799334
874	763876	667627624	29·5634910	9·561011	942	887364	835896888	30·6920185	9·802804
875	765625	669921875	29·5803989	9·564656	943	889249	838561807	30·7083051	9·806271
876	767376	672221376	29·5972972	9·568298	944	891136	841232384	30·7245830	9·809736
877	769129	674526133	29·6141858	9·571938	945	893025	843908625	30·7408523	9·813199
878	770884	676836152	29·6310648	9·575574	946	894916	846590536	30·7571130	9·816659
879	772641	679151439	29·6479342	9·579208	947	896809	849278123	30·7733651	9·820117
880	774400	681472000	29·6647939	9·582840	948	898704	851971392	30·7896086	9·823572
881	776161	683797841	29·6816442	9·586468	949	900601	854670349	30·8058436	9·827025
882	777924	686128968	29·6984848	9·590094	950	902500	857375000	30·8220700	9·830476
883	779689	688465387	29·7153159	9·593717	951	904401	860085351	30·8382879	9·833924
884	781456	690807104	29·7321375	9·597337	952	906304	862801408	30·8544972	9·837369
885	783225	693154125	29·7489496	9·600955	953	908209	865523177	30·8706981	9·840813
886	784996	695506456	29·7657521	9·604570	954	910116	868250664	30·8868904	9·844254
887	786769	697864103	29·7825452	9·608182	955	912025	870983875	30·9030743	9·847692
888	788544	700227072	29·7993289	9·611791	956	913936	873722816	30·9192497	9·851128
889	790321	702595369	29·8161030	9·615398	957	915849	876467493	30·9354166	9·854562
890	792100	704969000	29·8328678	9·619002	958	917764	879217912	30·9515751	9·857993
891	793881	707347971	29·8496231	9·622603	959	919681	881974079	30·9677251	9·861422
892	795664	709732288	29·8663690	9·626202	960	921600	884736000	30·9838668	9·864848
893	797449	712121957	29·8831056	9·629797	961	923521	887503681	31·0000000	9·868272
894	799236	714516984	29·8998328	9·633391	962	925444	890277128	31·0161248	9·871694
895	801025	716917375	29·9165506	9·636981	963	927369	893056347	31·0322413	9·875113
896	802816	719323136	29·9332591	9·640569	964	929296	895841344	31·0483494	9·878530
897	804609	721734273	29·9499583	9·644154	965	931225	898632125	31·0644491	9·881945
898	806404	724150792	29·9666481	9·647737	966	933156	901428696	31·0805405	9·885357
899	808201	726572699	29·9833287	9·651317	967	935089	904231063	31·0966236	9·888767
900	810000	729000000	30·0000000	9·654894	968	937024	907039232	31·1126984	9·892175
901	811801	731432701	30·0166620	9·658468	969	938961	909853209	31·1287648	9·895580
902	813604	733870808	30·0333148	9·662040	970	940900	912673000	31·1448230	9·898983
903	815409	736314327	30·0499584	9·665610	971	942841	915498611	31·1608729	9·902383
904	817216	738763264	30·0665928	9·669176	972	944784	918330048	31·1769145	9·905782
905	819025	741217625	30·0832179	9·672740	973	946729	921167317	31·1929479	9·909178
906	820836	743677416	30·0998339	9·676302	974	948676	924010424	31·2089731	9·912571
907	822649	746142643	30·1164407	9·679860	975	950625	926859375	31·2249900	9·915962
908	824464	748613312	30·1330383	9·683417	976	952576	929714176	31·2409987	9·919351
909	826281	751089429	30·1496269	9·686970	977	954529	932574833	31·2569992	9·922738
910	828100	753571000	30·1662063	9·690521	978	956484	935441352	31·2729915	9·926122
911	829921	756058031	30·1827765	9·694069	979	958441	938313739	31·2889757	9·929504
912	831744	758550528	30·1993377	9·697615	980	960400	941192000	31·3049517	9·932884
913	833569	761048497	30·2158899	9·701158	981	962361	944076141	31·3209195	9·936261
914	835396	763551944	30·2324329	9·704699	982	964324	946966168	31·3368792	9·939636
915	837225	766060875	30·2489669	9·708237	983	966289	949862087	31·3528308	9·943009
916	839056	768575296	30·2654919	9·711772	984	968256	952763904	31·3687743	9·946380
917	840889	771095213	30·2820079	9·715305	985	970225	955671625	31·3847097	9·949748
918	842724	773620632	30·2985148	9·718835	986	972196	958585256	31·4006369	9·953114
919	844561	776151559	30·3150128	9·722363	987	974169	961504803	31·4165561	9·956477
920	846400	778688000	30·3315018	9·725888	988	976144	964430272	31·4324673	9·959839
921	848241	781229961	30·3479818	9·729411	989	978121	967361669	31·4483704	9·963198
922	850084	783777448	30·3644529	9·732931	990	980100	970299000	31·4642654	9·966555
923	851929	786330467	30·3809151	9·736448	991	982081	973242271	31·4801525	9·969909
924	853776	788889024	20·3973683	9·739963	992	984064	976191488	31·4960315	9·973262
925	855625	791453125	30·4138127	9·743476	993	986049	979146657	31·5119025	9·976612
926	857476	794022776	30·4302481	9·746986	994	988036	982107784	31·5277655	9·979960
927	859329	796597983	30·4466747	9·750493	995	990025	985074875	31·5436206	9·983305
928	861184	799178752	30·4630924	9·753998	996	992016	988047936	31·5594677	9·986649
929	863041	801765089	30·4795013	9·757500	997	994009	991026973	31·5753068	9·989990
930	864900	804357000	30·4959014	9·761000	998	996004	994011992	31·5911380	9·993329
931	866761	806954491	30·5122926	9·764497	999	998001	997002999	31·6069613	9·996666
932	868624	809557568	30·5286750	9·767992	1000	1000000	1000000000	31·6227766	10·000000

END OF VOL. I.

GILBERT & RIVINGTON, Printers, St. John's Square, London.

ERRATA.

Page 160, l. 2, note, *for* $\left\{a + (n - a)d\right\}$ *read* $\left\{a + (n - 1)d\right\}$

— 163, two places, *for* s^n *read* s_n.

— 253, three places, *for* $\frac{1}{3M^3}$ *read* $\frac{1}{3N^3}$.

— 380, note, l. 10, b. *for* slip *read* step.

— 417, l. 4, bottom, *for* F′ *read* D′.

— 469, l. 3, *for* described, *read* escribed.

— 471, last line, *for* A *read* C.

— 472, note, *for* $a^2 + b^2 - c^2$, *read* $(a^2 + b^2 - c^2) \sin C$, in a few of the copies.

— 478, l. 8, bottom, *after* page , *insert* 471.